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A CONVERGENCE THEOREM FOR DIMENSIONALITIES OF MATRIX AND THE EVALUATION OF EIGENVALUE IN IDEMPOTENT ALGEBRA

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In the the case of finite matrices in idempotent algebra the inequalities, joining certain numerical functions and the eigenvalues of dimensionality of matrix, are obtained. These inequalities are used for proving a convergence theorem in studying the asymptotic behavior of dimensionality of matrix. It is shown that a general formula for the evaluation of eigenvalue can be obtained as a certain corollary of this theorem.

1. Introductions. One of the basic results of the spectral theory of matrices in the idempotent algebra is a convergence theorem, which establishes that for any matrix A the relation holds

$$\lim_{k\to\infty} \|A^k\|^{1/k} = \bigoplus_{m=1}^{n} [\mathrm{tr}(A^m)]^{1/m}, \tag{1}$$

where n is a dimensionality of matrix. This result was first obtained in the works [1, 2] in connection with the study of the asymptotic properties of solutions of the dynamic programming problem. The other proofs (1), which follows from the analysis of cyclic paths in the graph corresponding to matrix A, can be found in [3, 4, 5].

At the same time in the works [6, 7, 8] it was established that the value

$$\rho = \bigoplus_{m=1}^{n} [\mathrm{tr}(A^m)]^{1/m}, \tag{2}$$

is a unique single eigenvalue in the case of nonnegative matrix A, which satisfies the equation

$$A \otimes \boldsymbol{x} = \rho \otimes \boldsymbol{x}. \tag{3}$$

This result was generalized then in the works [3, 5, 9, 10] on the case of arbitrary matrices, which some eigenvalues can exist for. In this case (2) corresponds to the largest eigenvalue of the matrix A. Note that the convergence theorem turns out to be a direct analog of a classical result on the spectral radius of bounded linear operator.

The existence conditions of a unique single eigenvalue are considered in the works [5, 9, 10]. It was shown that if the matrix is nonanalyzable, then it has a unique single eigenvalue. In the most general form this result was obtained in [9] due to the developed in this works theory of the endomorphisms of semimodules under idempotent semirings and that in [10], using the Frobenius—Perron theorem on the eigenvalue of nonnegative matrix. The proof, given in [5], makes use of the analysis of cycles in the corresponding graph. In this work it is also shown that for nonanalyzable matrix the elements of any eigenvector are finite.

Note that the distinction of present approaches to the study of the asymptotic behavior of dimensionalities of matrix is the partition of the common problem into two parts: on the one hand the convergence of the sequence $\|A^k\|^{1/k}$ to the value (2) is proved and on the other hand it is established that (2) is eigenvalue. In this case for both proofs, in some way or other, the interpretation of the problem in terms of the theory of graphs is required, which somewhat complicates the reasonings and makes them less rigorous.

In the present work a simple proof is given for the theorem on the convergence as $k \to \infty$ of the values

$$\|A^k\|^{1/k}, \qquad [\mathrm{tr}(A^k)]^{1/k}$$

to the eigenvalue of the matrix A in the case when all elements of matrix are finite. The proof suggested is based on a number of general inequalities, obtained for the eigenvalues and dimensionalities of matrix and it does not use relation (2) for eigenvalue in explicit form. At the same time it is shown that general formula (2) can be constructed by the proved convergence theorem as a certain corollary of it.

The proof has an algebraic nature and, which essentially does not require the analysis of paths in the graph corresponding to the matrix.

2. Idempotent algebra. Denote by $\mathbb{R}_\varepsilon$ a set of real numbers, extended by the addition element $\varepsilon = -\infty$, and give the following operation

$$x \oplus y = \max(x, y), \qquad x \otimes y = x + y$$

for any $x, y \in \mathbb{R}_\varepsilon$ under the condition $x \otimes \varepsilon = \varepsilon \otimes x = \varepsilon$.

The set $\mathbb{R}_\varepsilon$ with the operations $\oplus$ and $\otimes$ is a commutative semiring with idempotent summation, the zero and unit elements of which are ε and 0, respectively. Such semiring are usually called an idempotent algebra [10] or (max, +)-algebra [3, 5].

In this algebra for any $x \in \mathbb{R}$ it is defined the invertible element x^{-1} under the operation $\otimes$, which is $-x$ in usual arithmetic. For any $x \in \mathbb{R}$ and rational q the dimensionality x^q can usually be defined, which is obviously equal to qx in standard representation. Below the notation of dimensionality will be use only in the sense of idempotent algebra. However, for simplicity the dimensionality of number will sometimes be changed by arithmetical multiplication.

We can easily check that for any $x_1, \dots, x_k \in \mathbb{R}_\varepsilon$ the following inequality (analog of the inequality for geometric and arithmetical means)

$$\bigotimes_{i=1}^{k} x_i \leq \left(\bigoplus_{i=1}^{k} x_i\right)^k = k \bigoplus_{i=1}^{k} x_i \tag{4}$$

is valid.

The idempotent algebra of $(n \times n)$-matrices on the set $\mathbb{R}_\varepsilon^{n\times n}$ can be introduced in the following way. For any two matrices $A, B \in \mathbb{R}_\varepsilon^{n\times n}$ we put

$$\{A \oplus B\}_{ij} = \{A\}_{ij} \oplus \{B\}_{ij}, \qquad \{A \otimes B\}_{ij} = \bigoplus_{k=1}^{n} \{A\}_{ik} \otimes \{B\}_{kj}.$$

Clearly, the matrix $\mathcal{E}$, all components of which are equal to ε, has the properties of zero element. The matrix $E = \mathrm{diag}(0, \dots, 0)$ with the nondiagonal elements equal to ε, play the role of the unit matrix.

The operations $\oplus$ and $\otimes$ have an obvious property of monotonicity: the componentwise inequalities $A \leq C$ and $B \leq D$ yield the inequalities $A \oplus B \leq C \oplus D$ and $A \otimes B \leq C \otimes D$.

Let be $A \neq \mathcal{E}$ and put $A^0 = E$ and $A^k \otimes A^l = A^{k+l}$ for any integer $k, l \geq 0$.

For any matrix $A = (a_{ij}) \in \mathbb{R}_\varepsilon^{n\times n}$ we can define the following values

$$\|A\| = \bigoplus_{1\leqslant i,j\leqslant n} a_{ij}, \qquad \mathrm{tr}(A) = \bigoplus_{i=1}^{n} a_{ii}.$$

It is easily seen that the inequality $A \leq B$ implies $\|A\| \leq \|B\|$ and $\mathrm{tr}(A) \leq \mathrm{tr}(B)$.

For any $A, B \in \mathbb{R}_\varepsilon^{n\times n}$ and $c \in \mathbb{R}_\varepsilon$ the following relations

$$\|c \otimes A\| = c \otimes \|A\|, \qquad \|A \otimes B\| \leq \|A\| \otimes \|B\|,$$
$$\mathrm{tr}(c \otimes A) = c \otimes \mathrm{tr}(A), \qquad \mathrm{tr}(A \otimes B) = \mathrm{tr}(B \otimes A)$$

are valid.

Consider the arbitrary matrix A and denote the i-th column of matrix by $\boldsymbol{a}_i$ and the j-th row by $\boldsymbol{a}^j$. For any $A, B \in \mathbb{R}_\varepsilon^{n\times n}$ we have

$$\|A \otimes B\| = \bigoplus_{i=1}^{n} \|\boldsymbol{a}_i\| \otimes \|\boldsymbol{b}^i\|.$$

3. Matrices with finite elements. Consider the matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$. All elements of such matrix satisfy the condition $a_{ij} > \varepsilon$, i.e. they are finite.

Define the matrix $A^- = (a_{ij}^-)$ with elements $a_{ij}^- = a_{ji}^{-1}$ for all $i,j = 1,\dots,n$. Then

$$\min_{1\le i,j\le n} a_{ij} = \|A^-\|^{-1}.$$

It is easy to check that for any $A, B \in \mathbb{R}^{n\times n}$ the following inequality

$$\operatorname{tr}(A \otimes B) \ge \|A^-\|^{-1} \otimes \|B\| \tag{5}$$

is satisfied.

As in the case of matrices, for any vector $\boldsymbol{x} = (x_1,\dots,x_n)^T \in \mathbb{R}^n$ we can introduce the vector $\boldsymbol{x}^- = (x_1^{-1},\dots,x_n^{-1})$.

By the obvious relation $\boldsymbol{x} \otimes \boldsymbol{x}^- \ge E$ for any $A \in \mathbb{R}_\varepsilon^{n\times n}$ and $\boldsymbol{x} \in \mathbb{R}^n$ we have

$$A \le A \otimes \boldsymbol{x} \otimes \boldsymbol{x}^-, \tag{6}$$

$$A \le \boldsymbol{x} \otimes \boldsymbol{x}^- \otimes A. \tag{7}$$

Finally, we can show that for any $A \in \mathbb{R}^{n\times n}$ and $\boldsymbol{x} \in \mathbb{R}^n$ the following inequalities

$$\|(A \otimes \boldsymbol{x})^-\| \le \|\boldsymbol{x}\|^{-1} \otimes \|A^-\|, \tag{8}$$

$$\|(A \otimes \boldsymbol{x})^- \otimes A\| \le \|\boldsymbol{x}\|^{-1} \otimes \|A^- \otimes A\| \tag{9}$$

are valid.

4. Inequalities for dimensionalities of matrix. Suppose, the matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ is nonanalyzable (i.e. it cannot be reduced to the block-triangle form by the interchange of like rows and columns). As is known (see, for example, [5]), for any nonanalyzable matrix there exists the unique single eigenvalue $\rho > \varepsilon$ and also, at least, the one eigenvector $\boldsymbol{x} \in \mathbb{R}^n$, which satisfy equation (3). Note that any matrix, all elements of which are finite, is the special case of nonanalyzable matrix.

Lemma 1. *For any nonanalyzable matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ and integer $k > 0$ the two-sided inequality holds*

$$\operatorname{tr}(A^k) \le \rho^k \le \|A^k\|. \tag{10}$$

Proof. Let $\boldsymbol{x}$ be an eigenvector of the matrix A, corresponding to ρ. Then the right-hand inequality results from equation (3), namely

$$\rho^k \otimes \|\boldsymbol{x}\| = \|\rho^k \otimes \boldsymbol{x}\| = \|A^k \otimes \boldsymbol{x}\| \le \|A^k\| \otimes \|\boldsymbol{x}\|.$$

(obviously, the inequality is valid for any matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$.)

Now we write inequality (6) in the form

$$A^k \le A^k \otimes \boldsymbol{x} \otimes \boldsymbol{x}^- = \rho^k \otimes \boldsymbol{x} \otimes \boldsymbol{x}^-.$$

Taking into account that $\boldsymbol{x}^- \otimes \boldsymbol{x} = 0$, we obtain the following left-hand inequality

$$\operatorname{tr}(A^k) \le \operatorname{tr}(\rho^k \otimes \boldsymbol{x} \otimes \boldsymbol{x}^-) = \rho^k \otimes \operatorname{tr}(\boldsymbol{x}^- \otimes \boldsymbol{x}) = \rho^k.$$

Lemma 2. *For any matrix $A \in \mathbb{R}^{n\times n}$ and integer $k > 0$ the following inequalities hold*

$$\|A^k\| \le \rho^{k+1} \otimes \|A^-\|, \tag{11}$$

$$\|A^k\| \le \rho^k \otimes \|A^- \otimes A\|, \tag{12}$$

$$\|A^k\| \le \operatorname{tr}(A^{k+1}) \otimes \|A^-\|. \tag{13}$$

Proof. We prove inequality (11). By (6)

$$A^k \le (\boldsymbol{x}^- \otimes A \otimes \boldsymbol{x})^{k-1} \otimes A \otimes \boldsymbol{x} \otimes \boldsymbol{x}^-.$$

Let $\boldsymbol{x}$ be the eigenvector of the matrix A, corresponding to ρ. Then the inequality can be represented as

$$A^k \le \rho^{k-1} \otimes A \otimes \boldsymbol{x} \otimes \boldsymbol{x}^- = \rho^{k+1} \otimes \boldsymbol{x} \otimes (\rho \otimes \boldsymbol{x})^- = \rho^{k+1} \otimes \boldsymbol{x} \otimes (A \otimes \boldsymbol{x})^-,$$

which by (8) gives

$$\|A^k\| \le \rho^{k+1} \otimes \|\boldsymbol{x} \otimes (A \otimes \boldsymbol{x})^-\| = \rho^{k+1} \otimes \|\boldsymbol{x}\| \otimes \|(A \otimes \boldsymbol{x})^-\| \le \rho^{k+1} \otimes \|A^-\|.$$

For proving (12) we make use of inequality (7):

$$A^k \le (\boldsymbol{x}^- \otimes A \otimes \boldsymbol{x})^{k-1} \otimes \boldsymbol{x} \otimes \boldsymbol{x}^- \otimes A.$$

If $\boldsymbol{x}$ is the eigenvector, corresponding to ρ, then we have

$$A^k \le \rho^{k-1} \otimes \boldsymbol{x} \otimes \boldsymbol{x}^- \otimes A = \rho^k \otimes \boldsymbol{x} \otimes (A \otimes \boldsymbol{x})^- \otimes A,$$

and, therefore,

$$\|A^k\| \le \rho^k \otimes \|\boldsymbol{x} \otimes (A \otimes \boldsymbol{x})^- \otimes A\| = \rho^k \otimes \|\boldsymbol{x}\| \otimes \|(A \otimes \boldsymbol{x})^- \otimes A\|.$$

Taking into account (9), we get inequality (12).

Finally, applying (5), we can easily check that inequality (13) holds

$$\mathrm{tr}(A^{k+1}) = \mathrm{tr}(A \otimes A^k) \ge \|A^-\|^{-1} \otimes \|A^k\|.$$

5. Convergence theorem. The asymptotic behavior of the dimensionalities of matrix with finite elements is described by the following

Theorem 1. *For any matrix* $A \in \mathbb{R}^{n\times n}$ *the relations hold*

$$\lim_{k\to\infty} \|A^k\|^{1/k} = \lim_{k\to\infty} \frac{1}{k}\|A^k\| = \rho, \tag{14}$$

$$\lim_{k\to\infty} [\mathrm{tr}(A^k)]^{1/k} = \lim_{k\to\infty} \frac{1}{k}\,\mathrm{tr}(A^k) = \rho, \tag{15}$$

where ρ *is an eigenvalue of matrix.*

Proof. Applying (10) and (12), we obtain the following inequality

$$\rho^k \le \|A^k\| \le \rho^k \otimes \|A^- \otimes A\|.$$

By the denotation $\Delta_1 = \|A^- \otimes A\|$ it can be represented as

$$\rho \le \frac{1}{k}\|A^k\| \le \rho + \frac{1}{k}\Delta_1.$$

Since the value Δ_1 under the condition $A \in \mathbb{R}^{n\times n}$ is bounded, the last inequality yields (14).

By inequalities (10) and (13) we have

$$\rho^k \otimes \|A\|^{-1} \otimes \|A^-\|^{-1} \le \mathrm{tr}(A^k) \le \rho^k.$$

Denoting $\Delta_2 = \|A\| \otimes \|A^-\| < \infty$, we obtain the following inequality

$$\rho - \frac{1}{k}\Delta_2 \le \frac{1}{k}\,\mathrm{tr}(A^k) \le \rho,$$

which results in (15).

Lemma 3. *For any matrix* $A \in \mathbb{R}^{n\times n}$ *the following relation holds*

$$\lim_{k\to\infty} \bigoplus_{m=1}^{k} [\mathrm{tr}(A^m)]^{1/m} = \lim_{k\to\infty} \bigoplus_{m=1}^{k} \frac{1}{m}\,\mathrm{tr}(A^m) = \rho.$$

The proof is by the same reasoning as in proving Theorem 1.

6. Evaluation of eigenvalue. We shall show how the general formula (2) for eigenvalue can be obtained on the basis of the Lemma 3.

Theorem 2. *For any matrix* $A \in \mathbb{R}^{n\times n}$ *the relation holds*

$$\rho = \bigoplus_{m=1}^{n} [\mathrm{tr}(A^m)]^{1/m} = \bigoplus_{m=1}^{n} \frac{1}{m}\,\mathrm{tr}(A^m),$$

where n *is a dimension of* A.

Proof. Taking into account Lemma 3, we can check that for all integer $k > 0$ the following inequality

$$\frac{1}{k}\,\mathrm{tr}(A^k) \le \bigoplus_{m=1}^{n} \frac{1}{m}\,\mathrm{tr}(A^m) \tag{16}$$

is satisfied.

Obviously, (16) is valid for all $k \le n$. We shall show, that this inequality is also valid for $k > n$.

Note first that for any k the relation holds

$$\mathrm{tr}(A^k) = \bigoplus_{i_1=1}^{n}\bigoplus_{i_2=1}^{n}\cdots\bigoplus_{i_k=1}^{n} a_{i_1i_2}\otimes a_{i_2i_3}\otimes\cdots\otimes a_{i_ki_1}. \tag{17}$$

We introduce

$$S(i_1,\dots,i_k) = a_{i_1i_2}\otimes a_{i_2i_3}\otimes\cdots\otimes a_{i_ki_1}.$$

By (17) for any set of indices $i_1,\dots,i_k$ we have

$$S(i_1,\dots,i_k) \le \mathrm{tr}(A^k). \tag{18}$$

If $k > n$, then in the sequence of indices $i_1,\dots,i_k$ there exist repeating values. In addition any continuous segment of the sequence, in which repetitions are lacking, consists of no more than n indices.

It is clear that in this case $S(i_1,\dots,i_k)$ can be represented as

$$S(i_1,\dots,i_k) = S(j_1,\dots,j_{m_1})\otimes S(j_{m_1+1},\dots,j_{m_2})\otimes\cdots\otimes S(j_{m_{r-1}+1},\dots,j_{m_r}),$$

where r, m_i are certain integer numbers such that $1 \le m_1 < \cdots < m_r = k$, $r > 1$, $m_1 \le n$ and also $m_{i+1} - m_i \le n$ for all $i = 1,\dots,r-1$.

By (18) we have $S(i_1,\dots,i_k) \le \mathrm{tr}(A^{m_1})\otimes\mathrm{tr}(A^{m_2-m_1})\otimes\cdots\otimes\mathrm{tr}(A^{m_r-m_{r-1}})$. After grouping the terms in the right-hand sides it can be represented as

$$S(i_1,\dots,i_k) \le \alpha_1\,\mathrm{tr}(A)\otimes\cdots\otimes\alpha_n\,\mathrm{tr}(A^n) = \bigotimes_{m=1}^{n}\alpha_m\,\mathrm{tr}(A^m),$$

where α_m are certain nonnegative integer numbers for all $m = 1,\dots,n$ and $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = k$.

Put $\alpha'_m = m\alpha_m$, and note that $\alpha'_1 + \cdots + \alpha'_n = k$. By (4) we have

$$S(i_1,\dots,i_k) \le \bigotimes_{m=1}^{n}\alpha_m\,\mathrm{tr}(A^m) = \bigotimes_{m=1}^{n}\frac{\alpha'_m}{m}\,\mathrm{tr}(A^m) \le k\bigoplus_{m=1}^{n}\frac{1}{m}\,\mathrm{tr}(A^m).$$

Taking into account that this inequality is valid, for any choice of the indices $i_1,\dots,i_k$ we can estimate the right-hand side of (17), namely

$$\mathrm{tr}(A^k) = \bigoplus_{i_1=1}^{n}\bigoplus_{i_2=1}^{n}\cdots\bigoplus_{i_k=1}^{n} S(i_1,\dots,i_k) \le k\bigoplus_{m=1}^{n}\frac{1}{m}\,\mathrm{tr}(A^m).$$

The last inequality is equal to inequality (16), which together with Lemma 3 proves theorem.

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