Bounds on the Mean Working Time for Queueing Networks with Random Topology*

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We consider a fork-join queueing network [1] with n nodes and customers of a single class. Each node of the network includes a server and a buffer which has infinite capacity. The evolution of a network presents a sequence of service cycles: the 1st cycle starts at the initial time, and it is completed as soon as the servers in all nodes of the network complete their 1st service of customers, the 2nd cycle is completed as soon as the servers complete their 2nd service, and so on.

The network topology is assumed to vary from one cycle to another at random. It is also assumed that with positive probability a topology may occur in each cycle k, which gives no way for the cycle to be completed. Suppose that for a network, the cycles from 1 to k were successfully completed, while cycle $k+1$ cannot be completed because of network topology. The completion time of the kth cycle is then said to be working time of the network.

Our main problem of interest is to derive bounds on the mean working time of a network under some given conditions on random service times at the network nodes as well as on the random network topology. We start with a representation of network dynamics based on the $(\max,+)$-algebra approach [2, 7] which has proved to be quite useful in describing queueing systems (see, e.g. [3, 4]).

Let τ_{ik} be a random variable which describes the kth service time in node i. We introduce the diagonal matrix $\mathcal{T}_k = \mathrm{diag}(\tau_{1k}, \ldots, \tau_{nk})$ with $-\infty$ as the off-diagonal elements, and the vector $\boldsymbol{x} = (x_1(k), \ldots, x_n(k))^T$ where $x_i(k)$ denotes the kth service completion time in node i. We suppose that the random matrices $\mathcal{T}_1, \mathcal{T}_2, \ldots$, are independent and identically distributed.

It is shown in [5, 6] that the dynamics of a network can be described by the implicit equation in $\boldsymbol{x}(k)$

$$\boldsymbol{x}(k) = \mathcal{T}_k \otimes \Gamma_k^T \otimes \boldsymbol{x}(k) \oplus \mathcal{T}_k \otimes \boldsymbol{x}(k-1), \tag{1}$$

where Γ_k denotes a random adjacency matrix of the graph describing the network topology for the kth cycle, $\oplus$ and $\otimes$ denotes the $(\max,+)$-algebra addition and multiplication, respectively. The random matrices $\Gamma_1, \Gamma_2, \ldots$, are assumed to be independent and identically distributed. We also suppose that these matrices are independent of $\mathcal{T}_1, \mathcal{T}_2, \ldots$.

We consider that the kth cycle can be completed if and only if the above implicit equation can be solved to produce the equation

$$\boldsymbol{x}(k) = A(k) \otimes \boldsymbol{x}(k-1) \tag{2}$$

with the matrix $A(k) = A(\mathcal{T}_k, \Gamma_k) = (E \oplus \mathcal{T}_k \otimes \Gamma_k^T)^{\otimes \eta_k} \otimes \mathcal{T}_k$, where η_k denotes the length of the longest path in the graph associated with the matrix Γ_k, and $E = \mathrm{diag}(0, \ldots, 0)$ with the off-diagonal elements equal to $-\infty$.

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Let $\mathcal{G} = \{G\}$ be the set of all deterministic adjacency matrices which allow for solving equation (1) in the explicit form of (2). With the symbol $\mathcal{Y}$ denoting the random working time of the network, we can represent the mean working time as

$$\mathbb{E}[\mathcal{Y}] = \mathbb{P}\{\Gamma \notin \mathcal{G}\} \sum_{k=1}^{\infty} \sum_{G_1,\ldots,G_k \in \mathcal{G}} \mathbb{E}\|A(\mathcal{T}_k, G_k) \otimes \cdots \otimes A(\mathcal{T}_1, G_1)\|_{\oplus} \prod_{i=1}^{k} \mathbb{P}\{\Gamma_i = G_i\},$$

where $\|\cdot\|_{\oplus}$ denotes the (max, +)-algebra norm (for any matrix $X = (x_{ij})$, its norm is defined as $\|X\|_{\oplus} = \max_{i,j} x_{ij}$).

It is shown that the next double inequality is valid

$$y_1 \leq \mathbb{E}[\mathcal{Y}] \leq y_2,$$

where

$$\begin{aligned} y_1 &= \mathbb{P}\{\Gamma \notin \mathcal{G}\}\mathbb{E}\left[\|A(\mathcal{T},\Gamma)\|_{\oplus} | \Gamma \in \mathcal{G}\right] + \frac{\mathbb{P}^2\{\Gamma \in \mathcal{G}\}\left(1 + \mathbb{P}\{\Gamma \notin \mathcal{G}\}\right) \|\mathbb{E}\left[\mathcal{T}\right]\|_{\oplus}}{\mathbb{P}\{\Gamma \notin \mathcal{G}\}}, \\ y_2 &= \frac{\mathbb{E}\left[\|A(\mathcal{T},\Gamma)\|_{\oplus} | \Gamma \in \mathcal{G}\right]}{\mathbb{P}\{\Gamma \notin \mathcal{G}\}}. \end{aligned}$$

We present results of calculating the bounds together with related results obtained through computer simulation for some stochastic network models. The behaviour of the bounds with respect to parameters of the models is also discussed.

References

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