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THE GROWTH RATE OF STATE VECTOR IN A GENERALIZED LINEAR DYNAMICAL SYSTEM WITH RANDOM TRIANGULAR MATRIX*

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A generalized linear dynamical system with triangular random matrix is considered. By assumption, the random elements of matrix have the arbitrary probability distributions with a finite mean and variance and do not have to be independent. It is shown that under rather general conditions, the mean growth rate of the state vector of the system is determined only by the mean values of diagonal elements of the matrix.

1. Introduction. For analyzing the technical, economical, production and other systems one often applies a generalized linear dynamical model of the following form

$$x(k) = A^T(k) \otimes x(k-1),$$

where $A(k)$ is a random matrix system, $x(k)$ is a state vector of system, $\otimes$ is an operation of multiplication of matrices, given in a certain idempotent algebra [1]. Such models turn out, in particular, to be highly convenient tools for description and consideration of certain classes of systems and queueing networks [2].

In studying real systems we often need to determine the mean growth rate of a state vector of system:

$$\gamma = \lim_{k\to\infty} \frac{1}{k}\|x(k)\|,$$

where $\|\cdot\|$ is a certain idempotent analog of usual vector norm. For analyzing stochastic models, in which the matrix of system is random, the exact solution of the above-mentioned problem often turns out rather hard [3]. The results obtained are restricted, in essence, to the case of the systems with a matrix of low dimension, the elements of which are independent and * identically distributed by exponential or normal law, and also the systems with matrices of special form. In particular, in the works [4, 5] it is shown how the mean growth rate of state vector can be computed for the system of arbitrary dimension with a triangle matrix of special form, which occurs in the description of the dynamics of a class of queueing networks.

In the present work a generalized linear dynamical system with arbitrary triangle random matrix is considered. It is shown that for rather general conditions the mean growth rate of a state vector of system is determined by the mean values of diagonal elements of matrix only. In this case it is required that the random elements of matrix have the probability distributions with bounded mean values and variance but their independence is unnecessary.

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2. Idempotent algebra. By the idempotent algebra we usually mean a certain semiring with idempotent summation [1]. We shall consider a semiring with operations of summation and multiplication, namely

$$x \oplus y = \max(x, y), \qquad x \otimes y = x + y,$$

which are given on an extended set of real numbers $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$, where $\varepsilon = -\infty$. However, it should be noted that the results, represented below, can easily be reformulated for other types of semirings, for example, for the semirings with the operations $x \oplus y = \min(x, y)$ and $x \otimes y = x + y$, which are defined for any $x, y \in \mathbb{R} \cup \{+\infty\}$.

In the idempotent algebra of matrices the matrix operations $\oplus$ and $\otimes$ are defined by means of their scalar analogs by the usual rules. For the solution of a number of practical problems it turns out to be useful to extend the available family of matrix operations. In particular, an approach was suggested which is based on the completion of the above-mentioned family on account of the addition operation of a standard arithmetical summation of matrices, which is related with the operations $\oplus$ and $\otimes$ by the following inequalities:

$$(A + B) \oplus (C + D) \leq A \oplus C + B \oplus D, \qquad (A + B) \otimes (C + D) \leq A \otimes C + B \otimes D, \tag{1}$$

where A, B, C, and D are any matrices of suitable size [4, 5].

We shall call any matrix with the elements 0 or ε a support matrix. It is clear that to any matrix $A = (a_{ij})$ can assign the certain support matrix G with the elements $g_{ij} = 0$ if $a_{ij} > \varepsilon$, and $g_{ij} = \varepsilon$ otherwise. It is easily seen that the inequality $A \leq \|A\| \otimes G$, where

$$\|A\| = \bigoplus_{i,j} a_{ij}$$

is always satisfied.

We shall say that a square matrix is diagonal if all its nondiagonal elements are equal to ε or are triangle if all elements below or above a diagonal are equal to ε. Note that in the idempotent algebra the matrix $E = \mathrm{diag}(0, \dots, 0)$ plays the role of unit matrix and the matrix $\mathcal{E}$, all elements of which are equal to ε, is zeroth.

If the elements of the diagonal matrix D, situated on the diagonal, are greater than ε, then there exists the inverse matrix D^{-1} such that $D \otimes D^{-1} = D^{-1} \otimes D = E$. In addition, for any diagonal matrices D_1 and D_2 the following relation $D_1 \otimes D_2 = D_1 + D_2$ is satisfied.

For any matrix $A \neq \mathcal{E}$ and integer $k, l \geq 0$ we assume that $A^k \otimes A^l = A^{k+l}$ and $A^0 = E$. Below, the denotation of degree will be used in the sense of idempotent algebra only. The matrix A is called nilpotent if there exists p such that $A^p = \mathcal{E}$. An example of a nilpotent matrix is an upper (lower) strictly triangular matrix such that all their elements, situated on a diagonal and below (above) it, are equal to ε.

3. Preliminary results. Suppose that $D(0), D(1), \dots, D(m)$ are diagonal matrices, G is a support matrix. In works [4, 5] the following inequality was obtained:

$$D(0) \otimes \bigotimes_{j=1}^{m} (G \otimes D(j)) \leq \sum_{j=0}^{m} G^j \otimes D(j) \otimes G^{m-j}. \tag{2}$$

We make use of inequality (2) for the estimation of the products of the matrices $A(i)$, $i = 1, \dots, k$, of the form

$$A(i) = D(i) \oplus T(i),$$

where the matrices $D(1), \dots, D(k)$ are assumed to be diagonal and the matrices $T(1), \dots, T(k)$ to be nilpotent of the index p with the common support matrix G.

We introduce the following denotation:

$$A_k = A(1) \otimes \cdots \otimes A(k), \qquad D_k = D(1) \otimes \cdots \otimes D(k),$$

and also $D(l, m) = D(l + 1) \otimes \cdots \otimes D(m)$ if $m > l$ and $D(l, m) = E$ otherwise.

Lemma 1. *The following inequality*

$$D_k \leq A_k \leq t_k \otimes B_k$$

is satisfied. Here $t_k = \left(\bigoplus_{i=1}^{k} \|T(i)\| \oplus 0\right)^{p-1}$, $\quad B_k = \bigoplus_{m=0}^{p-1} \sum_{j=0}^{m} G^j \otimes \left(\bigoplus_{0 \leq r < s \leq k} D(r, s)\right) \otimes G^{m-j}$.

Proof. The left-hand inequality results directly from the obvious relation $A(i) = D(i) \oplus T(i) \geq D(i)$ for all $i = 1, \dots, k$.

To check the right-hand inequality we consider the quantity

$$A_k = \bigotimes_{i=1}^{k} A(i) = \bigotimes_{i=1}^{k}(D(i) \oplus T(i)).$$

We transform the relation for A_k by making use of the distributive property of $\otimes$ with respect to $\oplus$. With provision for the nilpotent property of the matrices $T(i)$, assuming that $l_0 = 0$, $l_{m+1} = k+1$, we obtain

$$A_k = \bigoplus_{m=0}^{p-1} \bigoplus_{1 \le l_1 < \cdots < l_m \le k} D(l_0, l_1 - 1) \otimes \bigotimes_{j=1}^{m} (T(l_j) \otimes D(l_j, l_{j+1} - 1))$$
$$\le t_k \otimes \bigoplus_{m=0}^{p-1} \bigoplus_{1 \le l_1 < \cdots < l_m \le k} D(l_0, l_1 - 1) \otimes \bigotimes_{j=1}^{m} (G \otimes D(l_j, l_{j+1} - 1)).$$

Applying inequalities (2), (1), and the distributive law, we obtain

$$\bigoplus_{1 \le l_1 < \cdots < l_m \le k} D(l_0, l_1 - 1) \otimes \bigotimes_{j=1}^{m} (G \otimes D(l_j, l_{j+1} - 1))$$
$$\le \bigoplus_{1 \le l_1 < \cdots < l_m \le k} \sum_{j=0}^{m} G^j \otimes D(l_j, l_{j+1} - 1) \otimes G^{m-j} \le \sum_{j=0}^{m} G^j \otimes \left(\bigoplus_{0 \le r < s \le k} D(r, s) \right) \otimes G^{m-j}.$$

This implies the inequality $A_k \le t_k \otimes B_k$. □

4. Generalized linear dynamical system. Suppose that $A(k)$ is a random $(n \times n)$-matrix, $\boldsymbol{x}(k)$ is an n-dimension vector of states. Consider a system, the dynamics of which is described by the equation

$$\boldsymbol{x}(k) = A^T(k) \otimes \boldsymbol{x}(k-1).$$

We also assume that the matrices $A(k)$, $k = 1, 2, \ldots$, are identically distributed and independent, the mathematical expectation $\mathbb{E}\|A(1)\|$ and the variance $\mathbb{D}\|A(1)\|$ are finite and the coordinates of the initial vector $\boldsymbol{x}(0)$ are bounded with probability one.

The mean growth rate γ of the state vector $\boldsymbol{x}(k)$ can be determined in the following way:

$$\gamma = \lim_{k\to\infty} \frac{1}{k}\|\boldsymbol{x}(k)\| = \lim_{k\to\infty} \frac{1}{k}\|A_k\|,$$

where $A_k = A(1) \otimes \cdots \otimes A(k)$.

In the case when the matrix $A(1)$ is diagonal the obtaining of the quantity γ is a totally simple problem. In fact, in the considered case we have, as above, the following

$$A_k = \bigotimes_{i=1}^{k} A(i) = \sum_{i=1}^{k} A(i),$$

$\gamma = \lim_{k\to\infty} \|A_k\|/k = \|\mathbb{E}[A(1)]\|$. Here $\mathbb{E}[A(1)]$ is a matrix, obtained from $A(1)$ by means of the change of its elements to their mathematical expectation under the condition $\mathbb{E}[\varepsilon] = \varepsilon$.

5. The case of system triangular matrix. Consider a system, for which the matrices $A(1), A(2), \ldots$ are triangular. For each $i = 1, 2, \ldots$, we determine the diagonal matrix $D(i)$, the elements of which, situated on diagonal, coincide with the corresponding elements of the matrix $A(i)$. We also assume that $T(i)$ denotes a strictly triangular matrix, obtained from $A(i)$ by the change of all its diagonal elements to ε. It is clear that $A(i) = D(i) \oplus T(i)$.

Suppose that with probability one the support matrices for $D(1)$ and $T(1)$ are equal to E and $G \ne \mathcal{E}$, respectively, where G is a certain strictly triangular support matrix, which is obviously nilpotent with the index $p \le n$.

We assume first that all the diagonal elements of the matrix $A(1)$ have a zero mean value, i.e., in other words, the following relation $\mathbb{E}[D(1)] = E$ is satisfied.

Lemma 2. *If the condition* $\mathbb{E}[D(1)] = E$ *is satisfied, then* $\gamma = 0$ *with probability one.*

Proof. By Lemma 1 we have

$$\frac{1}{k}\|D_k\| \le \frac{1}{k}\|A_k\| \le \frac{1}{k} t_k \otimes \left\| \frac{1}{k} B_k \right\|,$$

where D_k, t_k and B_k are defined just in the same way as in the above-mentioned lemma.

Since $\lim_{k\to\infty} \|D_k\|/k = \|\mathbb{E}[D(1)]\| = \|E\| = 0$, then the left-hand inequality implies $\gamma \geq 0$. Now it remains to check the validity of opposite inequality.

Consider the quantity t_k. Note that $\|T(i)\|$, $i = 1, 2, \dots$, are the independent identically distributed random values with a finite mean value and variance. With increasing k their maximum increases not faster than $\sqrt{k}$ (see [6, 7]). Therefore we obtain

$$\lim_{k\to\infty} \frac{1}{k} t_k = \lim_{k\to\infty} \left(\frac{1}{k} \bigoplus_{i=1}^{k} \|T(i)\| \oplus 0 \right)^{p-1} = 0.$$

In addition, as was shown in [4, 8], for the independent and identically distributed diagonal matrices $D(i)$, $i = 1, 2, \dots$, such that $\mathbb{E}[D(1)] = E$ the following relation

$$\lim_{k\to\infty} \frac{1}{k} \bigoplus_{1\leq r<s\leq k} D(r,s) = E \quad \text{with probability one}$$

is satisfied. This implies that

$$\lim_{k\to\infty} \frac{1}{k} B_k = \lim_{k\to\infty} \bigoplus_{m=0}^{p-1} \sum_{j=0}^{m} G^j \otimes \left(\frac{1}{k} \bigoplus_{0\leq r<s\leq k} D(r,s) \right) \otimes G^{m-j} = \bigoplus_{m=0}^{p-1} G^m \quad \text{with probability one.}$$

Taking into account that in this case with probability one we have

$$\lim_{k\to\infty} \left\| \frac{1}{k} B_k \right\| = \left\| \bigoplus_{m=0}^{p-1} G^m \right\| = 0.$$

Thus, we arrive at the inequality $\gamma \leq 0$ with probability one. □

Finally, we consider the general case. Introduce the following denotation: $\bar{D} = \mathbb{E}[D(1)]$.

Theorem 1. *For a system with the triangular matrix $A(1)$ with probability one the following relation*

$$\gamma = \|\bar{D}\|$$

is satisfied.

Proof. In the same way as in Lemma 2, we can easily show that $\gamma \geq \|\bar{D}\|$.

To check the opposite inequality we determine for all $i = 1, 2, \dots$ the following relations:

$$A'(i) = D'(i) \oplus T'(i), \qquad D'(i) = \bar{D}^{-1} \otimes D(i), \qquad T'(i) = \bar{D}^{-1} \otimes T(i).$$

Further we assume that $A'_k = A'(1) \otimes \cdots \otimes A'(k)$ and also $\gamma' = \lim_{k\to\infty} \|A'_k\|/k$.

Note that the following inequality

$$A_k \leq \|\bar{D}\|^k \otimes A'_k$$

is satisfied. It follows that $\|A_k\|/k \leq \|\bar{D}\| \otimes \|A'_k\|/k$ and, therefore, $\gamma \leq \|\bar{D}\| \otimes \gamma'$.

Since by Lemma 2 with probability one the relation $\gamma' = 0$ is satisfied, the inequality $\gamma \leq \|\bar{D}\|$ holds with probability one. This implies the assertion of theorem. □

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