Proc. 4th St.Petersburg Workshop on Simulation, St.Petersburg, Russia, 2001, 304-309

Products of Random Matrices and Queueing System Performance Evaluation[1]

N.K. Krivulin[2]

St.Petersburg State University[3][4]

Abstract

We consider (max,+)-algebra products of random matrices, which arise from performance evaluation of acyclic fork-join queueing networks. A new algebraic technique to examine properties of the product and investigate its limiting behaviour is proposed based on an extension of the standard matrix (max,+)-algebra by endowing it with the ordinary matrix addition as an external operation. As an application, we derive bounds on the (max,+)-algebra maximal Lyapunov exponent which can be considered as the cycle time of the networks.

Keywords: (max,+)-algebra, product of random matrices, maximal Lyapunov exponent, acyclic fork-join networks, cycle time

1 Introduction

We consider $(\max, +)$-algebra products of random matrices arising from performance evaluation of acyclic fork-join queueing networks. The problem is to examine limiting behaviour of the product so as to evaluate its limiting matrix and the maximal Lyapunov exponent normally referred to as the system cycle time.

In order to investigate the products, we develop a pure algebraic technique similar to those involved in the conventional linear algebra. The technique is based on an extension of the standard matrix $(\max, +)$-algebra [8, 2, 1, 7] by endowing it with the ordinary matrix addition as an external operation. New properties of the extended algebra are then established in the form of inequalities, which may find

[1]The work was partially supported by the Russian Foundation for Basic Research, Grant #00-01-00760.

[2]E-mail: Nikolai.Krivulin@pobox.spbu.ru, http://www.math.spbu.ru/~krivulin

[3]Bibliotechnaya Sq. 2, Petrodvorets, 198904 St.Petersburg, Russia

[4]The writing of this paper was completed while the author visiting the Microcomputer Study Center at St. Cloud State University.

their applications beyond of the scope of the current topic. We conclude the paper with an example of application of the proposed technique to establish bounds on the cycle time and on its related limiting matrix in fork-join queueing networks.

In fact, there exist similar results on evaluation of the Lyapunov exponent (see, e.g., [1] and references therein). However, they are essentially based on the description of system dynamics and related proofs made in terms of either Petri nets or stochastic events graphs. On the contrary, we exploit a different approach (see [6] for farther details) based on pure algebraic techniques. It allows one to write and handle the dynamic equations directly without having recourse to an intermediate description in the Petri nets or in another tedious language.

2 Motivating Example and Algebraic Model

Consider a network of n nodes, with its topology described by an oriented acyclic graph. The nodes that have no predecessors are assumed to represent an infinite external arrival stream of customers. Each node without successors is considered as an output node which releases customers from the network.

Each node has a server and infinite buffer operating as a single-server queue under the first-come, first-served discipline. At the initial time, the servers and their buffers are assumed to be free of customers, except for the buffers in nodes with no predecessors, each assumed to have an infinite number of customers.

The operation of each node can include join and fork operations which are performed respectively before and after service. The join operation is actually thought to cause each customer which comes into a node not to enter the queue but to wait until at least one customer from all preceding nodes arrives. Upon arrival, these customers are replaced by a new customer which joins the queue.

The fork operation at a node is initiated every time the service of a customer is completed. It consists in replacing the customer by several new customers, each intended to go to one of the subsequent nodes.

For the queue at node i, we denote the kth departure epochs by $x_i(k)$, and the kth service time by τ_{ik}. We assume τ_{ik} to be a given nonnegative random variable (r.v.) for all $i = 1, \ldots, n$, and $k = 1, 2, \ldots$

We are interested in evaluating the limit

$$\gamma = \lim_{k\to\infty} \frac{1}{k} \max_i x_i(k),$$

which is normally referred to as the cycle time of the network.

In order to represent the network dynamics in a form suitable for further analysis, we exploit the idempotent $(\max,+)$-algebra based approach developed in [6].

The $(\max,+)$-algebra [8, 2, 1] presents a triple $\langle R_\varepsilon, \oplus, \otimes \rangle$ with $R_\varepsilon = R \cup \{\varepsilon\}$, $\varepsilon = -\infty$, and operations $\oplus$ and $\otimes$ defined for all $x, y \in R_\varepsilon$ as

$$x \oplus y = \max(x, y), \quad x \otimes y = x + y.$$

The $(\max,+)$-algebra of matrices is introduced in the ordinary way. The square matrix $\mathcal{E}$ with all its elements equal ε presents the null matrix, whereas the matrix $E = \mathrm{diag}(0, \ldots, 0)$ with ε as its off-diagonal elements is the identity.

Let us denote the vector of the kth customer departures from the network nodes by $\boldsymbol{x}(k) = (x_1(k), \dots, x_n(k))^T$, and introduce the matrix $\mathcal{T}_k = \operatorname{diag}(\tau_{1k}, \dots, \tau_{nk})$ with all its off-diagonal elements equal ε.

As it has been shown in [6], the dynamics of acyclic fork-join networks can be described by the stochastic difference equation

$$\boldsymbol{x}(k) = A(k) \otimes \boldsymbol{x}(k-1), \qquad A(k) = \bigoplus_{j=0}^{p} (\mathcal{T}_k \otimes G^T)^j \otimes \mathcal{T}_k, \tag{1}$$

where G is a matrix with the elements

$$g_{ij} = \begin{cases} 0, & \text{if there exists arc } (i,j) \text{ in the network graph,} \\ \varepsilon, & \text{otherwise,} \end{cases}$$

and p is the length of the longest path in the graph.

The matrix G is normally referred to as the support matrix of the network. Note that since the network graph is acyclic, we have $G^q = \mathcal{E}$ for all $q > p$.

Consider the service cycle time γ. Now we can represent it as

$$\gamma = \lim_{k\to\infty} \frac{1}{k} \|\boldsymbol{x}(k)\|,$$

where $\|\boldsymbol{x}(k)\| = \max_i x_i(k)$.

In order to get information about the growth rate of $\boldsymbol{x}(k)$, we will examine the limiting behaviour of the matrix

$$A_k = A(k) \otimes \cdots \otimes A(1) = \bigotimes_{i=1}^{k} \bigoplus_{j=0}^{p} (\mathcal{T}_k \otimes G^T)^j \otimes \mathcal{T}_k.$$

3 Distributivity Properties and Matrix Products

Let A_{ij} be $(n \times n)$-matrices for all $i = 1, \dots, k$ and $j = 1, \dots, m$. Distributivity of the operation $\otimes$ over $\oplus$ immediately gives the equality

$$\bigotimes_{i=1}^{k} \bigoplus_{j=1}^{m} A_{ij} = \bigoplus_{1 \le j_1, \dots, j_k \le m} A_{1j_1} \otimes \cdots \otimes A_{kj_k}, \tag{2}$$

which leads, in particular, to the inequality

$$\bigotimes_{i=1}^{k} \bigoplus_{j=1}^{m} A_{ij} \ge \bigoplus_{j=1}^{m} \bigotimes_{i=1}^{k} A_{ij}. \tag{3}$$

We consider the ordinary matrix addition $+$ as an external operation, and assume $\otimes$ and $\oplus$ to take precedence over $+$. In a similar way as above, we have

$$\sum_{i=1}^{k} \bigoplus_{j=1}^{m} A_{ij} = \bigoplus_{1 \le j_1, \dots, j_k \le m} (A_{1j_1} + \cdots + A_{kj_k}), \tag{4}$$

$$\sum_{i=1}^{k} \bigoplus_{j=1}^{m} A_{ij} \ge \bigoplus_{j=1}^{m} \sum_{i=1}^{k} A_{ij}. \tag{5}$$

Let G_1 and G_2 be support matrices. For any matrices A and B, we have

$$G_1 \otimes (A + B) \otimes G_2 \leq G_1 \otimes A \otimes G_2 + G_1 \otimes B \otimes G_2. \tag{6}$$

Assume D_1 and D_2 to be diagonal matrices with all off-diagonal elements equal ε. Then for any matrices A and B, it holds

$$D_1 \otimes (A+B) \otimes D_2 = D_1 \otimes A \otimes D_2 + B = D_1 \otimes A + B \otimes D_2 = A + D_1 \otimes B \otimes D_2. \tag{7}$$

Now we examine products of alternating diagonal and support matrices denoted respectively by D and G, which take the form

$$D \otimes \underbrace{(G \otimes D) \otimes \cdots \otimes (G \otimes D)}_{m \text{ times}} = D \otimes (G \otimes D)^m.$$

In order to simplify further formulas, we introduce the following notations

$$\Phi_j(D) = D \otimes (G \otimes D)^j, \qquad \Psi_i^j(D) = G^i \otimes D \otimes G^j.$$

First assume the diagonal matrices to have both positive and negative entries on the diagonal. The next lemma can be proved using (6) and induction on m.

Lemma 1. *It holds that*

$$\Phi_m(D) \leq \sum_{j=0}^{m} \Psi_j^{m-j}(D).$$

Furthermore, assuming D_i, $i = 1, \ldots, k$, to be diagonal matrices, one can obtain the next result based on Lemma 1 and inequality (6).

Lemma 2. *Let $m_1, \ldots, m_k$ be integers, and $m = m_1 + \cdots + m_k$. Then it holds*

$$\bigotimes_{i=1}^{k} \Phi_{m_i}(D_i) \leq \sum_{i=1}^{k} \sum_{j=M_{i-1}}^{M_i} \Psi_j^{m-j}(D_i)$$

with $M_0 = 0$, $M_i = m_1 + \cdots + m_i$, $i = 1, \ldots, k$.

Let the matrices $D_1, \ldots, D_k$ have only nonnegative elements on the diagonal. With (7) and (6), one can prove the next lemma.

Lemma 3. *Suppose that $m_1 + \cdots + m_r = m_{r+1} + \cdots + m_k = m$ with $m - m_r \leq m_{r+1}$ for some r. Then for any integer s such that $m - m_r \leq s \leq m_{r+1}$, it holds*

$$\bigotimes_{i=1}^{r} \Phi_{m_i}(D_i) + \bigotimes_{i=r+1}^{k} \Phi_{m_i}(D_i) \geq \bigotimes_{i=1}^{k} \Phi_{s_i}(D_i)$$

with $s_1 + \cdots + s_k = m$, and

$$s_i = \begin{cases} m_i, & \text{if } 1 \leq i < r, \\ s - m + m_r, & \text{if } i = r, \\ m_{r+1} - s, & \text{if } i = r + 1, \\ m_i, & \text{if } r + 1 < i \leq k. \end{cases}$$

4 Subadditivity Property and Algebraic Bounds

Consider the family $\{A_{lk}^T | l,k=0,1,\dots; l<k\}$ of matrices

$$A_{lk}^T = A^T(l+1)\otimes\cdots\otimes A^T(k), \quad A_{0k}^T = A_k^T.$$

The next lemma states that the family $\{A_{lk}^T\}$ possesses subadditivity property.

Lemma 4. *For all $l<r<k$, it holds*

$$A_{lk}^T \le A_{lr}^T + A_{rk}^T.$$

Proof: By applying (2) and (4), and then Lemma 3, we have

$$A_{lr}^T + A_{rk}^T = \bigotimes_{i=l+1}^{r}\bigoplus_{j=0}^{p}\Phi_j(\mathcal{T}_i) + \bigotimes_{i=r+1}^{k}\bigoplus_{j=0}^{p}\Phi_j(\mathcal{T}_i) \ge \bigoplus_{m=0}^{p}\bigoplus_{s_{l+1}+\cdots+s_k=m}\bigotimes_{i=l+1}^{k}\Phi_{s_i}(\mathcal{T}_i).$$

Finally, since $G^m=\mathcal{E}$ for all $m>p$, we get

$$A_{lr}^T + A_{rk}^T \ge \bigoplus_{0\le s_{l+1},\dots,s_k\le p}\bigotimes_{i=l+1}^{k}\Phi_{s_i}(\mathcal{T}_i) = \bigotimes_{i=l+1}^{k}\bigoplus_{j=0}^{p}\Phi_j(\mathcal{T}_i) = A_{lk}^T.$$

The next lemma offers bounds on A_k^T.

Lemma 5. *It holds that*

$$\bigoplus_{r=0}^{\lfloor p/k\rfloor}\bigotimes_{i=1}^{k}\Phi_r(\mathcal{T}_i) \le A_k^T \le \left\|\bigoplus_{i=1}^{k}\mathcal{T}_i\right\|\otimes\bigoplus_{r=1}^{p}G^r + \sum_{i=1}^{k}\bigoplus_{0\le r+s\le p}\Psi_r^s(\mathcal{T}_i),$$

where $\lfloor r\rfloor$ denotes the greatest integer equal to or less than r.

Proof: The lower bound is an immediate consequence from (3), and the condition that $G^m=\mathcal{E}$ if $m=kr>p$.

In order to derive the upper bound, we first apply (2) to write

$$A_k^T = \bigoplus_{0\le m_1,\dots,m_k\le p}\bigotimes_{i=1}^{k}\Phi_{m_i}(\mathcal{T}_i).$$

Application of Lemma 2 gives

$$\bigotimes_{i=1}^{k}\Phi_{m_i}(\mathcal{T}_i) \le \sum_{i=1}^{k}\sum_{j=M_{i-1}}^{M_i}\Psi_j^{m-j}(\mathcal{T}_i) = \sum_{i=1}^{k}\sum_{j=M_{i-1}+1}^{M_i}\Psi_j^{m-j}(\mathcal{T}_i) + \sum_{i=1}^{k}\Psi_{M_{i-1}}^{m-M_{i-1}}(\mathcal{T}_i).$$

With (5), we further obtain

$$A_k^T \le \bigoplus_{0\le m_1,\dots,m_k\le p}\sum_{i=1}^{k}\sum_{j=M_{i-1}+1}^{M_i}\Psi_j^{m-j}(\mathcal{T}_i) + \bigoplus_{0\le m_1,\dots,m_k\le p}\sum_{i=1}^{k}\Psi_{M_{i-1}}^{m-M_{i-1}}(\mathcal{T}_i).$$

It remains to replace the first sum with its obvious upper bound, and then apply (4) and (5) to the second sum so as to get the desired result.

5 Evaluation of Bounds on the Cycle Time

The next statement follows from the classical result in [5], combined with Lemma 4.

Theorem 1. *If $\tau_{i1}, \tau_{i2}, \ldots$, are i.i.d. r.v.'s with $\mathrm{E}[\tau_{i1}] < \infty$ for each $i = 1, \ldots, n$, then there exists a fixed matrix A such that with probability 1,*

$$\lim_{k\to\infty} A_k^T/k = A^T, \quad \textit{and} \quad \lim_{k\to\infty} \mathrm{E}[A_k^T]/k = A^T.$$

Furthermore, application of Lemma 5 together with asymptotic results in [3, 4] gives us the next theorem.

Theorem 2. *If in addition to the conditions of Theorem 1, $\mathrm{D}[\tau_{i1}] < \infty$ for each $i = 1, \ldots, n$, then it holds*

$$\mathrm{E}[\mathcal{T}_1] \leq A^T \leq \mathrm{E}\left[\bigoplus_{0\leq r+s\leq p} G^r \otimes \mathcal{T}_1 \otimes G^s\right]. \tag{8}$$

As a consequence, we have the next lemma.

Lemma 6. *Under the conditions of Theorem 2, for any finite vector $\boldsymbol{x}(0)$, it holds*

$$\|\mathrm{E}[\mathcal{T}_1]\| \leq \gamma \leq \left\|\mathrm{E}\left[\bigoplus_{0\leq r+s\leq p} G^r \otimes \mathcal{T}_1 \otimes G^s\right]\right\|.$$

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