Computation of the first Lyapunov quantity for the second-order dynamical system
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Abstract. A new direct method for the computation of Lyapunov quantities (Lyapunov values or coefficients, Poincare-Lyapunov constants, focus values) for the second-order dynamical system, permitting us to narrow the requirements on a smoothness of system, is obtained.

Keywords: Lyapunov value or coefficient, Poincare-Lyapunov constant, focus value, symbolic computations, small limit cycle, 16th Hilbert problem.

1 Introduction
The classical method for the computation of Lyapunov quantities involves the introduction of the polar coordinates and the reduction of original system to normal form [Lyapunov, 1892; Bautin, 1962; Lloyd & Pearson, 1990; Yu, 1998; Lynch, 2005]. In the present work the substantially different method, not requiring the direct reduction to normal form, is proposed. The quality of this method is ideological simplicity and visualization. In this method a less smoothness of the right-hand sides of differential equations in comparison with classical consideration is required. We follow here ideas, developed in [Leonov 2006, 2007].

2 Computation of Lyapunov quantity
Consider a system
\[
\begin{align*}
\dot{x} &= -y + u_f(t), \\
\dot{y} &= x + u_g(t).
\end{align*}
\]  
(2.1)
The solution of this system with initial data \(x(0) = 0, y(0) = 0\) is as follows
\[
\begin{align*}
x &= u_g(0) \cos(t) + \\
&\quad + \cos(t) \int_0^t \cos(\tau)(u'_{g}(\tau) + u_f(\tau))d\tau + \\
&\quad + \sin(t) \int_0^t \sin(\tau)(u'_{g}(\tau) + u_f(\tau))d\tau - u_g(t) \\
y &= u_g(0) \sin(t) + \\
&\quad + \sin(t) \int_0^t \cos(\tau)(u'_{g}(\tau) + u_f(\tau))d\tau - \\
&\quad - \cos(t) \int_0^t \sin(\tau)(u'_{g}(\tau) + u_f(\tau))d\tau
\end{align*}
\]  
(2.2)
Consider now the equations

\[
\begin{align*}
\dot{x} &= -y + f(x, y) \\
y &= x + g(x, y)
\end{align*}
\]  

(2.3)

Here \(f(0, 0) = g(0, 0) = 0\) and in a certain neighborhood of the point \((x, y) = (0, 0)\) the functions \(f(\cdot, \cdot)\) and \(g(\cdot, \cdot)\) have partial derivatives up to the order 2 and \(f_x'(0, 0) = f_y'(0, 0) = g_x'(0, 0) = g_y'(0, 0) = 0\).

Further we shall use a smoothness of the functions \(f\) and \(g\) and follow the first Lyapunov method on finite time interval [Lefschetz, 1957; Cesari, 1959].

Since the functions \(f\) and \(g\) are assumed to be smooth, we can write

\[
\begin{align*}
f(x, y) &= f_{20}x^2 + f_{11}xy + f_{02}y^2 + o((|x| + |y|)^2) = \\
&= f_2(x, y) + o((|x| + |y|)^2), \\
g(x, y) &= g_{20}x^2 + g_{11}xy + g_{02}y^2 + o((|x| + |y|)^2) = \\
&= g_2(x, y) + o((|x| + |y|)^2).
\end{align*}
\]  

(2.4)

in a certain neighborhood of the point \((0,0)\).

Consider the solution

\[
x(t, h) = x(t, x(0), y(0)), \quad y(t, h) = y(t, x(0), y(0))
\]

of system (2.3) with the initial data

\[
\begin{align*}
x(0, x(0), y(0)) &= 0, \\
y(0, x(0), y(0)) &= h.
\end{align*}
\]  

(2.5)

From the equations

\[
\begin{align*}
\dot{x}_1 &= -y_1, \quad x_1(0, h) = 0, \\
\dot{y}_1 &= x_1, \quad y_1(0, h) = h
\end{align*}
\]  

(2.6)

for the first approximation \(x_1(t, h), y_1(t, h)\) of the solution \(x(t, x(0), y(0)), y(t, x(0), y(0))\) we have

\[
x_1(t, h) = -h \sin(t), \quad y_1(t, h) = h \cos(t).
\]

By the assumption that \(f, g\) is smooth, we obtain that the right-hand side of system (2.3) has 2 continuous partial derivatives with respect to \(x\) and \(y\). Then [Hartman, 1964] the solution of system (2.3), i.e. \(x(t, h), y(t, h)\), has partial derivatives up to the order 2 with respect to the initial data \(h\).

We shall seek sequential approximations for \(x(t, h), y(t, h)\) in the form of the sums

\[
\begin{align*}
x_2(t, h) &= x_1(t)h + x_2(t)h^2, \quad x_2(0) = 0, \\
y_2(t, h) &= y_1(t)h + y_2(t)h^2, \quad y_2(0) = 0.
\end{align*}
\]  

(2.7)

Here, in according to the local Taylor formula, at the fixed moment of time \(t = t^*\) the following representation holds

\[
\begin{align*}
x(t^*, h) &= x_2(t^*, h) + o(h^2), \\
y(t^*, h) &= y_2(t^*, h) + o(h^2).
\end{align*}
\]  

(2.8)

Substituting (2.7) in (2.4) and then in (2.3) and determining the coefficients \(u_2^f(t)\) and \(u_2^g(t)\) of \(h^2\) in \(f(x_1(t, h), y_1(t, h))\) and \(g(x_1(t, h), y_1(t, h))\), respectively, we obtain the approximations

\[
\begin{align*}
u_2^f(t, h) &= u_2^f(t)h^2, \\
u_2^g(t, h) &= u_2^g(t)h^2.
\end{align*}
\]  

(2.9)

For determining \(x_2(t), y_2(t)\) we have the equations

\[
\begin{align*}
\dot{x}_2 &= -y_2 + u_2^f(t) \\
\dot{y}_2 &= x_2 + u_2^g(t).
\end{align*}
\]  

(2.10)
Taking into account (2.2), we can find solutions of these equations. They take the form
\[
x_2(t) = \frac{1}{3} \left[-(\cos(t) - 1)^2 g_{20} - \sin(t)(\cos(t) - 1)g_{11} + \\
+ (\cos(t) + \cos(t)^2 - 2)g_{02} - (\sin(2t) - 2 \sin(t))f_{20} - \\
- (\cos(t) - \cos(2t))f_{11} + (\sin(2t) + \sin(t))f_{02} \right] \\
y_2(t) = \frac{1}{3} \left[-(\sin(2t) - 2 \sin(t))g_{20} - (\cos(t) - \cos(2t))g_{11} + \\
+ (\sin(2t) + \sin(t))g_{02} + (\cos(t) - 1)^2 f_{20} + \\
+ \sin(t)(\cos(t) - 1)f_{11} - (\cos(t) - 2 + \cos(t)^2) f_{02} \right]
\]

Here \(x_2(0) = y_2(0) = x_2(2\pi) = y_2(2\pi) = 0\).

**Lemma.** Suppose
\[
\begin{align*}
x_1(2\pi) &= 0, \\
y_1(2\pi) &= 1, \\
x_2(2\pi) &= y_2(2\pi) = 0.
\end{align*}
\] (2.11)

Then on a phase plane for sufficiently small \(h\), the solution \(x(t, h), y(t, h)\) crosses the half-line \((x = 0, y > 0)\) at time
\[
T = 2\pi + o(h).
\] (2.12)

**Proof.**
Since \(x_2(2\pi, h) = 0\) and \(y_2(2\pi, h) = h\), we conclude that on a phase plane at time \(t = 2\pi\) the trajectory \((x(t, h), y(t, h))\) lies in the neighborhood, of radius \(o(h^2)\) of the point \((x = 0, y = h)\) (2.8).

At the fixed moment of time \(t = t^*\), we have [Hartman, 1964 and (2.8)]
\[
\dot{x}(t^*, h) = -h \cos t^* + o(h).
\]

Since \(\dot{x}(t, h)\) is bounded with respect to \(h\) and \(t\) in a certain neighborhood of \((x = 0, y = h)\) and \(t = 2\pi\), respectively, for \(t\) in a certain neighborhood of the point \(2\pi\) and sufficiently small \(h\) we obtain the following inequality
\[
\dot{x}(t, h) \leq -ch
\]
for a certain \(c > 0\).

Hence
\[
T = 2\pi + o(h).
\]

Consider now a function
\[
V(x, y) = x^2 + y^2.
\] (2.13)

We remark that for the derivative of the function \(V\) along the solutions of system (2.3), the relation
\[
\dot{V}(x, y) = 2xf(x, y) + 2yg(x, y)
\] (2.14)
is valid.

Introduce the following notation
\[
L = V(x(T, h), y(T, h)) - V(x(0, h), y(0, h)).
\] (2.15)
Integrating (2.14) from 0 to $T = 2\pi + o(h)$, we obtain

$$L = \int_0^T \dot{V}(x(t,h), y(t,h)) dt = \int_0^{2\pi} \dot{V}(x(t,h), y(t,h)) dt + o(h^4).$$

Substituting (2.14) in this relation, we have

$$L = \int_0^{2\pi} 2x_2(t,h)f_2(x_2(t,h), y_2(t,h)) + 2y_2(t,h)g_2(x_2(t,h), y_2(t,h)) dt + o(h^4).$$

(2.16)

Substituting $x_2(t,h), y_2(t,h)$ in $f_2(x,y), g_2(x,y)$ and then in (2.16) and using terms grouping up to the order $h^4$, we find

$$L = L_1 h^4 + o(h^4),$$

(2.17)

where $L_1/2$ is the 1th Lyapunov quantity $L_1$:

$$L_1 = \frac{\pi}{4} (f_{11}f_{02} + 2f_{02}g_{02} - 2f_{20}g_{20} - g_{11}g_{20} - g_{11}g_{02} + f_{11}f_{20})$$

Here the sign $L_1$ characterizes an unwinding or a twisting of trajectory of the system $(x(t,h), y(t,h))$ on a phase plane.

We stress that for the computation of $L_1$ it is sufficient that in the neighborhood of considered stationary point the relation $f, g \in C^2$ is satisfied, what is one less than conventional assumptions on a smoothness [Marsden & McCracken, 1976].

3 Conclusion

In conclusion, we note that there is a wide class of polynomial systems, for which by the proposed technique small cycles can be constructed (see, for example, [Bautin, 1952; Leonov, 1998; Lloyd & Pearson, 1997; Lynch, 2005; Yu & Han, 2005] and others).

References


