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Evaluation of the Mean Interdeparture Time in Tandem Queueing Systems[1]

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Abstract

The problem of exact evaluation of the mean customers interdeparture time in tandem systems of single-server queues with both infinite and finite buffers is considered. We give some general conditions for the interdeparture time to exist, and show how it can be calculated.

Keywords: tandem queueing systems, mean interdeparture time, recursive equations, independent random variables

1 Introduction

We consider tandem systems of single-server queues with both infinite and finite buffers. The interarrival and service times of customers are assumed to form two sequences of independent and identically distributed (i.i.d.) random variables (r.v.'s). Given the mean values of interarrival and service times, we are interested in evaluating the mean interdeparture time of customers from the system as the number of customers tends to infinity.

In this paper, we give general conditions for the mean interdeparture time in tandem queueing systems to exist, and show how it can be calculated. In Section 2, we introduce some notations, and consider recursive equations describing the dynamics of tandem queueing systems with infinite buffers. Section 3 presents preliminary results including an existence theorem and some inequalities. Our main result which provides general existence conditions and a simple expression for calculating the mean interdeparture time is included in Section 4. Finally, in Section 5, we consider evaluation of the mean interdeparture time for particular tandem systems with finite buffers.

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2 Tandem Queues with Infinite Buffers

We consider a series of M queues with infinite buffers. Each customer arriving into the system is placed in the buffer at the 1^{st} server and then has to pass through all the queues one after the other. Upon completion of his service at server i, the customer goes to queue $i+1$, $i=1,\ldots,M-1$, and occupies the $(i+1)^{\text{st}}$ server provided that it is free. If the customer finds this server busy, he enters the buffer so as to wait until the service of all his predecessors is completed.

Denote the random time between the arrivals of the n^{th} customer and his predecessor by τ_{0n}, and the random service time of the n^{th} customer at server i by τ_{in}, $i=1,\ldots,M$, $n=1,2,\ldots$. Furthermore, let $D_0(n)$ be the n^{th} arrival epoch to the system, and $D_i(n)$ be the n^{th} departure epoch from the i^{th} server.

With the natural condition $D_i(0)=0$ for all $i=0,\ldots,M$, the recursive equations representing the system dynamics can be written for all $n=1,2,\ldots$ as

$$\begin{aligned} D_0(n) &= D_0(n-1)+\tau_{0n}, \\ D_m(n) &= \max(D_{m-1}(n), D_m(n-1))+\tau_{mn}, \quad m=1,\ldots,M. \end{aligned}$$

The above recursions can be resolved for each $m=1,\ldots,M$, to get

$$D_m(n) = \max_{1\le k_1\le\cdots\le k_m\le n}\left\{\sum_{j=1}^{k_1}\tau_{0j}+\sum_{j=k_1}^{k_2}\tau_{1j}+\cdots+\sum_{j=k_m}^{n}\tau_{mj}\right\}. \qquad (1)$$

We are interested in evaluating the mean interdeparture time of customers from the system, which can be defined as

$$\gamma = \lim_{n\to\infty}\frac{1}{n}D_M(n). \qquad (2)$$

3 Preliminary Results

In order to examine the existence of the mean interdeparture time for the tandem system, we will apply the next classical theorem which has been proved in [3].

Theorem 1. *Let $\{\zeta_{ln}|\, l,n=0,1,\ldots;\, l<n\}$ be a family of r.v.'s, such that*

1) $\zeta_{ln}\le\zeta_{lk}+\zeta_{kn}$ for all $l<k<n$;

2) the joint distributions are the same for both families $\{\zeta_{ln}|l<n\}$ and $\{\zeta_{l+1,n+1}|l<n\}$;

3) for all $n=1,2,\ldots$, there exists $E[\zeta_{0n}]\ge -cn$ for some constant $c>0$.

Then there exists a constant γ, such that with probability one (w.p.1),

$$\lim_{n\to\infty}\zeta_{0n}/n=\gamma, \quad \textit{and} \quad \lim_{n\to\infty}E[\zeta_{0n}]/n=\gamma.$$

Let us now consider some useful inequalities. In what follows, we assume $\xi_1,\ldots,\xi_n$ to be independent r.v.'s. We start with a classical result presented in [4], which provides an upper bound on the mean value of the maximum of sums

$$\zeta_k=\xi_1+\cdots+\xi_k$$

of independent r.v.'s with zero means.

Lemma 1. *If* $E[\xi_k] = 0$, *and* $E|\xi_k|^p < \infty$ *for some* $p > 1$, $k = 1, \dots, n$, *then it holds*

$$E\left[\max_{1\le k\le n} |\zeta_k|\right]^p \le 2\left(\frac{p}{p-1}\right)^p E|\zeta_n|^p.$$

The next inequality has been derived in [5] under somewhat weaker conditions than that of independence between the r.v.'s $\xi_1, \dots, \xi_n$.

Lemma 2. *If* $E[\xi_k] = 0$, *and* $E|\xi_k|^p < \infty$ *for some* p, $1 \le p \le 2$, $k = 1, \dots, n$, *then it holds*

$$E\,|\zeta_n|^p \le \left(2 - \frac{1}{n}\right)\sum_{k=1}^n E|\xi_k|^p.$$

With Lemmas 1 and 2, one can prove the next result.

Lemma 3. *If* $E[\xi_k] = 0$, *and* $E[\xi_k^2] < \infty$, $k = 1, \dots, n$, *then it holds*

$$E\left[\max_{1\le k\le n} \zeta_k\right] \le 2\sqrt{\frac{2(2n-1)}{n}}\left(\sum_{k=1}^n E[\xi_k^2]\right)^{1/2}.$$

Now assume $\xi_1, \dots, \xi_n$ to be i.i.d. r.v.'s. The above inequality takes the form

$$E\left[\max_{1\le k\le n} \zeta_k\right] \le 2\sqrt{2(2n-1)E[\xi_1^2]}.$$

The next result has been obtained in [1, 2].

Lemma 4. *If* $E[\xi_1] < \infty$ *and* $D[\xi_1] < \infty$, *then it holds*

$$E\left[\max_{1\le k\le n} \xi_k\right] \le E[\xi_1] + \frac{n-1}{\sqrt{2n-1}}\sqrt{D[\xi_1]}.$$

Assuming $\xi_1, \dots, \xi_n$ to be i.i.d. r.v.'s, let us introduce the notation

$$\zeta_{lk} = \xi_l + \xi_{l+1} + \cdots + \xi_k$$

with $1 \le l \le k \le n$. It is not difficult to verify the following statement.

Lemma 5. *If* $E[\xi_1] = a \le 0$, *and* $D[\xi_1] < \infty$, *then it holds*

$$E\left[\max_{1\le l\le k\le n} \zeta_{lk}\right] \le E[\xi_1] + \left(4\sqrt{2(2n-1)} + \frac{n-1}{\sqrt{2n-1}}\right)\sqrt{D[\xi_1]}.$$

4 Evaluation of the Mean Interdeparture Time

We are now in a position to prove the next theorem.

Theorem 2. *Suppose that* $\{\tau_{in} |\ n = 1, 2, \dots\}$, $i = 0, 1, \dots, M$, *are mutually independent sequences of i.i.d. r.v.'s with* $0 \le E[\tau_{i1}] < \infty$.

Then the limit at (2) exists w.p.1, and if $D[\tau_{i1}] < \infty$, *it is given by*

$$\gamma = \max_{0\le i\le M} E[\tau_{i1}]. \tag{3}$$

Proof. In order to verify the existance of the limit at (2), let us denote

$$\zeta_{ln} = \max_{l<k_1\leq\cdots\leq k_M\leq n} \left\{ \sum_{j=l+1}^{k_1} \tau_{0j} + \sum_{j=k_1}^{k_2} \tau_{2j} + \cdots + \sum_{j=k_M}^{n} \tau_{Mj} \right\} \qquad (4)$$

for each $l, n,\ 0 \leq l < n$, and note that we can now write $D_M(n) = \zeta_{0n}$.

Clearly, the family $\{\zeta_{ln} |\ l < n\}$ defined by (4) satisfies the conditions of Theorem 1. Therefore, the limit at (2) exists w.p.1, and it can be calculated as

$$\gamma = \lim_{n\to\infty} \frac{1}{n} E[D_M(n)].$$

Suppose that the maximum at (3) is achieved at some $i = m$. Consider the completion time $D_M(n)$ and represent it in the form

$$D_M(n) = \max_{1\leq k_1\leq\cdots\leq k_M\leq n} \left\{ \sum_{j=1}^{k_1} \tau_{0j} + \sum_{j=k_1}^{k_2} \tau_{1j} + \cdots + \sum_{j=k_M}^{n} \tau_{Mj} \right\} = \sum_{j=1}^{n} \tau_{mj} + \mu,$$

where

$$\begin{aligned} \mu \quad = \quad & \max_{1\leq k_1\leq\cdots\leq k_M\leq n} \left\{ \sum_{j=1}^{k_1} (\tau_{0j} - \tau_{mj}) + \sum_{j=k_1}^{k_2} (\tau_{1j} - \tau_{mj}) + \cdots + \sum_{j=k_M}^{n} (\tau_{Mj} - \tau_{mj}) \right. \\ & \left. + \tau_{mk_1} + \cdots + \tau_{mk_M} \right\}. \end{aligned} \qquad (5)$$

Now we can write

$$\frac{1}{n} E[D_M(n)] = E[\tau_{m1}] + \frac{1}{n} E[\mu].$$

Let us examine the expected value $E[\mu]$. With $k_1 = \cdots = k_m = 1$, and $k_{m+1} = \cdots = k_M = n$, we have from (5)

$$\mu \geq \tau_{01} + \tau_{11} + \cdots + \tau_{m-1,1} + \tau_{m+1,n} + \cdots + \tau_{Mn} \geq 0,$$

and so $E[\mu] \geq 0$. On the other hand, we have

$$\begin{aligned} \mu \quad \leq \quad & M \max(\tau_{m1}, \ldots, \tau_{mn}) + \max_{1\leq k_1\leq n} \sum_{j=1}^{k_1} (\tau_{0j} - \tau_{mj}) \\ & + \max_{1\leq k_1\leq k_2\leq n} \sum_{j=k_1}^{k_2} (\tau_{1j} - \tau_{mj}) + \cdots + \max_{1\leq k_M\leq n} \sum_{j=k_M}^{n} (\tau_{Mj} - \tau_{mj}). \end{aligned}$$

With the condition that $E(\tau_{i1} - \tau_{m1}) \leq 0$ for all $i = 0, 1, \ldots, M$, one can apply Lemma 5 to the first $M + 1$ terms on the right-hand side, and then Lemma 4 to the last one so as to get an upper bound:

$$E[\mu] \leq \sum_{\substack{i=0 \\ i\neq m}}^{M} E[\tau_{i1}] + O(\sqrt{n}).$$

Finally, we have the double inequality

$$E[\tau_{m1}] \le \frac{1}{n} E[D_M(n)] \le E[\tau_{m1}] + \frac{1}{n} \sum_{\substack{i=0 \\ i \ne m}}^{M} E[\tau_{i1}] + \frac{O(\sqrt{n})}{n},$$

and with $n \to \infty$, immediately arrive at (5).

5 Tandem Queues with Finite Buffers

In this section, we show how the above approach can be applied to the analysis of tandem systems which include queues with finite buffers. Because of limited buffer capacity, servers in the systems may be blocked according to a blocking rule. Below we present examples of systems with manufacturing blocking and communication blocking which are most commonly encountered in practice.

Let us consider a system which consists of two queues in tandem, and suppose that the buffer at the first server is infinite, while that at the second server is finite.

First we assume the system to operate under the manufacturing blocking rule. With this type of blocking, if upon completion of a service, the first server sees the buffer of the second server full, it has to remain busy until the second server completes its current service to provide a free space in its buffer.

Let the finite buffer have capacity 0. With the notations introduced above, one can represent the dynamics of the system by the equations

$$\begin{aligned} D_0(n) &= D_0(n-1) + \tau_{0n}, \\ D_1(n) &= \max(\max(D_0(n), D_1(n-1)) + \tau_{1n}, D_2(n-1)), \\ D_2(n) &= \max(D_1(n), D_2(n-1)) + \tau_{2n}. \end{aligned} \tag{6}$$

Note that from the second equation, we have $D_1(n) \ge D_2(n-1)$. Therefore, the third equation can be reduced to

$$D_2(n) = D_1(n) + \tau_{2n},$$

and both $E[D_1(n)]/n$ and $E[D_2(n)]/n$ have a common limit γ as $n \to \infty$.

By resolving the recursive equations, we get

$$D_1(n) = \max_{1 \le k \le n} \left\{ \sum_{j=1}^{k} \tau_{0j} + \tau_{1k} + \sum_{j=k}^{n-1} \max(\tau_{1,j+1}, \tau_{2j}) \right\}.$$

As it is easy to verify, $D_1(n)$ satisfies the double inequality

$$L(n) - \max(\tau_{1,n+1}, \tau_{2n}) \le D_1(n) \le U(n), \tag{7}$$

where

$$L(n) = \max_{1 \le k \le n} \left\{ \sum_{j=1}^{k} \tau_{0j} + \sum_{j=k}^{n} \max(\tau_{1,j+1}, \tau_{2j}) \right\},$$

$$U(n) \quad = \quad \max_{1\le k\le n}\left\{\sum_{j=1}^{k}\tau_{0j}+\sum_{j=k}^{n}\max(\tau_{1j},\tau_{2,j-1})\right\}.$$

Taking into account that both $L(n)$ and $U(n)$ actually have the form of (1), one can see that, under the same conditions as in Theorem 2, it holds

$$\lim_{n\to\infty}\frac{1}{n}E[L(n)]=\lim_{n\to\infty}\frac{1}{n}E[U(n)]=\max(E[\tau_{01}],E\max(\tau_{11},\tau_{21})).$$

Finally, proceeding to mean value in both sides of (7), divided by n, we conclude that the mean interdeparture time is given by

$$\gamma=\max(E[\tau_{01}],E\max(\tau_{11},\tau_{21})).$$

Let us assume the system to follow the communication blocking rule. This type of blocking requires a server not to initiate service of a customer if the buffer of the next server is full. With the finite buffer having capacity 0, we have the same recursions as above, except for equation (6) which now takes the form

$$D_1(n)=\max(D_0(n),D_1(n-1),D_2(n-1))+\tau_{1n}.$$

Resolving the recursive equations leads us to the expression

$$D_2(n)=\max_{1\le k\le n}\left\{\sum_{j=1}^{k}\tau_{0j}+\sum_{j=k}^{n}(\tau_{1j}+\tau_{2j})\right\}.$$

Under the same conditions as in Theorem 2, we get

$$\gamma=\max(E[\tau_{01}],E[\tau_{11}]+E[\tau_{21}]).$$

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