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SOLUTION OF GENERALIZED LINEAR VECTOR EQUATIONS IN IDEMPOTENT ALGEBRA

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The problem on the solutions of homogeneous and nonhomogeneous generalized linear vector equations in idempotent algebra is considered. For the study of equations, an idempotent analog of matrix determinant is introduced and its properties are investigated. In the case of irreducible matrix, existence conditions are found and the general solutions of equations are obtained. The results are extended to the case of arbitrary matrix. As a consequence the solutions of homogeneous and nonhomogeneous inequalities are presented.

1. Introduction. For analysis of different technical, economical, and engineering systems the problems are often occurred which require the solution of vector equations linear in a certain idempotent algebra [1–5]. As a basic object of idempotent algebra one usually regards a commutative semiring with an idempotent summation, a zero, and a unity. At once many practical problems give rise to idempotent semiring, in which any nonzero (in the sense of idempotent algebra) element has the inverse one by multiplication. Taking into account a group property of multiplications, such a semiring are called sometimes idempotent semifield.

Note that in passing from idempotent semrings to semifields, the idempotent algebra takes up an important common property with a usual linear algebra. In this case it is naturally expected that the solution of certain problems of idempotent algebra can be obtained by a more simple way and in a more conventional form, in particular, due to the applications of idempotent analogs of notions and results of usual algebra.

Consider, for example, the problem on the solution with respect to the unknown vector $\boldsymbol{x}$ the equation $A \otimes \boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}$, where A is a certain matrix, $\boldsymbol{b}$ is a vector, $\oplus$ and $\otimes$ are the signs of operations of summation and multiplication of algebra. Different approaches to the solution of this equation were happily developed in the work [3–7] and the others. However many of these works consider a general case of idempotent semiring and, therefore, the represented in them results have often too general theoretical nature and are not always convenient for practical application. In a number of works it is mainly considered existence conditions of solution of equations and only some its partial (for example, minimal) solution is suggested in explicit form.

In the present work a new method for the solution of linear equations in the case of idempotent semiring with the inverse one by multiplication (a semifield) is suggested which can be used for obtaining the results in compact form convenient for as their realization in the form of computational procedures as a formal analysis. For the proof of certain assertions the approaches, developed in [1, 2, 4], are used.

In the work there is given first a short review of certain basic notions of idempotent algebra [2, 4, 5, 8], involving the generalized linear vector spaces and the elements of matrix calculus, and a number of auxiliary

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results. Then a certain function, given on a set of square matrices, is determined which is regarded as a certain idempotent analog of a determinant of matrix, and the properties of matrices, concerning to the value of this function, are studied.

The function, mentioned above, is introduced in such a way that it is a (idempotent) polynomial of matrix elements and can be used in studying linear equations, where possible, as a usual determinant in arithmetical space. Such analog of determinant is more convenient tools for analysis of equations than another similar constructions, well-known in the literature [2, 9].

Further, the equations $A\otimes \boldsymbol{x} = \boldsymbol{x}$ and $A\otimes\boldsymbol{x}\oplus\boldsymbol{b} = \boldsymbol{x}$ are considered which in idempotent algebra play the role of homogeneous and nonhomogeneous equations in the sense that a general solution of nonhomogeneous equation can be represented as a sum of its minimal partial solution and a general solution of homogeneous equation [4].

In the case of the irreducible matrix A existence conditions and a general solution of homogeneous and nonhomogeneous equations are obtained. The obtained results are used then for determining existence conditions and constructing the general solutions of equations with decomposable matrix. Then the solutions of the homogeneous $A \otimes \boldsymbol{x} \leq \boldsymbol{x}$ and nonhomogeneous $A \otimes \boldsymbol{x} \oplus \boldsymbol{b} \leq \boldsymbol{x}$ inequalities are obtained. Finally, certain remarks concerning to the dimension of a space of solutions of equations and inequalities and also the forms of representation of solutions themselves are given.

2. Idempotent algebra. Consider an extended set of real numbers $\mathbb{R}_\varepsilon = \mathbb{R}\cup\{\varepsilon\}$, where $\varepsilon = -\infty$, on which the operations of summation $\oplus$ and multiplication $\otimes$ are defined such that $x \oplus y = \max(x, y)$ and $x \otimes y = x + y$ for any x, y from $\mathbb{R}_\varepsilon$.

The set $\mathbb{R}_\varepsilon$ together with the mentioned above operations makes up an idempotent semifield, i.e. a semiring with an idempotent summation, a zero, and a unity, in which for any nonzero element there exists one inverse element by multiplication.

Note that together with the semiring $\mathbb{R}_\varepsilon$ another semirings often occur which possess the same properties, for example, a semiring with a pair of operations $(\min, +)$, given on the set $\mathbb{R}\cup\{+\infty\}$ and also the semiring with the operations $(\max, \times)$ and $(\min, \times)$, given on $\mathbb{R}_+$. Taking into account that these semirings are isomorphic $\mathbb{R}_\varepsilon$, the results, represented below, can be extended to the case of any of them.

It is clear that in the semiring $\mathbb{R}_\varepsilon$ a zero is ε and a unity is the number 0. For any $x \in \mathbb{R}$ the inverse element x^{-1}, which is equal to $-x$ in usual arithmetic, is determined. If $x = \varepsilon$, then we suppose that $x^{-1} = \varepsilon$.

For any $x, y \in \mathbb{R}$ the degree x^y, the value of which corresponds to the arithmetical product xy, is introduced in the usual way.

The notion of degree is used below in the sense of idempotent algebra only. However, for the sake of simplicity, for notation of relations the usual arithmetical operations will be used in place of an exponent of power. For any numbers $x_i \in \mathbb{R}$, $\alpha_i \geq 0$, $i = 1, \dots, m$, the following inequality

$$x_1^{\alpha_1} \otimes \cdots \otimes x_n^{\alpha_m} \leq (x_1 \oplus \cdots \oplus x_m)^{\alpha_1+\cdots+\alpha_m}$$

is satisfied.

For $\alpha_1 = \cdots = \alpha_m = 1/m$ we have an idempotent analog of inequality for geometric and arithmetic averages

$$(x_1 \otimes \cdots \otimes x_m)^{1/m} \leq x_1 \oplus \cdots \oplus x_m.$$

3. Matrix algebra. For any matrices $A, B \in \mathbb{R}_\varepsilon^{m\times n}$, $C \in \mathbb{R}_\varepsilon^{n\times l}$ and the number $x \in \mathbb{R}_\varepsilon$, the operations of summation and multiplication of matrices and the multiplication of a matrix by a number are defined

$$\{A \oplus B\}_{ij} = \{A\}_{ij} \oplus \{B\}_{ij}, \quad \{B \otimes C\}_{ij} = \bigoplus_{k=1}^{n} \{B\}_{ik} \otimes \{C\}_{kj}, \quad \{x \otimes A\}_{ij} = x \otimes \{A\}_{ij}.$$

The operations $\oplus$ and $\otimes$ have a monotonicity property, i.e. for any matrices A, B, C , and D of suitable order the inequalities $A \leq C$ and $B \leq D$ imply that

$$A \oplus B \leq C \oplus D, \qquad A \otimes B \leq C \otimes D.$$

A square matrix is said to be diagonal if all of its nondiagonal elements are equal to ε and to be triangular if all of its elements above (below) diagonal are equal to ε.

The matrix $\mathcal{E}$, all elements of which are equal to ε, is called zero. The square matrix $E = \mathrm{diag}(0, \dots, 0)$ is called unit.

The matrix A^- is said to be pseudoinverse for the matrix A if the following conditions (see also [1])

$$\{A^-\}_{ij} = \begin{cases} \{A\}_{ji}^{-1}, & \text{iIf } \{A\}_{ji} > \varepsilon, \\ \varepsilon, & \text{otherwise} \end{cases}$$

are satisfied.

The square matrix A is said to be decomposable if it can be represented in block-triangular form by means of the interchange of rows together with the same interchange of columns and to be irreducible if not.

4. Linear vector space. For any two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}_\varepsilon^n$, $\boldsymbol{a} = (a_1, \dots, a_n)^T$, $\boldsymbol{b} = (b_1, \dots, b_n)^T$ and the number $x \in \mathbb{R}_\varepsilon$ there are defined the following operations

$$\boldsymbol{a} \oplus \boldsymbol{b} = (a_1 \oplus b_1, \dots, a_n \oplus b_n)^T, \qquad x \otimes \boldsymbol{a} = (x \otimes a_1, \dots, x \otimes a_n)^T.$$

A zero vector is the vector $\boldsymbol{\varepsilon} = (\varepsilon, \dots, \varepsilon)^T$.

A set of the vectors $\mathbb{R}_\varepsilon^n$ with the operations $\oplus$ and $\otimes$ is called a generalized linear vector space (or, simply, a linear vector space) [1, 2].

Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$. Then the inequality $\boldsymbol{a} \leq \boldsymbol{b}$ results in $\boldsymbol{a}^- \geq \boldsymbol{b}^-$. Besides, it is easy to check that $\boldsymbol{a} = \boldsymbol{b}$ if and only if $\boldsymbol{b}^- \otimes \boldsymbol{a} \oplus \boldsymbol{a}^- \otimes \boldsymbol{b} = 0$.

The vector $\boldsymbol{b} \in \mathbb{R}_\varepsilon^n$ is said to be linearly dependent on the vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m \in \mathbb{R}_\varepsilon^n$ if it is of their linear combination, i.e. $\boldsymbol{b} = x_1 \otimes \boldsymbol{a}_1 \oplus \dots \oplus x_m \otimes \boldsymbol{a}_m$, where $x_1, \dots x_m \in \mathbb{R}_\varepsilon$.

The zero vector $\boldsymbol{\varepsilon}$ depends linearly on any system of the vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$.

Two systems of vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ and $\boldsymbol{b}_1, \dots, \boldsymbol{b}_k$ are said to be equivalent if either vector of one of system depends linearly on the vectors of another system.

The system of vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ is said to be linearly dependent if at least one of vectors of system depends linearly on the rest of vectors and to be linearly independent if not.

Let $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m \in \mathbb{R}_\varepsilon^n$ be certain nonzero vectors. Denote by A a matrix with the columns $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$. We have (see also [2]) the following

Lemma 1. *The vector $\boldsymbol{b} \in \mathbb{R}^n$ depends linearly on the vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ if and only if $(A \otimes (\boldsymbol{b}^- \otimes A)^-)^- \otimes \boldsymbol{b} = 0$.*

Proof. A linear dependence of the vector $\boldsymbol{b}$ on $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ is equivalent to the existence of the solution $\boldsymbol{x} \in \mathbb{R}^m$ of the equation $A \otimes \boldsymbol{x} = \boldsymbol{b}$. As shown in [8], this equation has a solution if and only if $(A \otimes (\boldsymbol{b}^- \otimes A)^-)^- \otimes \boldsymbol{b} = 0$. □

We assume that the vector $\boldsymbol{b}$ has the coordinates equal to ε (the zero coordinates). Denote an index set of the zero coordinates of the vector $\boldsymbol{b}$ by I.

Suppose, $\boldsymbol{b}'$ is a vector, obtained from $\boldsymbol{b}$ by means of the deleting of all zero coordinates, A' is a matrix, obtained from A by means of the deleting of all rows with the index $i \in I$ and the columns with the index j, which is $a_{ij} \neq \varepsilon$ at least for one of $i \in I$. If the set I is empty, then $A' = A$ and $\boldsymbol{b}' = \boldsymbol{b}$.

Lemma 2. *The vector $\boldsymbol{b} \in \mathbb{R}_\varepsilon^n$, $\boldsymbol{b} \neq \boldsymbol{\varepsilon}$ depends linearly on the vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ if and only if $(A' \otimes (\boldsymbol{b}'^- \otimes A')^-)^- \otimes \boldsymbol{b}' = 0$.*

Proof. For any $i \in I$ from the equation $A \otimes \boldsymbol{x} = \boldsymbol{b}$ we have $a_{i1} \otimes x_1 \oplus \dots \oplus a_{in} \otimes x_n = \varepsilon$. This implies $x_j = \varepsilon$ if $a_{ij} \neq \varepsilon$.

Fix the values $x_j = \varepsilon$ for all indices j such that $a_{ij} \neq \varepsilon$ at least for one of $i \in I$. From the system of equations $A \otimes \boldsymbol{x} = \boldsymbol{b}$ it is possible to eliminate all equations, which correspond to the indices $i \in I$ and also all unknown $x_j = \varepsilon$. Then we obtain the equation $A' \otimes \boldsymbol{x}' = \boldsymbol{b}'$ with respect to the vector $\boldsymbol{x}'$, which is of smaller dimension.

Since the vector $\boldsymbol{b}'$ has no the zero coordinates, applying the equation of the previous lemma, we obtain the required assertion. □

Let $A_{(i)}$ be a matrix, obtained from A by means of the elimination of the column $\boldsymbol{a}_i$. Consider, as above, the index set of the zero coordinates $\boldsymbol{a}_i$, and determine then the vector $\boldsymbol{a}_i'$ and the matrix $A_{(i)}'$ for all $i = 1, \dots, m$. Then we have the following

Proposition 1. *The system of vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ is linearly independent if and only if $(A_{(i)}' \otimes (\boldsymbol{a}_i'^- \otimes A_{(i)}')^-)^- \otimes \boldsymbol{a}_i' \neq 0$ for all $i = 1, \dots, m$.*

Corollary 1. *To construct a linearly independent subsystem equivalent to the system $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$, it is sufficient to eliminate sequentially from this system each vector $\boldsymbol{a}_i$, $i = 1, \dots, m$ such that $(A_{(i)}'' \otimes (\boldsymbol{a}_i'^- \otimes A_{(i)}'')^-)^- \otimes \boldsymbol{a}_i' = 0$, where the matrix $A_{(i)}''$ is composed only of the columns $A_{(i)}'$, which are still not be eliminated.*

5. Square matrices. Let $A = (a_{ij}) \in \mathbb{R}_{\varepsilon}^{n\times n}$ be an arbitrary square matrix. It is clear that any such matrix gives a certain (generalized) linear operator, acting in the linear space $\mathbb{R}_{\varepsilon}^{n}$, i.e. endomorphism.

The integer nonnegative degree of the matrix A is determined from the relations $A^0 = E$, $A^{k+l} = A^k \otimes A^l$ for any $k, l = 1, 2, \dots$

We introduce a certain (idempotent) analogs of a spur and a determinant of matrix. Taking into account that further a spur and a determinant of matrix will be regarded only in the sense of their idempotent analogs, we will save usual notions for these analogs.

A sum of diagonal elements of the matrix A is called a spur and is denoted by

$$\operatorname{tr} A = \bigoplus_{i=1}^{n} a_{ii}.$$

A product of elements of the matrix A of the form $a_{i_0 i_1} \otimes \cdots \otimes a_{i_{m-1} i_m}$, where $i_0 = i_m$, is called cyclic.

A sum of all cyclic products of the matrix A is called its determinant and is denoted by

$$\det A = \bigoplus_{m=1}^{n} \bigoplus_{i_0,\dots,i_{m-1}} a_{i_0 i_1} \otimes \cdots \otimes a_{i_{m-1} i_0} = \bigoplus_{m=1}^{n} \operatorname{tr} A^m.$$

Consider certain properties of matrices, concerning the value of their determinant.

Lemma 3. *For any matrix A and the vector $\boldsymbol{x} = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ the following assertions are valid:*
1) if $\det A \le 0$, then $\boldsymbol{x}^- \otimes A \otimes \boldsymbol{x} \ge \det A$;
2) if $\det A > 0$, then $\boldsymbol{x}^- \otimes A \otimes \boldsymbol{x} \ge (\det A)^{1/n}$.

Proof. We introduce the notion $\varphi(\boldsymbol{x}; A) = \boldsymbol{x}^- \otimes A \otimes \boldsymbol{x}$ and consider

$$\varphi(\boldsymbol{x}; A) = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n} x_i^{-1} \otimes a_{ij} \otimes x_j.$$

For any sequence of indices $i_0, \dots, i_m$, where $i_0 = i_m$, $1 \le m \le n$, applying the inequality for arithmetical and geometric average, we have

$$\varphi(\boldsymbol{x}; A) \ge (x_{i_0}^{-1} \otimes a_{i_0 i_1} \otimes x_{i_1}) \oplus \cdots \oplus (x_{i_{m-1}}^{-1} \otimes a_{i_{m-1} i_m} \otimes x_{i_m}) \ge (a_{i_0 i_1} \otimes \cdots \otimes a_{i_{m-1} i_m})^{1/m}.$$

This implies the inequality $\varphi(\boldsymbol{x}; A) \ge \operatorname{tr}^{1/m}(A^m)$, which is valid for all m.

Then for $\det A \le 0$ we obtain

$$\varphi(\boldsymbol{x}; A) \ge \operatorname{tr} A \oplus \cdots \oplus \operatorname{tr}^{1/n}(A^n) \ge \operatorname{tr} A \oplus \cdots \oplus \operatorname{tr} A^n = \det A.$$

In the case when $\det A > 0$ we have $\varphi(\boldsymbol{x}; A) \ge (\det A)^{1/n}$. □

For any matrix A we define the matrices A^+ and $A^\times$ as

$$A^+ = E \oplus A \oplus \cdots \oplus A^{n-1}, \qquad A^\times = A \otimes A^+ = A \oplus \cdots \oplus A^n.$$

If $\det A = \varepsilon$, then it is easily shown (see, for example, [4]) that $A^m = \mathcal{E}$ for the certain $m < n$ and, therefore, $A^k \le A^+$ for all $k \ge 0$.

Lemma 4. *If $\det A \ne \varepsilon$, then for any integer $k \ge 0$ the following assertions are valid:*
1) if $\det A \le 0$, then $A^k \le (\det A)^{(k+1)/n-1} \otimes A^+$;
2) if $\det A > 0$, then $A^k \le (\det A)^k \otimes A^+$.

Proof. We prove that the inequalities are satisfied for $k < n$.

Let $k \ge n$. We shall show that the inequalities are valid for the corresponding elements a_{ij}^k and a_{ij}^+ of the matrices A^k and A^+, respectively. Assuming that $i_0 = i$ and $i_k = j$, we represent a_{ij}^k as

$$a_{ij}^k = \bigoplus_{i_1,\dots,i_{k-1}} a_{i_0 i_1} \otimes \cdots \otimes a_{i_{k-1} i_k}.$$

Consider the product $S\,S_{ij} = a_{i_0 i_1} \otimes \cdots \otimes a_{i_{k-1} i_k}$. If among the multipliers S_{ij} there is the number ε, then $S_{ij} = \varepsilon$ and we have $S_{ij} \le (\det A)^\alpha \otimes a_{ij}^+$ for any $\alpha > 0$.

Let $S_{ij} > \varepsilon$. We regroup the multipliers of the product S_{ij} in the following way. We combine first all cyclic products, consisting of $m = 1$ multipliers. Let $\alpha_1 \ge 0$ be the number of such products. From the rest of them we choose cyclic products of $m = 2$ multipliers and denote the number of them by α_2. We continue then this procedure for all subsequent values of $m \le n$.

Taking into account that the cyclic product of m multipliers is less than or equal to $\operatorname{tr} A^m$, we have

$$S_{ij} \le \bigotimes_{\substack{i=1\\ \alpha_i>0}}^{n} \operatorname{tr}^{\alpha_i}(A^i) \otimes S'_{ij} \le (\det A)^{\alpha_1+\cdots+\alpha_n} \otimes S'_{ij},$$

where S'_{ij} is a product without cycles, which consists of no more than $n-1$ multipliers. Obviously, $S'_{ij} \le a^+_{ij}$ and, in addition, $k-n+1 \le \alpha_1 + 2\alpha_2 + \cdots + n\alpha_n \le k$. This implies that $(k-n+1)/n \le \alpha_1+\cdots+\alpha_n \le k$.

In this case if $\det A \le 0$, then $S_{ij} = a_{i_0 i_1} \otimes \cdots \otimes a_{i_{k-1} i_k} \le (\det A)^{(k+1)/n-1} \otimes a^+_{ij}$ for any set of indices $i_1,\ldots,i_{k-1}$ and, therefore, $a^k_{ij} \le (\det A)^{(k+1)/n-1} \otimes a^+_{ij}$.

If $\det A > 0$, then $S_{ij} \le (\det A)^k \otimes a^+_{ij}$, and, therefore, $a^k_{ij} \le (\det A)^k \otimes a^+_{ij}$. □

Corollary 2. *If* $\det A \le 0$, *then the following relations are satisfied*

$$A^+ = E \oplus A^\times, \qquad A^+ \otimes A^+ = A^+.$$

Proof. Taking into account that $A^k \le A^+$ for all $k \ge n$, we obtain the first relation $E \oplus A^\times = A^+ \oplus A^n = A^+$. The second relation is verified similarly. □

From the relation $A^+ = E \oplus A^\times$ it follows that $A^\times \le A^+$, in which case the corresponding elements a^+_{ij} and $a^\times_{ij}$ of these matrices coincide except for, perhaps, diagonal elements, which satisfy the conditions $a^+_{ii} = 0$ and $a^\times_{ii} \le 0$.

Denote by $\boldsymbol{a}^+_i$ and $\boldsymbol{a}^\times_i$ the columns with the index i of the matrices A^+ and $A^\times$ and by a^m_{ii} the diagonal elements of the matrix A^m. Below, a determinant properties are used for obtaining the analog of the assertion established in the works [2–4].

Proposition 2. *If* $\det A = 0$, *then the matrices* A^+ *and* $A^\times$ *have common like columns, which coincide, and in this case the relation* $\boldsymbol{a}^+_i = \boldsymbol{a}^\times_i$ *is satisfied if and only if* $a^m_{ii} = 0$ *for a certain* $m = 1,\ldots,n$.

Proof. If $\det A = 0$, then the nondiagonal elements of the matrices A^+ and $A^\times$ coincide. In addition, the relation $\det A = 0$ is equivalent to that $\operatorname{tr} A^m = 0$ for a certain $m = 1,\ldots,n$. The latter occurs only if $a^m_{ii} = 0$ for the certain index i. Taking into account that in this case $a^\times_{ii} = 0$, we have $a^\times_{ii} = a^+_{ii} = 0$, i.e. $\boldsymbol{a}^+_i = \boldsymbol{a}^\times_i$. □

For any matrix A such that $\det A = 0$, denote by A^* the matrix of the same order, the columns of which satisfy the condition $\boldsymbol{a}^*_i = \boldsymbol{a}^+_i$ if $\boldsymbol{a}^+_i = \boldsymbol{a}^\times_i$ and $\boldsymbol{a}^*_i = \boldsymbol{\varepsilon}$ if $\boldsymbol{a}^+_i \ne \boldsymbol{a}^\times_i$, $i = 1,\ldots,n$. If $\det A \ne 0$, we put $A^* = \mathcal{E}$.

6. Homogeneous and nonhomogeneous linear equations. Let $A \in \mathbb{R}^{n\times n}_\varepsilon$ be a certain given matrix, $\boldsymbol{x} \in \mathbb{R}^n_\varepsilon$ be an unknown vector. We shall say that a homogeneous equation with respect to $\boldsymbol{x}$ is the equation

$$A \otimes \boldsymbol{x} = \boldsymbol{x}. \tag{1}$$

Let $\boldsymbol{b} \in \mathbb{R}^n_\varepsilon$ be a certain given vector. We shall say that a nonhomogeneous equation with respect to $\boldsymbol{x}$ is the equation

$$A \otimes \boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}. \tag{2}$$

The solution $\boldsymbol{x} = \boldsymbol{\varepsilon}$ of equations (1) and (2) is called trivial.

The solution $\boldsymbol{x}_0$ of equation is said to be minimal if for any solution $\boldsymbol{x}$ of this equation the following relation $\boldsymbol{x}_0 \le \boldsymbol{x}$ is satisfied.

All the solutions of homogeneous equation make up a linear space.

Proposition 3. *If* $\det A = 0$, *then the solution of homogeneous equation* (1) *is the vector* $\boldsymbol{x} = A^* \otimes \boldsymbol{v}$ *for any* $\boldsymbol{v} \in \mathbb{R}^n_\varepsilon$.

Proof. Let $\det A = 0$. Then the matrices A^+ and $A^\times$ have the common columns $\boldsymbol{a}^+_i = \boldsymbol{a}^\times_i$. Since $A^\times = A \otimes A^+$, we have $A \otimes \boldsymbol{a}^+_i = \boldsymbol{a}^\times_i = \boldsymbol{a}^+_i$, i.e. $\boldsymbol{a}^+_i$ satisfies equation (1). Taking into account that all such columns, and only they, are nonzero columns of the matrix A^*, we conclude that any vector $\boldsymbol{x} = A^* \otimes \boldsymbol{v}$, where $\boldsymbol{v}$ is a vector, is a solution of (1). □

7. Irreducible matrices. We shall seek existence conditions of solution and a general solution of equations (1) and (2) under the assumption that the matrix A is irreducible. We prove first certain auxiliary assertions.

Proposition 4. *If* A *is an irreducible matrix, then any nontrivial solution* $\boldsymbol{x}$ *of equations* (1) *and* (2) *has no zero coordinates.*

Proof. Let $\boldsymbol{x}$ be a nontrivial solution of equation (1) (equation (2) is considered similarly). We shall show that all coordinates of the vector $\boldsymbol{x}$ are nonzero.

Assume that there is one coordinate $x_i = \varepsilon$ while $x_j > \varepsilon$ for all $j \neq i$. From the relation $a_{i1} \otimes x_1 \oplus \cdots \oplus a_{in} \otimes x_n = \varepsilon$ it follows that $a_{ij} = \varepsilon$ if $j \neq i$. In this case, by inverting the rows 1 and i and then the columns with the same indices, it is possible to reduce the matrix A to triangular form, what is in contrast to the condition of indecomposability.

The assumption that the vector $\boldsymbol{x}$ has any number $l < n$ of zero coordinates is considered similarly. □

Proposition 5. *Homogeneous equation* (1) *with the irreducible matrix A has a nontrivial solution if and only if* $\det A = 0$.

Proof. A sufficiency of the condition $\det A = 0$ results from Proposition 3.

We verify a necessity, using the same reasonings as those in the work [1]. Let $\boldsymbol{x}$ be a nontrivial solution of equation. We shall show that $\det A = 0$. Consider any sequence of indices $i_0, \ldots, i_m$ such that $i_m = i_0$, $1 \leq m \leq n$. Equation (1) yields the inequalities

$$a_{i_0 i_1} \otimes x_{i_1} \leq x_{i_0}, \quad a_{i_1 i_2} \otimes x_{i_2} \leq x_{i_1}, \quad \ldots \quad a_{i_{m-1} i_m} \otimes x_{i_m} \leq x_{i_{m-1}}.$$

Multiplying the left- and right-hand sides of these inequalities and reducing by the quantity $x_{i_1} \otimes \cdots \otimes x_{i_m} \neq \varepsilon$, we arrive at the inequality $a_{i_0 i_1} \otimes \cdots \otimes a_{i_{m-1} i_m} \leq 0$.

Taking into account an arbitrary choice of the indices $i_0, \ldots, i_m$, for all $m = 1, \ldots, n$ we have $\operatorname{tr} A^m \leq 0$. Therefore, $\det A = \operatorname{tr} A \oplus \cdots \oplus \operatorname{tr} A^n \leq 0$.

In addition, from (1) it follows that for any index i there exists the index j such that $a_{ij} \otimes x_j = x_i$. Take the arbitrary index i_0, and determine sequentially the indices $i_1, i_2, \ldots$ such that the following relations

$$a_{i_0 i_1} \otimes x_{i_1} = x_{i_0}, \quad a_{i_1 i_2} \otimes x_{i_2} = x_{i_1}, \quad \ldots,$$

are satisfied down the first repetition of indices. From the obtained sequence of indices we choose the subsequence $i_l, i_{l+1}, \ldots, i_{l+m}$, where $i_l = i_{l+m}$, $l \geq 0$, $1 \leq m \leq n$.

Multiplying the relations, corresponding to the subsequence, and reducing by $x_{i_l} \otimes \cdots \otimes x_{i_{l+m}} \neq \varepsilon$, we have $a_{i_l i_{l+1}} \otimes \cdots \otimes a_{i_{l+m-1} i_{l+m}} = 0$. This implies that $\det A \geq \operatorname{tr} A^m \geq 0$. Since at a time the inequality $\det A \leq 0$ is satisfied, we conclude that $\det A = 0$. □

Now we find a general solution of homogeneous equation. We have the following

Lemma 5. *Suppose, $\boldsymbol{x}$ is a general solution of homogeneous equation* (1) *with the irreducible matrix A. Then the following assertions are valid:*

1) if $\det A = 0$, *then* $\boldsymbol{x} = A^* \otimes \boldsymbol{v}$ *for all* $\boldsymbol{v} \in \mathbb{R}^n_\varepsilon$*;*

2) if $\det A \neq 0$, *then we have the trivial solution* $\boldsymbol{x} = \boldsymbol{\varepsilon}$ *only.*

Proof. Obviously, $\boldsymbol{x} = \boldsymbol{\varepsilon}$ is a solution of homogeneous equation (1). In this case if $\det A \neq 0$, then from Proposition 5 it follows that another solutions is lacking.

Put $\det A = 0$. Note that in this case $A^+ \otimes A^+ = A^+$ and the matrices A^+ and $A^\times$ have common columns. For the sake of simplicity, we assume that the first m columns of these matrices coincide. Represent the matrices A^+, $A^\times$ and A^* and the vector $\boldsymbol{x}$ in block form:

$$A^+ = \begin{pmatrix} B & C \\ D & F \end{pmatrix}, \quad A^\times = \begin{pmatrix} B & C \\ D & G \end{pmatrix}, \quad A^* = \begin{pmatrix} B & \mathcal{E} \\ D & \mathcal{E} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix},$$

where B is of order $m \times m$, F and G is of order $(n-m) \times (n-m)$, the vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are of dimensions m and $n-m$, respectively, and $\mathcal{E}$ denotes zero matrices of the corresponding order.

We establishes certain relations for blocks. it is easy to see that $F \geq G$, in which case $\det F = \operatorname{tr} F = 0$ and $\det G = \operatorname{tr} G < 0$. In addition, the relation

$$A^+ \otimes A^+ = \begin{pmatrix} B^2 \oplus C \otimes D & B \otimes C \oplus C \otimes F \\ D \otimes B \oplus F \otimes D & D \otimes C \oplus F^2 \end{pmatrix} = \begin{pmatrix} B & C \\ D & F \end{pmatrix} = A^+$$

implies, in particular, the inequalities $D \geq F \otimes D \geq G \otimes D$ and $B \geq C \otimes D$.

We assume that $\boldsymbol{x}$ is a nontrivial solution of equation (1). In this case $\boldsymbol{x}$ is a solution of the homogeneous equation $A^\times \otimes \boldsymbol{x} = \boldsymbol{x}$. Now we write the last equation in the form

$$\begin{aligned} \boldsymbol{x}_1 &= B \otimes \boldsymbol{x}_1 \oplus C \otimes \boldsymbol{x}_2, \\ \boldsymbol{x}_2 &= D \otimes \boldsymbol{x}_1 \oplus G \otimes \boldsymbol{x}_2. \end{aligned}$$

Taking into account the inequalities, obtained above, by means of iterations for any integer $k \geq 1$ from the second equation we obtain $\boldsymbol{x}_2 = D \otimes \boldsymbol{x}_1 \oplus G^k \otimes \boldsymbol{x}_2$.

Since $\det G < 0$, by Lemma 4 we have $G^k \leq (\det G)^{(k+1)/(n-m)-1} \otimes G^+$. Now for any vector $\boldsymbol{x} \in \mathbb{R}^n$, the number k can always be chosen such that the following condition

$$D \otimes \boldsymbol{x}_1 \geq (\det G)^{(k+1)/(n-m)-1} \otimes G^+ \otimes \boldsymbol{x}_2 \geq G^k \otimes \boldsymbol{x}_2$$

is satisfied. This implies that the second equation has actually the form $\boldsymbol{x}_2 = D \otimes \boldsymbol{x}_1$.

We substitute $\boldsymbol{x}_2 = D \otimes \boldsymbol{x}_1$ in the first equation. Taking into account that $B \geq C \otimes D$, we arrive at the equation $\boldsymbol{x}_1 = B \otimes \boldsymbol{x}_1$. Thus, we have

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} = \begin{pmatrix} B & \mathcal{E} \\ D & \mathcal{E} \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} = A^* \otimes \boldsymbol{x}.$$

This means that any nontrivial solution of equation (1) has the form $\boldsymbol{x} = A^* \otimes \boldsymbol{v}$, where $\boldsymbol{v}$ is a certain vector. It remains to verify that $\boldsymbol{x} = A^* \otimes \boldsymbol{v}$ is a solution of (1) for any vector $\boldsymbol{v}$. The latter was established in Proposition 3. □

We proceed to the study of nonhomogeneous equation.

Lemma 6. *Nonhomogeneous equation* (2) *with the irreducible matrix* A *has a solution if and only if at least one of the following conditions holds:*

1) $\det A \leq 0$*;*

2) $\boldsymbol{b} = \boldsymbol{\varepsilon}$.

In this case $\boldsymbol{x} = A^+ \otimes \boldsymbol{b}$ *is a minimal partial solution of* (2).

Proof. Suppose, $\det A \leq 0$. Then by iterations with applying Lemma 4, equation (2) can be reduced to the form $A^n \otimes \boldsymbol{x} \oplus A^+ \otimes \boldsymbol{b} = \boldsymbol{x}$. This implies, in particular, inequality $\boldsymbol{x} \geq A^+ \otimes \boldsymbol{b}$. By direct substitution we obtain that the vector $\boldsymbol{x} = A^+ \otimes \boldsymbol{b}$ is a solution of equation (2) and in virtue of previous inequality it is its minimal solution.

Suppose, $\det A > 0$. We shall show that equation (2) under this condition has no the nontrivial solutions. Really, in virtue of Lemma 3 for any $\boldsymbol{x} \in \mathbb{R}^n$ we have $\boldsymbol{x}^- \otimes (A \otimes \boldsymbol{x} \oplus \boldsymbol{b}) \geq \boldsymbol{x}^- \otimes A \otimes \boldsymbol{x} \geq (\det A)^{1/n} > 0$, which gives $A \otimes \boldsymbol{x} \oplus \boldsymbol{b} \neq \boldsymbol{x}$.

This implies that the solution $\boldsymbol{x} = \boldsymbol{\varepsilon}$ exists if and only if $\boldsymbol{b} = \boldsymbol{\varepsilon}$. □

To prove the following lemma we apply the approach, suggested in [4].

Lemma 7. *A general solution of nonhomogeneous equation* (2) *with the irreducible matrix* A *has the form* $\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}$, *where* $\boldsymbol{u}$ *is a minimal partial solution of equation* (2), $\boldsymbol{v}$ *is a general solution of equation* (1).

Proof. We assume that $\boldsymbol{u}$ is any solution of equation (2) and $\boldsymbol{v}$ is any solution of equation (1). Then $\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}$ is also a solution of (2) since

$$A \otimes \boldsymbol{x} \oplus \boldsymbol{b} = A \otimes (\boldsymbol{u} \oplus \boldsymbol{v}) \oplus \boldsymbol{b} = (A \otimes \boldsymbol{u} \oplus \boldsymbol{b}) \oplus (A \otimes \boldsymbol{v}) = \boldsymbol{u} \oplus \boldsymbol{v} = \boldsymbol{x}.$$

Suppose, $\boldsymbol{x}$ is an arbitrary solution of equation (2). We shall show that it can be represented in the form $\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}$, where $\boldsymbol{u}$ is a minimal solution of (2) and $\boldsymbol{v}$ is a certain solution of (1). Note first that equation (1) under the condition $\det A \neq 0$ has a trivial solution only and, therefore, $\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}$, where $\boldsymbol{u} = \boldsymbol{x}$, $\boldsymbol{v} = \boldsymbol{\varepsilon}$.

Suppose, $\det A = 0$. We assume that $\boldsymbol{u} = A^+ \otimes \boldsymbol{b}$ is a minimal solution of (2). By inequality $\boldsymbol{x} \geq A^+ \otimes \boldsymbol{b} = \boldsymbol{u}$ the vector $\boldsymbol{v}'$ can always be found such that $\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}'$.

Since $A \otimes \boldsymbol{x} = A \otimes (\boldsymbol{u} \oplus \boldsymbol{v}') = A \otimes A^+ \otimes \boldsymbol{b} \oplus A \otimes \boldsymbol{v}'$, by (2) we have $\boldsymbol{x} = A \otimes \boldsymbol{x} \oplus \boldsymbol{b} = A^+ \otimes \boldsymbol{b} \oplus A \otimes \boldsymbol{v}'$. It follows that for the vector $\boldsymbol{v} = A \otimes \boldsymbol{v}'$ the relation $\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}$ remains true. This relation occurs for each vector $\boldsymbol{v} = A^m \otimes \boldsymbol{v}'$ for all integer $m \geq 0$ and, therefore, for the vectors $A^+ \otimes \boldsymbol{v}'$ and $A^\times \otimes \boldsymbol{v}'$.

Take the vector $\boldsymbol{v}'$ with the coordinates $v_i' = x_i$ if $u_i < x_i$ and $v_i' = \varepsilon$ if $u_i = x_i$, $i = 1, \ldots, n$. We have $\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}'$ and, in addition, $\boldsymbol{v}' \leq \boldsymbol{v}$ for any vector $\boldsymbol{v}$ such that $\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}$. Whence it follows that the inequality $\boldsymbol{v}' \leq A \otimes \boldsymbol{v}'$ and, therefore, the inequality $A^+ \otimes \boldsymbol{v}' \leq A^\times \otimes \boldsymbol{v}'$ are satisfied. Since the opposite inequality $A^+ \otimes \boldsymbol{v}' \geq A^\times \otimes \boldsymbol{v}'$, is always satisfied, we conclude that $A^+ \otimes \boldsymbol{v}' = A^\times \otimes \boldsymbol{v}'$.

it remains to put $\boldsymbol{v} = A^+ \otimes \boldsymbol{v}'$. Then $\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}$ is a solution of equation (2), in which case $A \otimes \boldsymbol{v} = A^\times \otimes \boldsymbol{v}' = \boldsymbol{v}$, i.e. the vector $\boldsymbol{v}$ is a solution of (1). □

By Lemmas 5 and 7 we have the following

Theorem 1. *Suppose, the solution of nonhomogeneous equation* (2) *with the irreducible matrix* A *exists,* $\boldsymbol{x}$ *is a general solution of* (2).

Then the following assertions hold:

1) if $\det A < 0$, *then we have a unique solution* $\boldsymbol{x} = A^+ \otimes \boldsymbol{b}$*;*

2) if $\det A = 0$, *then* $\boldsymbol{x} = A^+ \otimes \boldsymbol{b} \oplus A^* \otimes \boldsymbol{v}$ *for all* $\boldsymbol{v} \in \mathbb{R}_\varepsilon^n$*;*

3) if $\det A > 0$, *then we have the trivial solution* $\boldsymbol{x} = \boldsymbol{\varepsilon}$ *only.*

It is easy to verify that for $A=\mathcal{E}$ this theorem is also valid.

8. Decomposable matrices. We assume now that the matrix A is decomposable. By the interchange of rows together with the same interchange of columns it can be represented in block-triangular normal form

$$A=\begin{pmatrix} A_1 & \mathcal{E} & \dots & \mathcal{E} \\ A_{21} & A_2 & & \mathcal{E} \\ \vdots & \vdots & \ddots & \\ A_{s1} & A_{s2} & \dots & A_s \end{pmatrix}, \tag{3}$$

where A_i is either irreducible, either zero $n_i\times n_i$-matrix, A_{ij} is an arbitrary $n_i\times n_j$-matrix for all $j<i$, $i=1,\dots,s$, under the condition $n_1+\dots+n_s=n$, and $\mathcal{E}$ denotes zero matrices of the corresponding order.

We shall say that the family of rows (columns) of the matrix A, which correspond to each diagonal block A_i is a horizontal (vertical) series of matrix.

We assume that the matrix A is reduced to normal form (3). Note that then $\det A=\det A_1\oplus\dots\oplus\det A_s$.

Denote by I_0 a set of the indices i such that the relation $\det A_i=0$ is satisfied and by I_1 a set of indices such that $\det A_i>0$.

We assume first that $I_1=\emptyset$. The matrix A can be represented in the form $A=T\oplus D$, where T is a block strictly triangular matrix and D is a block-diagonal matrix,

$$T=\begin{pmatrix} \mathcal{E} & \dots & \dots & \mathcal{E} \\ A_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ A_{s1} & \dots & A_{s,s-1} & \mathcal{E} \end{pmatrix}, \qquad D=\begin{pmatrix} A_1 & & \mathcal{E} \\ & \ddots & \\ \mathcal{E} & & A_s \end{pmatrix}.$$

Determine the following auxiliary matrices:

$$D^+=\operatorname{diag}(A_1^+,\dots,A_s^+), \qquad C=D^+\otimes T, \qquad D^*=\operatorname{diag}(A_1^*,\dots,A_s^*).$$

We have $C^s=\mathcal{E}$ and, therefore, $C^+=E\oplus C\oplus\dots\oplus C^{s-1}$. Note that the matrix C^+ has a lower block-triangular form with the blocks C_i^+ and C_{ij}^+. The order of these blocks coincides with the order of the corresponding blocks A_i and A_{ij} of the matrix A.

If $I_1\ne\emptyset$, then we consider the matrix $\bar{A}$, which is obtained from A by means of the interchange of all blocks of its vertical rows $i\in I_1$ by zero matrices. Denote the diagonal blocks of the matrix $\bar{A}$ by $\bar{A}_i$ and the subdiagonal blocks by $\bar{A}_{ij}$.

Represent the matrix $\bar{A}$ in the form $\bar{A}=\bar{T}\oplus\bar{D}$, where $\bar{T}$ is a block strictly triangular matrix and $\bar{D}$ is a block-diagonal matrix, and put

$$\bar{D}^+=\operatorname{diag}(\bar{A}_1^+,\dots,\bar{A}_s^+), \qquad \bar{C}=\bar{D}^+\otimes\bar{T}, \qquad \bar{D}^*=\operatorname{diag}(\bar{D}_1^*,\dots,\bar{D}_s^*),$$

where $\bar{D}_j^*=\mathcal{E}$ if the conditions $j\in I_0$ and $\bar{C}_{ij}^+\ne\mathcal{E}$ are satisfied at least for one of $i\in I_1$, and $\bar{D}_j^*=\bar{A}_j^*=A^*$ otherwise.

Consider equations (1) and (2). For each $i=1,\dots,s$, denote by $\boldsymbol{x}_i$ and $\boldsymbol{b}_i$ the vectors of order n_i, which are made up by the coordinates of the vectors $\boldsymbol{x}$ and $\boldsymbol{b}$, corresponding to the horizontal row i of the matrix A.

We consider first homogeneous equation (1).

Lemma 8. *Suppose, $\boldsymbol{x}$ is a general solution of homogeneous equation* (1) *with the matrix A, represented in form* (3). *Then the following assertions are valid:*

1) if $\det A<0$, then we have the trivial solution $\boldsymbol{x}=\boldsymbol{\varepsilon}$ only;

2) if $\det A=0$, then $\boldsymbol{x}=C^+\otimes D^*\otimes\boldsymbol{v}$ for all $\boldsymbol{v}\in\mathbb{R}_\varepsilon^n$;

3) if $\det A>0$, then $\boldsymbol{x}=\bar{C}^+\otimes\bar{D}^*\otimes\boldsymbol{v}$ for all $\boldsymbol{v}\in\mathbb{R}_\varepsilon^n$, in which case we have only the trivial solution $\boldsymbol{x}=\boldsymbol{\varepsilon}$ if $I_0=\emptyset$.

Proof. Equation (1) can be represented as a system of equations, which correspond to the horizontal series $i=1,\dots,s$:

$$A_{i1}\otimes\boldsymbol{x}_1\oplus\dots\oplus A_{i,i-1}\otimes\boldsymbol{x}_{i-1}\oplus A_i\otimes\boldsymbol{x}_i=\boldsymbol{x}_i. \tag{4}$$

If $\det A=\det A_1\oplus\dots\oplus A_s\le 0$, then by Theorem 1 the solution $\boldsymbol{x}_i$ of each equation exists and all the vectors $\boldsymbol{x}_i$ can be sequentially determined from the following equations

$$\boldsymbol{x}_1=A_1^*\otimes\boldsymbol{v}_1, \qquad \boldsymbol{x}_i=A_i^+\otimes(A_{i1}\otimes\boldsymbol{x}_1\oplus\dots\oplus A_{i,i-1}\otimes\boldsymbol{x}_{i-1})\oplus A_i^*\otimes\boldsymbol{v}_i, \quad i>1,$$

where $\boldsymbol{v}_i$ is any vector of dimension n_i, $i=1,\dots,s$.

Assuming that $\boldsymbol{v} = (\boldsymbol{v}_1^T, \ldots, \boldsymbol{v}_s^T)^T$, these equations can be represented in the form of one equation

$$\boldsymbol{x} = C \otimes \boldsymbol{x} \oplus D^* \otimes \boldsymbol{v},$$

the solution of which by means of iterations gives $\boldsymbol{x} = C^s \otimes \boldsymbol{x} \oplus C^+ \otimes D^* \otimes \boldsymbol{v} = C^+ \otimes D^* \otimes \boldsymbol{v}$. In particular, for $\det A < 0$ we have $D^* = \mathcal{E}$ and, therefore, $\boldsymbol{x} = \boldsymbol{\varepsilon}$.

Suppose, $\det A > 0$. Consider equation (4) for any series $i \in I_1$. By Theorem 1 if the solution $\boldsymbol{x}_i$ of such equation exists, then $\boldsymbol{x}_i = \boldsymbol{\varepsilon}$.

We assume that $\boldsymbol{x}_i = \boldsymbol{\varepsilon}$ for all $i \in I_1$. The solution of equation (1) under this condition is not changed if we assume that all elements of vertical series $i \in I_1$ of the matrix A are equal to ε. Then for each $i = 1, \ldots, s$ equations (4) take the form

$$\bar{A}_{i1} \otimes \boldsymbol{x}_1 \oplus \cdots \oplus \bar{A}_{i,i-1} \otimes \boldsymbol{x}_{i-1} \oplus \bar{A}_i \otimes \boldsymbol{x}_i = \boldsymbol{x}_i.$$

Taking into account that $\det \bar{A} \leq 0$ and that $\bar{A}_i^* = A_i^*$ for all $i = 1, \ldots, s$, a general solution of this system of equations is $\boldsymbol{x} = \bar{C}^+ \otimes D^* \otimes \boldsymbol{v}$ for all $\boldsymbol{v} \in \mathbb{R}_\varepsilon^n$.

For the obtained solution to be satisfied the condition $\boldsymbol{x}_i = \boldsymbol{\varepsilon}$ for all $i \in I_1$, it is necessary that for each such i the following relation

$$\bar{C}_{i1}^+ \otimes A_1^* \otimes \boldsymbol{v}_1 \oplus \cdots \oplus \bar{C}_{is}^+ \otimes A_s^* \otimes \boldsymbol{v}_s = \boldsymbol{\varepsilon}$$

is satisfied for any $\boldsymbol{v} \in \mathbb{R}_\varepsilon^n$.

Since $A_j^* = \mathcal{E}$ for each $j \notin I_0$, we conclude that for the above relations to be satisfied it is necessary and sufficient that $A_j^* \otimes \boldsymbol{v}_j = \boldsymbol{\varepsilon}$ for the indices $j \in I_0$ such that $\bar{C}_{ij}^+ \neq \mathcal{E}$ at least for one of $i \in I_1$. The latter occurs in the case of a formal change of all such matrices A_j^* to zero. Since this is equivalent to the passage from D^* to $\bar{D}^*$, we obtain a general solution of equation (1) in the form $\boldsymbol{x} = \bar{C}^+ \otimes \bar{D}^* \otimes \boldsymbol{v}$ for all $\boldsymbol{v} \in \mathbb{R}_\varepsilon^n$.

It remains to note that $\boldsymbol{x} = \boldsymbol{\varepsilon}$ if $I_0 = \emptyset$. □

In the case of nonhomogeneous equation we have the following

Lemma 9. *Nonhomogeneous equation* (2) *with the matrix* A *in form* (3) *has a solution if and only if, at least, one of conditions hold:*

1) $\det A \leq 0$*;*

2) $\boldsymbol{b}_i = \boldsymbol{\varepsilon}$ *for all* $i \in I_1$ *and* $\boldsymbol{b}_j = \boldsymbol{\varepsilon}$ *for each* $j \notin I_1$ *such that* $\bar{A}_{ij}^+ \neq \mathcal{E}$ *at least for one of* $i \in I_1$.

In this case $\boldsymbol{x} = \bar{A}^+ \otimes \boldsymbol{b}$ *is a minimal partial solution of* (2).

Proof. If $\det A \leq 0$, then like the proof of Lemma 6 we can show that the solution of equation (2) exists, in which case a minimal solution is the vector $\boldsymbol{x} = A^+ \otimes \boldsymbol{b} = \bar{A}^+ \otimes \boldsymbol{b}$.

Suppose, $\det A > 0$. Represent equation (2) in the form of the system of equations, which correspond to the series $i = 1, \ldots, s$,

$$A_{i1} \otimes \boldsymbol{x}_1 \oplus \cdots \oplus A_{i,i-1} \otimes \boldsymbol{x}_{i-1} \oplus A_i \otimes \boldsymbol{x}_i \oplus \boldsymbol{b}_i = \boldsymbol{x}_i. \tag{5}$$

Obviously, for each $i \in I_1$ there is a uniquely possible solution of equation (5), namely $\boldsymbol{x}_i = \boldsymbol{\varepsilon}$, existence condition of which is $\boldsymbol{b}_i = \boldsymbol{\varepsilon}$. Then, like the proof of Lemma 5, it is possible to change the matrix A to $\bar{A}$.

Since $\det \bar{A} \leq 0$, a minimal solution of equation (2) with the matrix $\bar{A}$ is $\boldsymbol{x} = \bar{A}^+ \otimes \boldsymbol{b}$. In this case for each $i = 1, \ldots, s$ we have the vector

$$\boldsymbol{x}_i = \bar{A}_{i1}^+ \otimes \boldsymbol{b}_1 \oplus \cdots \oplus \bar{A}_{i,i-1}^+ \otimes \boldsymbol{b}_{i-1} \oplus \bar{A}_i^+ \otimes \boldsymbol{b}_i,$$

which has to satisfy the condition $\boldsymbol{x}_i = \boldsymbol{\varepsilon}$ if $i \in I_1$. This is equivalent to $\boldsymbol{b}_j = \boldsymbol{\varepsilon}$ for each $j \notin I_1$ such that $\bar{A}_{ij}^+ \neq \mathcal{E}$, at least, for one of $i \in I_1$. □

Like the proof of Lemma 7 we obtain the following result.

Lemma 10. *A general solution of nonhomogeneous equation* (2) *with the matrix* A*, represented in form* (3)*, has the form* $\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}$*, where* $\boldsymbol{u}$ *is a minimal partial solution of nonhomogeneous equation* (2)*,* $\boldsymbol{v}$ *is a general solution of homogeneous equation* (1).

By Lemma 8 and 10 it is easy to prove the following

Theorem 2. *Suppose, the solution of nonhomogeneous equation* (2) *with the matrix* A*, represented in form* (3)*, exists,* $\boldsymbol{x}$ *is a general solution of* (2).

Then the following assertions are valid:

1) if $\det A < 0$*, then we have a unique solution* $\boldsymbol{x} = A^+ \otimes \boldsymbol{b}$*;*

2) if $\det A = 0$*, then* $\boldsymbol{x} = A^+ \otimes \boldsymbol{b} \oplus C^+ \otimes D^* \otimes \boldsymbol{v}$ *for all* $\boldsymbol{v} \in \mathbb{R}_\varepsilon^n$*;*

3) if $\det A > 0$*, then* $\boldsymbol{x} = \bar{A}^+ \otimes \boldsymbol{b} \oplus \bar{C}^+ \otimes \bar{D}^* \otimes \boldsymbol{v}$ *for all* $\boldsymbol{v} \in \mathbb{R}_\varepsilon^n$*, in which case we have a unique solution* $\boldsymbol{x} = \bar{A}^+ \otimes \boldsymbol{b}$ *if* $I_0 = \emptyset$.

9. Homogeneous and nonhomogeneous linear inequalities. The inequality of the form

$$A \otimes \boldsymbol{x} \leq \boldsymbol{x} \tag{6}$$

with respect to the unknown vector $\boldsymbol{x}$ is called homogeneous and the inequality of the form

$$A \otimes \boldsymbol{x} \oplus \boldsymbol{b} \leq \boldsymbol{x} \tag{7}$$

is called nonhomogeneous.

We shall show how the obtained results can be applied to solve inequalities (6) and (7). We assume first that the matrix A is irreducible.

Lemma 11. *Suppose, $\boldsymbol{x}$ is a general solution of homogeneous inequality (6) with the irreducible matrix A. Then the following assertions are true:*

1) if $\det A < 0$, then $\boldsymbol{x} = A^+ \otimes \boldsymbol{u}$ for all $\boldsymbol{u} \in \mathbb{R}^n_\varepsilon$;

2) if $\det A = 0$, then $\boldsymbol{x} = A^+ \otimes \boldsymbol{u} \oplus A^* \otimes \boldsymbol{v}$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n_\varepsilon$;

3) if $\det A > 0$, then we have the trivial solution $\boldsymbol{x} = \boldsymbol{\varepsilon}$ only.

Proof. The set of solutions of inequality (6) coincides with the set of solutions $\boldsymbol{x}$ of the equation $A \otimes \boldsymbol{x} \oplus \boldsymbol{u} = \boldsymbol{x}$ with respect to two unknown $\boldsymbol{x}$ and $\boldsymbol{u}$ for all possible values of $\boldsymbol{u}$. Applying Lemma 6 and Theorem 1, we obtain the required result. □

The validity of the following assertion is verified similarly.

Lemma 12. *Nonhomogeneous inequality (7) with the irreducible matrix A has a solution if and only if it is valid at least one of conditions:*

1) $\det A \leq 0$;

2) $\boldsymbol{b} = \boldsymbol{\varepsilon}$.

In this case $\boldsymbol{x} = A^+ \otimes \boldsymbol{b}$ is minimal solution (7).

Theorem 3. *Suppose, a solution of nonhomogeneous inequality (7) with the irreducible matrix A exists, $\boldsymbol{x}$ is a general solution of (7).*

Then the following assertions are valid:

1) if $\det A < 0$, then $\boldsymbol{x} = A^+ \otimes \boldsymbol{b} \oplus A^+ \otimes \boldsymbol{u}$ for all $\boldsymbol{u} \in \mathbb{R}^n_\varepsilon$;

2) if $\det A = 0$, then $\boldsymbol{x} = A^+ \otimes (\boldsymbol{b} \oplus \boldsymbol{u}) \oplus A^* \otimes \boldsymbol{v}$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n_\varepsilon$;

3) if $\det A > 0$, then we have the trivial solution $\boldsymbol{x} = \boldsymbol{\varepsilon}$ only.

We assume now that A is a decomposable matrix. By Lemma 9 and Theorem 2, as in the case of irreducible matrix, we can obtain the following results.

Lemma 13. *Suppose, $\boldsymbol{x}$ is a general solution of homogeneous inequality (6) with the matrix A, represented in form (3). Then it is valid the following assertions:*

1) if $\det A < 0$, then $\boldsymbol{x} = A^+ \otimes \boldsymbol{u}$ for all $\boldsymbol{u} \in \mathbb{R}^n_\varepsilon$;

2) if $\det A = 0$, then $\boldsymbol{x} = A^+ \otimes \boldsymbol{u} \oplus C^+ \otimes D^* \otimes \boldsymbol{v}$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n_\varepsilon$;

3) if $\det A > 0$, then $\boldsymbol{x} = \bar{A}^+ \otimes \boldsymbol{u} \oplus \bar{C}^+ \otimes \bar{D}^* \otimes \boldsymbol{v}$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n_\varepsilon$, in which case $\boldsymbol{x} = \bar{A}^+ \otimes \boldsymbol{u}$ if $I_0 = \emptyset$.

Lemma 14. *Nonhomogeneous inequality (7) with the matrix A in form (3) has a solution if and only if it is valid, at least, one of conditions:*

1) $\det A \leq 0$;

2) $\boldsymbol{b}_i = \boldsymbol{\varepsilon}$ for all $i \in I_1$ and $\boldsymbol{b}_j = \boldsymbol{\varepsilon}$ for each $j \notin I_1$ such that $\bar{A}^+_{ij} \neq \mathcal{E}$, at least, for one of $i \in I_1$.

In this case $\boldsymbol{x} = \bar{A}^+ \otimes \boldsymbol{b}$ is a minimal solution of (7).

Theorem 4. *Suppose, the solution of nonhomogeneous inequality (7) with the matrix A, represented in form (3), exists, $\boldsymbol{x}$ is a general solution of (7).*

Then the following assertions are valid:

1) if $\det A < 0$, then $\boldsymbol{x} = A^+ \otimes \boldsymbol{b} \oplus A^+ \otimes \boldsymbol{u}$ for all $\boldsymbol{u} \in \mathbb{R}^n_\varepsilon$;

2) if $\det A = 0$, then $\boldsymbol{x} = A^+ \otimes (\boldsymbol{b} \oplus \boldsymbol{u}) \oplus C^+ \otimes D^* \otimes \boldsymbol{v}$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n_\varepsilon$;

3) if $\det A > 0$, then $\boldsymbol{x} = \bar{A}^+ \otimes (\boldsymbol{b} \oplus \boldsymbol{u}) \oplus \bar{C}^+ \otimes \bar{D}^* \otimes \boldsymbol{v}$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n_\varepsilon$, in which case $\boldsymbol{x} = \bar{A}^+ \otimes \boldsymbol{b} \oplus \bar{A}^+ \otimes \boldsymbol{u}$ if $I_0 = \emptyset$.

10. Improvement of dimension of a space of solutions. Note that in the previous divisions general solutions of equations and inequalities in the space $\mathbb{R}^n_\varepsilon$ were represented, for sake of simplicity, by means of endomorphisms of the same space.

For example, a general solution of homogeneous equation with the irreducible matrix A has the form $\boldsymbol{x} = A^* \otimes \boldsymbol{v}$ for all $\boldsymbol{v} \in \mathbb{R}^n_\varepsilon$, i.e. makes up the subspace of the vectors $\boldsymbol{x} = v_1 \otimes \boldsymbol{a}^*_1 \oplus \cdots \oplus v_n \otimes \boldsymbol{a}^*_n$, where $\boldsymbol{a}^*_i$ are the columns of the matrix A^*, $i = 1, \ldots, n$. However among the vectors $\boldsymbol{a}^*_1, \ldots, \boldsymbol{a}^*_n$ can be linearly dependent and, therefore, the subspace, mentioned above, has a dimension, smaller than n.

We assume that $\tilde{\boldsymbol{a}}_1^*, \dots, \tilde{\boldsymbol{a}}_k^*$ is a linearly independence subsystem of vectors, which is equivalent to the system $\boldsymbol{a}_1^*, \dots, \boldsymbol{a}_n^*$, $k \leq n$. Such a subsystem can be constructed, applying, for example, the procedure, which is based on the result of Proposition 1 and its corollary.

Denote by $\tilde{A}^*$ a matrix with the columns $\tilde{\boldsymbol{a}}_1^*, \dots, \tilde{\boldsymbol{a}}_k^*$. Then a general solution of homogeneous equation can be represented in the form $\boldsymbol{x} = \tilde{A}^* \otimes \tilde{\boldsymbol{v}}$ for all $\tilde{\boldsymbol{v}} \in \mathbb{R}_\varepsilon^k$.

Similarly we can specify the form of representation of general solution of all above-considered equations and inequalities.

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References

1. N. N. Vorob'ev, *Elektronische Informationsverarbeitung und Kybernetik,* Bd. 3, No. 1, S. 39–72, 1967.
2. R. A. Cuninghame-Green, *Minimax Algebra,* Berlin, 1979.
3. U. Zimmermann, *Linear and Combinatorial Optimization in Ordered Algebraic Structures*, Amsterdam, 1981.
4. F. Baccelli, G. Cohen, G. J. Olsder, and J.-P. Quadrat, *Synchronization and Linearity*, Chichester, 1992.
5. V. P. Maslov and V. N. Kolokol'tsov, *Idempotent Analysis and its Application to Optimal Control* [in Russian], Moscow, 1994.
6. B. Mahr, *Annals of Discrete Mathematics*, Vol. 19, pp. 229–256, 1984.
7. P. I. Dudnikov and S. N. Samborskii, *Endomorphisms of Semimodules over Semirings with Idempotent Operation*[in Russian], Kiev, 1987.
8. N. K. Krivulin, *Mathematical Models. Theory and Application* [in Russian], Issue 5. pp. 105–113, 2004.
9. G. J. Olsder and C. Roos, *Linear Algebra and its Applications*, Vol. 101, pp. 87–108, 1988.

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