Vestnik St. Petersburg University
Mathematics
Vol. 38, No. 2, pp. 45-54, 2005

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Universiteta. Matematika
UDC 519.63

EVALUATION OF BOUNDS ON THE MEAN RATE OF GROWTH OF THE STATE VECTOR OF A LINEAR DYNAMICAL STOCHASTIC SYSTEM IN IDEMPOTENT ALGEBRA*

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A dynamical system which is described in terms of an idempotent algebra by means of a vector equation with random irreducible matrix is considered. An approach based on approximation of the matrix of the system by means of matrices of simple structure is applied to evaluate bounds on the mean rate of growth of the state vector of the system. The process of constructing the approximations is reduced to the solution of problems of minimization of certain numerically valued functions. Examples that illustrate the evaluation of bounds on the mean rate of growth of the state vector for a system with matrix of dimension 2 are presented.

1. Introduction. The process of arriving at exact results in the course of solving practical problems using the apparatus and methods of idempotent algebra [1, 2] often represents a rather difficult problem. Let us consider as an example a stochastic dynamical system which is described in terms of an idempotent algebra by means of vector equations of the form

$$x(k) = A^T(k) \otimes x(k-1),$$

where $x(k)$ is a state vector and $A(k)$ a random matrix of the system.

In analyzing these systems it is often necessary to determine the asymptotic (mean) rate of growth of the state vector of the system $x(k)$ as $k \to \infty$. However, an exact solution of this problem is known only for a small number of special cases of systems with matrix of dimension 2 as well as with triangular matrix of arbitrary dimension (cf. [1, 3]). Thus, the development of effective procedures for finding approximate solutions and evaluating bounds would be of particular interest [1, 4].

One possible approach to the approximate solution of this problem, which was proposed in [4], is based on a study of products of matrices of the system $A(1) \otimes \ldots \otimes A(k)$ and consists in replacing each matrix by its bound, the latter being constructed by means of matrices of simple structure of the form $u \otimes v^T$, where u and v are certain vectors. By means of such a substitution arithmetic sums of scalar products of the corresponding vectors may be considered in place of products of matrices and, thus, the solution simplified.

The objective of the present study is to further develop this approach for the case of systems with irreducible matrix. In the article certain useful relations for matrices and vectors in idempotent algebra are first presented and a single extremal property for the spectral radius of a matrix is considered. Bounds are then constructed for arbitrary irreducible matrices using matrices of simple structure. The process of constructing the bounds is reduced to the

*The present study was completed with the support of the Russian Foundation for Basic Research, Grant No. 04-01-00840).

solution of problems of minimization of certain numerically valued functions of matrices. In conclusion examples are presented that show how the results that have been obtained may be applied for the calculation of upper and lower bounds of the mean rate of growth of the state vector of a system with matrix of dimension 2.

2. Preliminary remarks. We will consider an idempotent algebra (idempotent commutative semiring with zero and unit) with addition operation $x \oplus y = \max(x, y)$ and multiplication operation $x \otimes y = x + y$, which are defined for any x and y from an augmented set of real numbers $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$, where $\varepsilon = -\infty$.

Obviously ε is the zero element in the semiring being considered here and the number 0 the unit. Moreover, for any $x \in \mathbb{R}$ an inverse x^{-1} which is equal to $-x$ in ordinary arithmetic may be defined, as well as a power x^a the value of which corresponds to the arithmetic product ax for all $a \in \mathbb{R}$.

The operation of multiplication $\otimes$ is introduced on a set of matrices $\mathbb{R}_\varepsilon^{n\times n}$ usually by means of the scalar operations $\oplus$ and $\otimes$. The entire positive degree of the matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ is determined from the relation $A^k \otimes A^l = A^{k+l}$ for any integers $k, l \geq 1$. Below, the notations for degrees will be used only in the sense of an idempotent algebra.

The operation of multiplication of matrices $\otimes$ is monotonic, i.e., from the componentwise inequalities $A \leq B$ and $C \leq D$ there follows the inequality $A \otimes C \leq B \otimes D$.

For any matrix $A = (a_{ij}) \in \mathbb{R}_\varepsilon^{n\times n}$ we introduce the matrix A^- with elements $a^-_{ij} = a_{ji}^{-1}$ if $a_{ji} > \varepsilon$ and $a^-_{ij} = \varepsilon$ otherwise. It is clear that for any matrices $A, B \in \mathbb{R}_\varepsilon^{n\times n}$ it follows that $A^- \geq B^-$ from the inequality $A \leq B$, and conversely.

In addition to the operation $\otimes$ we will use the ordinary arithmetic addition of matrices. We define the arithmetic difference of matrices $A, B \in \mathbb{R}_\varepsilon^{n\times n}$ in terms of addition thus:

$$A - B = A + (B^-)^T.$$

It is easily verified that with any vector $x \in \mathbb{R}^n$ the following inequality is satisfied for all matrices $A \in \mathbb{R}_\varepsilon^{n\times n}$:

$$A \leq x \otimes x^- \otimes A, \tag{1}$$

while in the case in which $A \in \mathbb{R}^{n\times n}$ the following inequality is also satisfied:

$$A \geq x \otimes (A^- \otimes x)^-. \tag{2}$$

For any matrix $A = (a_{ij}) \in \mathbb{R}_\varepsilon^{n\times n}$ the following quantities may be determined:

$$\|A\| = \bigoplus_{1\leq i,j\leq n} a_{ij}, \qquad \mathrm{tr}(A) = \bigoplus_{1\leq i\leq n} a_{ii}.$$

Suppose that $A, B \in \mathbb{R}_\varepsilon^{n\times n}$. Then, if $A \leq B$, it is then obvious that $\|A\| \leq \|B\|$ and $\mathrm{tr}(A) \leq \mathrm{tr}(B)$. Moreover, the following equality always holds:

$$\mathrm{tr}(A \otimes B) = \|A + B^T\|.$$

The matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ will be called a matrix of simple structure if vectors $x, y \in \mathbb{R}_\varepsilon^n$ may be found such that $A = x \otimes y^T$.

Note that for any vectors $x, y \in \mathbb{R}_\varepsilon^n$,

$$\mathrm{tr}(x \otimes y^T) = y^T \otimes x, \; (x \otimes y^-)^- = y \otimes x^-.$$

A matrix is irreducible if it cannot be reduced to block-triangular form by interchanging rows and matrices of the same index. A matrix is said to be upper (respectively, lower) triangular if all of its elements below (respectively, above) the diagonal are equal to ε.

3. Spectral radius. The vector $x \in \mathbb{R}_\varepsilon^n$, $x \neq \varepsilon = (\varepsilon, \ldots, \varepsilon)^T$, is an eigenvector for the matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ if it satisfies the equality

$$A \otimes x = \rho \otimes x,$$

where ρ is the eigenvalue of the matrix corresponding to the vector x.

It may be proved (cf. [1]) that in the case of an irreducible matrix there exists a unique eigenvalue, while any eigenvector x of the matrix has bounded coordinates, i.e., $x \in \mathbb{R}^n$.

The maximal eigenvalue ρ, which is calculated from the formula [6, 7, 5],

$$\rho = \bigoplus_{m=1}^{n} \mathrm{tr}^{1/m}(A^m), \tag{3}$$

will be called the spectral radius of the matrix A.

We next define the function

$$\phi(x; A) = x^- \otimes A \otimes x,$$

where $A \in \mathbb{R}_\varepsilon^{n\times n}$ is some matrix and $x \in \mathbb{R}^n$.

Lemma 1. The equality

$$\min_{x\in\mathbb{R}^n} \phi(x; A) = \rho,$$

where $\rho = \rho(A)$ is the spectral radius of the matrix A, is satisfied for any irreducible matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$; moreover, a minimum is attained on the eigenvector A which corresponds to ρ.

Proof. Let us first note that by virtue of equality (1), for any vector $x \in \mathbb{R}^n$ and integer $m > 0$,

$$A^m \le (x^- \otimes A \otimes x)^{m-1} \otimes x \otimes x^- \otimes A,$$

and, consequently,

$$\mathrm{tr}(A^m) \le (x^- \otimes A \otimes x)^{m-1} \otimes \mathrm{tr}(x \otimes x^- \otimes A) = (x^- \otimes A \otimes x)^m.$$

Thus, for all $m > 0$, the inequality

$$\phi(x; A) = x^- \otimes A \otimes x \ge \mathrm{tr}^{1/m}(A^m),$$

is satisfied, whence it may be concluded that

$$\phi(x; A) \ge \bigoplus_{m=1}^{n} \mathrm{tr}^{1/m}(A^m) = \rho.$$

On the other hand, if x is an eigenvector of the matrix A corresponding to the eigenvalue ρ, then, in light of the obvious equality $x^- \otimes x = 0$, we will have

$$\phi(x; A) = x^- \otimes A \otimes x = \rho \otimes (x^- \otimes x) = \rho. \quad \square$$

4. Linear dynamical system. Let us consider a dynamical system the evolution of which is described by the equation

$$x(k) = A^T(k) \otimes x(k-1), \tag{4}$$

where $x(k)$ is a state vector and $A(k)$ a random matrix of the system.

We will suppose that the matrices $A(k)$, $k = 1, 2, \ldots$, are identically distributed and independent and that the mathematical expectation $\mathbb{E}\|A(1)\|$ is finite.

One of the important characteristics of the system is the mean rate of growth of the state vector $x(k)$, which is defined thus:

$$\lambda = \lim_{k \to \infty} \|x(k)\|^{1/k}.$$

Let us suppose that, with probability 1, the coordinates of the initial vector $x(0)$ are bounded and, using the notation

$$A_k = A(1) \otimes \ldots \otimes A(k),$$

represent the mean rate of growth of the state vector of the system in the form

$$\lambda = \lim_{k \to \infty} \|A_k\|^{1/k}. \tag{5}$$

Relying, for example, on the ergodic theorem in [8], it may be shown that in the present case the limit (5) exists with probability 1, and that the limit

$$\lim_{k \to \infty} \mathbb{E}\|A_k\|^{1/k} = \lambda$$

also exists.

The following technique [4] may be used to evaluate bounds on the quantity λ. Suppose that for all $k = 1, 2, \ldots$ the inequality $A(k) \leq u(k) \otimes v^T(k)$ is satisfied with probability 1, where $u(k)$ and $v(k)$ are certain random vectors. Then for the matrix A_k we will have with probability 1 the following inequality:

$$A_k \leq u(1) \otimes \bigotimes_{i=1}^{k-1} v^T(i) \otimes u(i+1) \otimes v^T(k)$$

Let us suppose that the augmented vectors $w(k) = (u^T(k), v^T(k))^T$ are identically distributed for all $k = 1, 2, \ldots$ and that the mathematical expectations $\mathbb{E}\|u(1)\|$ and $\mathbb{E}\|v(1)\|$ are finite. Then from the last inequality there follows the upper bound

$$\lambda = \lim_{k \to \infty} \mathbb{E}\|A_k\|^{1/k} \leq \mathbb{E}[v^T(1) \otimes u(2)]. \tag{6}$$

A lower bound may be obtained in analogous fashion by selecting $u(k)$ and $v(k)$ so that the inequality $A(k) \geq (u(k) \otimes v^T(k))$ is fulfilled for all $k = 1, 2, \ldots$.

5. Bounds for matrices. Suppose that $A \in \mathbb{R}_\varepsilon^{n \times n}$ is an irreducible matrix. Let us consider the problem of estimating the matrix A by means of matrices of simple structure L and U such that

$$L \leq A \leq U.$$

5.1. Upper bounds. We set $U = u \otimes v^T$ under the condition $u, v \in \mathbb{R}^n$. Consider the inequality

$$A \leq u \otimes v^T \tag{7}$$

and note that for all $A \in \mathbb{R}_\varepsilon^{n \times n}$ it is equivalent to the inequality

$$u^- \otimes A \leq v^T. \tag{8}$$

In fact, it is easily seen that (8) follows directly from (7). On the other hand, from inequality (8) we have $u \otimes u^- \otimes A \leq u \otimes v^T$, whence in combination with (1) we obtain (7).

It is clear that any two vectors u and v such that $v^T \geq u^- \otimes A$ will satisfy inequality (7). Recalling that for fixed u the elements of the matrix $U = u \otimes v^T$ have least values under the selection

$$v^T = u^- \otimes A, \tag{9}$$

we will next consider only upper bounds for which $U = u \otimes u^- \otimes A$.

The construction of the bounds (7) may now be related to the problem of finding a vector u such that the matrix $U = u \otimes u^- \otimes A$ will possess certain definite useful properties, for example, preserve the eigenvalue and vector of the matrix A or have least possible value of the quantity $\| U \|$.

In many cases it is natural to consider the bound (7) to be better, the smaller is the difference of the values of the elements of U from the corresponding elements of A. Then the vector u may be found as the solution of the problem

$$\min_{u \in \mathbb{R}^n} \varphi(u; A)$$

for some appropriate numerical criterion φ which reflects to some degree the degree of proximity between the elements of matrix A and its bound $U = u \otimes u^- \otimes A$.

5.2. Lower bounds. Under the additional condition $A \in \mathbb{R}^{n \times n}$ the problem of constructing lower bounds may be considered in analogous fashion and may be represented in the same form as in the case of upper bounds.

Let $L = u \otimes v^T$, where $u, v \in \mathbb{R}^n$. As before, using (2) we may prove that the inequality $u \otimes v^T \leq A$ is equivalent $v^T \leq (A^- \otimes u)^-$. Then, setting

$$v^T = (A^- \otimes u)^-, \tag{10}$$

only lower bounds with matrix $L = u \otimes (A^- \otimes u)^-$ may be considered by selecting the vector u according to the corresponding numerical criterion.

6. Construction of bounds for matrices. Let us consider a number of upper and lower bounds for A that are determined by different methods of selecting the vector u.

6.1. Upper bounds. Let us first note that setting u equal to the eigenvector of A that corresponds to its spectral radius ρ we obtain

$$U \otimes u = u \otimes u^- \otimes A \otimes u = \rho \otimes u \otimes (u^- \otimes u) = \rho \otimes u,$$

i.e., we arrive at a matrix U that preserves the eigenvalue and vector of A.

Let us now consider two methods of specifying the criterion φ. A natural measure of proximity of the matrices A and U is obviously the function

$$\varphi_1(u; A) = \| U - A \| = \| u \otimes u^- \otimes A - A \|. \tag{11}$$

Lemma 2. The inequality

$$\min_{u \in \mathbb{R}^n} \varphi_1(u; A) = \rho,$$

where $\rho = \rho(A \otimes A^-)$ is the spectral radius of the matrix $A \otimes A^-$, is satisfied for any irreducible matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$, moreover, a minimum is attained on the eigenvector of this matrix that corresponds to ρ.

Proof. To verify the assertion of the lemma it is sufficient to note that

$$\varphi_1(u; A) = \| u \otimes u^- \otimes A + (A^-)^T \| = \mathrm{tr}(u \otimes u^- \otimes A \otimes A^-) = u^- \otimes (A \otimes A^-) \otimes u,$$

and to then apply Lemma 1. $\square$

Still another simple criterion may be introduced in the following way. Consider the quantity $\|U\| = \|u \otimes u^- \otimes A\|$ and represent it in the form

$$\|U\| = \|A\| \oplus \bigoplus_{i=1}^{n} \bigoplus_{j \neq i} u_i \otimes u_j^{-1} \otimes \|a_j\|,$$

where u_i denotes the coordinate i of the vector u and a_j row j of matrix A.

We define the function

$$\varphi_2(u; A) = \bigoplus_{i=1}^{n} \bigoplus_{j \neq i} u_i \otimes u_j^{-1} \otimes \|a_j\|. \tag{12}$$

Then $\|U\| = \|A\| \oplus \varphi_2(u; A) \geq \|A\|$, and it may be expected that the bound will be, in general, sharper, the less is the difference between the maximal elements of U and A, i.e., the difference between the quantities $\|U\|$ and $\|A\|$, respectively. It is clear that the difference $\|U\| - \|A\|$ decreases or, at least, does not grow as the function φ_2 decreases.

Lemma 3. The equality

$$\min_{u \in \mathbb{R}^n} \varphi_2(u; A) = \left(\bigoplus_{i=1}^{n} \bigoplus_{j > i} \|a_i\| \otimes \|a_j\| \right)^{1/2},$$

is satisfied for any irreducible matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$, moreover, a minimum is attained on the vector u for which $u_i = \|a_i\|^{1/2}$, $i = 1, \dots, n$.

Proof. We write the function φ_2 in the form

$$\varphi_2(u; A) = \bigoplus_{i=1}^{n} \bigoplus_{j > i} u_i \otimes u_j^{-1} \otimes \|a_j\| \oplus u_i^{-1} \otimes u_j \otimes \|a_i\|.$$

Recalling that the function $f(z) = z \otimes c_1 \oplus z^{-1} \otimes c_2 = \max\{c_1 + z, c_2 - z\}$ for any $c_1, c_2 \in \mathbb{R}$, has the minimum $(c_1 + c_2)/2 = (c_1 \otimes c_2)^{1/2}$, we obtain the inequality

$$u_i \otimes u_j^{-1} \otimes \|a_j\| \oplus u_i^{-1} \otimes u_j \otimes \|a_i\| \geq (\|a_i\| \otimes \|a_j\|)^{1/2},$$

whence it follows that

$$\varphi_2(u; A) \geq \left(\bigoplus_{i=1}^{n} \bigoplus_{j > i} \|a_i\| \otimes \|a_j\| \right)^{1/2}.$$

Now it remains for us to verify that the latter inequality turns into an equality for the vector u with coordinates $u_i = \|a_i\|^{1/2}$ for all $i = 1, \dots, n$. □

6.2. Lower bounds. Now let us suppose that $A \in \mathbb{R}^{n \times n}$.

We will define a function which for lower bounds fulfills the same role that criterion (11) does for upper bounds:

$$\psi_1(u; A) = \|A - L\| = \|A - u \otimes (A^- \otimes u)^-\|. \tag{13}$$

It is easily seen that, in fact, $\psi_1(u; A) = \varphi_1(u; A)$. In fact,

$$\psi_1(u; A) = \|A + (A^- \otimes u \otimes u^-)^T\| = \mathrm{tr}(A \otimes A^- \otimes u \otimes u^-) = u^- \otimes A \otimes A^- \otimes u.$$

Thus, by criteria (11) and (13) the choice of an eigenvector of the matrix $A \otimes A^-$ as u is optimal for the upper and lower bounds simultaneously. It may be proved that such a vector also ensures a minimum of the function

$$\delta(u; A) = \|U - L\| = \|u \otimes u^- \otimes A - u \otimes (A^- \otimes u)^-\|.$$

To verify the latter result it is sufficient to note that $\delta(u; A) = \varphi_1(u; A)$:

$$\delta(u; A) = \mathrm{tr}(u \otimes u^- \otimes A \otimes A^- \otimes u \otimes u^-) = u^- \otimes A \otimes A^- \otimes u.$$

In conclusion, consider the matrix $L^- = A^- \otimes u \otimes u^- \geq A^-$. Recalling that

$$\|L^-\| = \|A^-\| \oplus \bigoplus_{i=1}^{n} \bigoplus_{j \neq i} \|a_i^-\| \otimes u_i \otimes u_j^{-1},$$

we introduce the function

$$\psi_2(u; A) = \bigoplus_{i=1}^{n} \bigoplus_{j \neq i} \|a_i^-\| \otimes u_i \otimes u_j^{-1}. \tag{14}$$

It is easily seen that for the lower bound this function constitutes a certain analog of criterion (12). And as in the proof of Lemma 3, it may be proved that

$$\min_{u \in \mathbb{R}^n} \psi_2(u; A) = \left(\bigoplus_{i=1}^{n} \bigoplus_{j > i} \|a_i^-\| \otimes \|a_j^-\| \right)^{1/2},$$

moreover, a minimum is attained for the vector u for which $u_i = \|a^-_i\|^{-1/2}$, $i = 1, \ldots, n$.

7. Examples. Let us show how the results that have been obtained here may be applied in constructing a bound on the mean rate of growth of the state vector of the dynamical system (4).

We will consider a second-order system with matrix

$$A(k) = \begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix}.$$

Suppose that $\{\alpha_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$, and $\{\delta_k\}$ are sequences of independent random variables having exponential distribution with mean 1 and also suppose that the random variables α_k, β_k, γ_k, and δ_k are independent for any k, $k = 1, 2, \ldots$. Note that the matrices $A(k)$ thus defined are irreducible.

It is known (cf. [1]) that for this system the mean rate of growth of the state vector $\lambda = 407/228 \approx 1.7851$.

Example 1. Let us find the upper bound (6) for λ which corresponds to a choice of an eigenvector of the matrix $A(k)$ as $u(k)$. From formula (3) we determine the spectral radius of the matrix:

$$\rho_k = \alpha_k \oplus (\beta_k \otimes \gamma_k)^{1/2} \oplus \delta_k.$$

It is easily verified that the eigenvector $u(k)$ corresponding to ρ_k has the form

$$u(k) = \begin{cases} \begin{pmatrix} \alpha_k \oplus (\beta_k \otimes \gamma_k)^{1/2} \\ \gamma_k \end{pmatrix} & \text{if} \quad \alpha_k \geq \delta_k, \\ \begin{pmatrix} \beta_k \\ (\beta_k \otimes \gamma_k)^{1/2} \oplus \delta_k \end{pmatrix} & \text{if} \quad \alpha_k < \delta_k. \end{cases}$$

Now let us find the vector $v(k)$ using (9):

$$v(k) = \begin{cases} \begin{pmatrix} 0 \\ \beta_k \otimes (\alpha_k \oplus (\beta_k \otimes \gamma_k)^{1/2})^{-1} \oplus \gamma_k^{-1} \otimes \delta_k \end{pmatrix} & \text{if} \quad \alpha_k \geq \delta_k, \\ \begin{pmatrix} \alpha_k \otimes \beta_k^{-1} \oplus ((\beta_k \otimes \gamma_k)^{1/2} \oplus \delta_k)^{-1} \otimes \gamma_k \\ 0 \end{pmatrix} & \text{if} \quad \alpha_k < \delta_k. \end{cases}$$

To determine the bound (6) it remains for us to find the value of $\mathbb{E}[v^T(1) \otimes u(2)]$. This may be done, for example, by means of a construction of two-dimensional probability distributions for each of the vectors $u(k)$ and $v(k)$ and then the distributions of the random variable $v^T(1) \otimes u(2)$. Performing all the required operations, we obtain

$$\lambda \leq \mathbb{E}[v^T(1) \otimes u(2)] = \frac{61021}{30240} \approx 2{,}0179.$$

Example 2. To find the bounds in accordance with criteria (11) and (13) we first consider the matrix

$$A(k) \otimes A^-(k) = \begin{pmatrix} 0 & \alpha_k \otimes \gamma_k^{-1} \oplus \beta_k \otimes \delta_k^{-1} \\ \alpha_k^{-1} \otimes \gamma_k \oplus \beta_k^{-1} \otimes \delta_k & 0 \end{pmatrix}$$

and determine its spectral radius:

$$\varrho_k = \left((\alpha_k \otimes \gamma_k^{-1} \oplus \beta_k \otimes \delta_k^{-1}) \otimes (\alpha_k^{-1} \otimes \gamma_k \oplus \beta_k^{-1} \otimes \delta_k)\right)^{1/2} = |\alpha_k \otimes \gamma_k^{-1} \otimes \beta_k^{-1} \otimes \delta_k|^{1/2}.$$

It is easily verified that the eigenvector

$$u(k) = \begin{pmatrix} (\alpha_k \otimes \beta_k)^{1/2} \\ (\gamma_k \otimes \delta_k)^{1/2} \end{pmatrix}.$$

corresponds to the number ϱ_k.

Applying (9) we find the vector $v(k)$ for the upper bound:

$$v(k) = \begin{pmatrix} (\alpha_k \otimes \beta_k^{-1} \oplus \gamma_k \otimes \delta_k^{-1})^{1/2} \\ (\alpha_k^{-1} \otimes \beta_k \oplus \gamma_k^{-1} \otimes \delta_k)^{1/2} \end{pmatrix},$$

and then the quantity

$$v^T(1) \otimes u(2) = ((\alpha_1 \otimes \beta_1^{-1} \oplus \gamma_1 \otimes \delta_1^{-1}) \otimes \alpha_2 \otimes \beta_2 \oplus$$

$$\oplus (\alpha_1^{-1} \otimes \beta_1 \oplus \gamma_1^{-1} \otimes \delta_1) \otimes \gamma_2 \otimes \delta_2)^{1/2}.$$

After calculating the mathematical expectation we find

$$\lambda \leq \mathbb{E}[v^T(1) \otimes u(2)] = \frac{123}{64} \approx 1{,}9219.$$

Let us determine the lower bound. The vector $v(k)$ is now calculated from formula (10) and assumes the form

$$v(k) = \begin{pmatrix} (\alpha_k^{-1} \otimes \beta_k \oplus \gamma_k^{-1} \otimes \delta_k)^{-1/2} \\ (\alpha_k \otimes \beta_k^{-1} \oplus \gamma_k \otimes \delta_k^{-1})^{-1/2} \end{pmatrix}.$$

Recalling that

$$v^T(1) \otimes u(2) = ((\alpha_1^{-1} \otimes \beta_1 \oplus \gamma_1^{-1} \otimes \delta_1)^{-1} \otimes \alpha_2 \otimes \beta_2 \oplus$$

$$\oplus (\alpha_1 \otimes \beta_1^{-1} \oplus \gamma_1 \otimes \delta_1^{-1})^{-1} \otimes \gamma_2 \otimes \delta_2)^{1/2},$$

we obtain the bound

$$\lambda \geq \mathbb{E}[v^T(1) \otimes u(2)] = \frac{75}{64} \approx 1{,}1719.$$

Example 3. Let us evaluate the bounds in accordance with criteria (12) and (14). For the upper bound we have the vectors

$$u(k) = \begin{pmatrix} (\alpha_k \oplus \beta_k)^{1/2} \\ (\gamma_k \oplus \delta_k)^{1/2} \end{pmatrix}, \qquad v(k) = \begin{pmatrix} \alpha_k \otimes (\alpha_k \oplus \beta_k)^{-1/2} \oplus \gamma_k \otimes (\gamma_k \oplus \delta_k)^{-1/2} \\ \beta_k \otimes (\alpha_k \oplus \beta_k)^{-1/2} \oplus \delta_k \otimes (\gamma_k \oplus \delta_k)^{-1/2} \end{pmatrix},$$

as well as the quantity

$$v^T(1) \otimes u(2) = (\alpha_1 \otimes (\alpha_1 \oplus \beta_1)^{-1/2} \oplus \gamma_1 \otimes (\gamma_1 \oplus \delta_1)^{-1/2}) \otimes (\alpha_2 \oplus \beta_2)^{1/2} \oplus$$

$$\oplus (\beta_1 \otimes (\alpha_1 \oplus \beta_1)^{-1/2} \oplus \delta_1 \otimes (\gamma_1 \oplus \delta_1)^{-1/2}) \otimes (\gamma_2 \oplus \delta_2)^{1/2}.$$

Calculation of the mathematical expectation yields the bound

$$\lambda \leq \mathbb{E}[v^T(1) \otimes u(2)] = \frac{21601}{11340} \approx 1{,}9049.$$

For the lower bound the vectors $u(k)$ and $v(k)$ assume the form

$$u(k) = \begin{pmatrix} (\alpha_k^{-1} \oplus \beta_k^{-1})^{-1/2} \\ (\gamma_k^{-1} \oplus \delta_k^{-1})^{-1/2} \end{pmatrix},$$

$$v(k) = \begin{pmatrix} (\alpha_k^{-1} \otimes (\alpha_k^{-1} \oplus \beta_k^{-1})^{-1/2} \oplus \gamma_k^{-1} \otimes (\gamma_k^{-1} \oplus \delta_k^{-1})^{-1/2})^{-1} \\ (\beta_k^{-1} \otimes (\alpha_k^{-1} \oplus \beta_k^{-1})^{-1/2} \oplus \delta_k^{-1} \otimes (\gamma_k^{-1} \oplus \delta_k^{-1})^{-1/2})^{-1} \end{pmatrix},$$

whence it follows that

$$v^T(1) \otimes u(2) = (\alpha_1^{-1} \otimes (\alpha_1^{-1} \oplus \beta_1^{-1})^{-1/2} \oplus \gamma_1^{-1} \otimes (\gamma_1^{-1} \oplus \delta_1^{-1})^{-1/2})^{-1} \otimes (\alpha_2^{-1} \oplus \beta_2^{-1})^{-1/2} \oplus$$

$$\oplus (\beta_1^{-1} \otimes (\alpha_1^{-1} \oplus \beta_1^{-1})^{-1/2} \oplus \delta_1^{-1} \otimes (\gamma_1^{-1} \oplus \delta_1^{-1})^{-1/2})^{-1} \otimes (\gamma_2^{-1} \oplus \delta_2^{-1})^{-1/2}.$$

Calculating the mathematical expectation we arrive at the bound

$$\lambda \geq \mathbb{E}[v^T(1) \otimes u(2)] = \frac{223}{270} \approx 0{,}8259.$$

These examples show that for the second-order system being considered here, bounds that have been obtained in accordance with criteria (11) and (12) yield the best results in calculating the upper bounds of the mean rate of growth of the state vector. Moreover, these bounds are also sharper by comparison with the bounds that were proposed in [4]. At the same time, it may be concluded from these results that the lower bounds prove to be less sharp than the corresponding bounds in [4].

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Received 23 November 2004