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ESTIMATION OF THE RATE OF GROWTH OF THE STATE VECTOR OF A GENERALIZED LINEAR DYNAMICAL SYSTEM WITH RANDOM MATRIX

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A stochastic model of a generalized linear dynamical system with random matrix the elements of which may have arbitrary distributions and are not independent is considered. Based on use of the apparatus and methods of idempotent algebra, new lower and upper bounds of the mean rate of growth of the state vector of the system are obtained for the model. Examples of a numerical computation of the bounds are presented.

1. Introduction. We wish to consider a model of a dynamical system the evolution of which may be described by means of a vector equation of the form

$$x(k) = A(k) \otimes x(k-1).$$

which is linear in some idempotent algebra. Such generalized linear dynamical models have found applications in the analysis of production system, business processes, and computational systems and networks [1]. In particular, these types of models prove to be an extremely useful tool in the description of certain classes of systems and networks with queues [2].

One of the important characteristics of the system is the mean rate of growth of the state vector of the system $x(k)$. The vector may be defined as the quantity

$$\lambda = \lim_{k\to\infty} \frac{1}{k} \|x(k)\|,$$

where $\|\cdot\|$ is some idempotent analog of the ordinary vector norm. The quantity λ is often called the generalized Lyapunov constant (exponent) [1, 3].

In the case of deterministic systems the problem of finding λ was successfully solved in [4, 5]. At the same time, an exact determination of the value of λ for stochastic systems for which the matrix $A(k)$ is random usually proves to be a rather complex problem. Existing results in this area are limited to the case of systems with matrix of low dimension the elements of which are independent and have exponential or normal distribution [3, 6, 7]. In this case the development of effective methods of estimating the generalized Lyapunov constant for a broad class of models of systems acquires special importance. A number of results related to the construction of bounds is presented in [1, 8, 9].

In the present study we will consider a stochastic model of a generalized linear dynamical system with random matrix the elements of which may have arbitrary distributions and are not independent. Based on the use of the apparatus and methods of idempotent algebra, new lower and upper bounds of the mean rate of growth of the state vector of the system are obtained for the model. Examples of a numerical computation of the bounds are presented.

2. Idempotent algebra. We denote by $\mathbb{R}_\varepsilon$ the set of real numbers, complemented with the addition of the element $\varepsilon = -\infty$, and specify the operations

$$x \oplus y = \max(x, y), x \otimes y = x + y$$

for any $x, y \in \mathbb{R}_\varepsilon$ under the condition that $x \otimes \varepsilon = \varepsilon \otimes x = \varepsilon$.

The set $\mathbb{R}_\varepsilon$ with the operations $\oplus$ and $\otimes$ is a commutative semiring with idempotent addition the zero and unit elements of which are ε and 0, respectively. Such a semiring is usually called an idempotent algebra [1, 10].

In such an algebra, for every $x \neq \varepsilon$ an element x^{-1} the inverse relative to the operation $\otimes$, which constitutes $-x$ in ordinary arithmetic, is defined.

For any two $(n \times n)$-matrices $A = (a_{ij})$ and $B = (b_{ij})$, we have

$$\{A \oplus B\}_{ij} = a_{ij} \oplus b_{ij}, \qquad \{A \otimes B\}_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes b_{kj}.$$

The matrix $\mathcal{E} = (\varepsilon)$ and the matrix $E = \mathrm{diag}(0, \ldots, 0)$ with elements equal to ε outside the diagonal perform the functions of the zero and unit matrix, respectively.

For any matrix $A \neq \mathcal{E}$ and integers k and $l \geq 0$, we set $A^0 = E$ and $A^k \otimes A^l = A^{k+l}$.

In addition to the matrix operations $\otimes$ and $\oplus$ we will use the operation of the ordinary arithmetic addition of matrices. In writing out the arithmetic operations we will suppose that in any sequence of operations, arithmetic addition is executed following the operations $\otimes$ and $\oplus$.

It is easily verified that for any matrices A, B, C, and D, the following inequality is valid:

$$(A + B) \otimes (C + D) \leq A \otimes C + B \otimes D. \tag{1}$$

For any matrix $A = (a_{ij})$, the following quantities may be defined:

$$\|A\| = \bigoplus_{1 \leqslant i,j \leqslant n} a_{ij}, \qquad \mathrm{tr}(A) = \bigoplus_{i=1}^{n} a_{ii}.$$

Suppose A and B are certain matrices. It is clear that it follows from the componentwise inequality $A \leq B$ that $\|A\| \leq \|B\|$. Moreover, the following obvious relations hold:

$$\|A \oplus B\| = \|A\| \oplus \|B\|, \ \|A \otimes B\| \leq \|A\| \otimes \|B\|, \ \|A + B\| \leq \|A\| + \|B\|.$$

For any number $c > 0$, we have $\|cA\| = c\|A\|$ under the condition $c\varepsilon = \varepsilon$.

Let us consider an arbitrary matrix A. The eigenvalue λ and eigenvector x corresponding to it of the matrix A satisfy the equality

$$A \otimes x = \lambda \otimes x.$$

The following result holds [4] (cf. [5]).

Theorem 1. For any matrix A the following limit exists:

$$\lim_{k \to \infty} \frac{1}{k}\|A^k\| = \rho(A) = \bigoplus_{i=1}^{n} \frac{1}{i}\mathrm{tr}(A^i),$$

where $\rho(A)$ is the maximal eigenvalue of the matrix.

Let us suppose that the elements of A are random variables. We denote by $\mathbb{E}[A]$ the matrix obtained as a result of the application of the operator of the mathematical expectation to each element of A under the condition that $\mathbb{E}[\varepsilon] = \varepsilon$.

Let A and B be random matrices. The following inequalities hold:

$$\mathbb{E}[A \oplus B] \geq \mathbb{E}[A] \oplus \mathbb{E}[B], \quad \mathbb{E}[A \otimes B] \geq \mathbb{E}[A] \otimes \mathbb{E}[B], \quad \mathbb{E}\|A\| \geq \|\mathbb{E}[A]\|.$$

Moreover, if A and B are independent,

$$\mathbb{E}[A \oplus B] \geq \mathbb{E}[A \oplus \mathbb{E}[B]], \qquad \mathbb{E}\|A \otimes B\| \geq \mathbb{E}\|A \otimes \mathbb{E}[B]\|.$$

3. Generalized linear systems. Let us consider a system the dynamic behavior of which is described by the generalized linear equation

$$x(k) = A^T(k) \otimes x(k-1),$$

where $A(k)$ is a random $(n \times n)$-matrix of the system and $x(k)$ the k-dimensional state vector of the system. We will suppose that the matrices $A(k)$, $k = 1, 2, \ldots$, are identically distributed and independent, and that the mathematical expectation $\mathbb{E}\|A(1)\|$ is finite.

One of the most important characteristic of the system is the mean rate of growth of the state vector $x(k)$, which is determined thus:

$$\lambda = \lim_{k \to \infty} \frac{1}{k}\|x(k)\|.$$

Provided that this limit exists, the quantity λ is often called the generalized Lyapunov constant for the given system [1, 3].

We introduce the matrix

$$A_k = A(1) \otimes \ldots \otimes A(k).$$

Supposing that the coordinates of the initial vector $x(0)$ are bounded, the generalized Lyapunov constant may be represented in the form

$$\lambda = \lim_{k \to \infty} \frac{1}{k}\|A_k\|.$$

The ergodic theorem, which is proved in [11], is a useful tool for verifying the existence of this limit. From this theorem it follows that for the system we are considering, this limit exists with probability 1 and that the following limit also exists:

$$\lim_{k \to \infty} \frac{1}{k}\mathbb{E}\|A_k\| = \lambda.$$

Simple upper and lower bounds for λ may be obtained in the following manner. Since for $\|A_k\|$,

$$\|A_k\| \leq \|A(1)\| \otimes \ldots \otimes \|A(k)\| = \|A(1)\| + \ldots + \|A(k)\|,$$

is satisfied, it is self-evident that

$$\lambda = \lim_{k \to \infty} \frac{1}{k}\|A_k\| \leq \mathbb{E}\|A(1)\|. \tag{2}$$

On the other hand, the inequalities

$$\mathbb{E}\|A_k\| \geq \|\mathbb{E}[A_k]\| \geq \|\mathbb{E}[A(1)] \otimes \cdots \otimes \mathbb{E}[A(k)]\| = \|(\mathbb{E}[A(1)])^k\|,$$

hold, whence, by Theorem 1, there follows the bound

$$\lambda \geq \rho(\mathbb{E}[A(1)]). \tag{3}$$

Note that inequalities (2) and (3) are exact in the sense that systems may be indicated for which they turn into equalities.

Example 1. Let $\{\alpha_k\}$ be a sequence of independent identically distributed random variables. Consider the system with matrix

$$A(k) = \begin{pmatrix} \alpha_k & 0 \\ 0 & \alpha_k \end{pmatrix}.$$

It is easily verified that

$$\|A_k\| = \bigotimes_{i=1}^{k}(0 \oplus \alpha_i) = \sum_{i=1}^{k}(0 \oplus \alpha_i),$$

whence

$$\lambda = \lim_{k\to\infty} \frac{1}{k}\|A_k\| = \mathbb{E}[0 \oplus \alpha_1] = \mathbb{E}\|A(1)\|.$$

Example 2. Let $\{\alpha_k\}$ and $\{\beta_k\}$ be two sequences of independent identically distributed random variables. Define the matrix of $A(k)$ in the form

$$A(k) = \begin{pmatrix} \alpha_k & \varepsilon \\ \varepsilon & \beta_k \end{pmatrix}.$$

Since

$$A_k = \begin{pmatrix} \alpha_1 \otimes \cdots \otimes \alpha_k & \varepsilon \\ \varepsilon & \beta_1 \otimes \cdots \otimes \beta_k \end{pmatrix},$$

we have

$$\|A_k\| = \bigotimes_{i=1}^{k} \alpha_i \oplus \bigotimes_{i=1}^{k} \beta_i = \left(\sum_{i=1}^{k} \alpha_i\right) \oplus \left(\sum_{i=1}^{k} \beta_i\right).$$

Computation of the limit leads to the following result:

$$\lambda = \mathbb{E}[\alpha_1] \oplus \mathbb{E}[\beta_1] = \rho(\mathbb{E}[A(1)]).$$

Example 3. Let $\{\alpha_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$, and $\{\delta_k\}$ be sequences of independent random variables having exponential distribution with mean 1. We will suppose that α_k, β_k, γ_k, and δ_k are independent for any k, $k = 1, 2, \ldots$.

Let us consider the system with matrix

$$A(k) = \begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix}. \tag{4}$$

Note that an exact value of the Lyapunov constant is known for this system [3, 7]: $\lambda = 407/228 \approx 1.7851$.

Let us determine bounds for λ in accordance with (3) and (2). Since $\mathbb{E}[A(1)] = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, we will have for the lower bound $\rho(\mathbb{E}[A(1)]) = 1$.

Computation of the upper bound yields the quantity

$$\mathbb{E}\|A(1)\| = \mathbb{E}[\alpha_1 \oplus \beta_1 \oplus \gamma_1 \oplus \delta_1] = \frac{25}{12} \approx 2{,}0833.$$

4. Matrices of simple structure

Suppose that some matrix A may be represented in the form

$$A = \boldsymbol{u} \otimes \boldsymbol{v}^T,$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are certain vectors. A matrix for which there exists such a representation will be called a matrix of simple structure.

It is easily verified that for A, we have $\|A\| = \|\boldsymbol{u}\| \otimes \|\boldsymbol{v}\|$.

Let us consider two matrices of simple structure $A = \boldsymbol{u} \otimes \boldsymbol{v}^T$ and $B = \mathbf{r} \otimes \boldsymbol{s}^T$. It is clear that

$$\|A \otimes B\| = \|(\boldsymbol{u} \otimes \boldsymbol{v}^T) \otimes (\boldsymbol{r} \otimes \boldsymbol{s}^T)\| = (\boldsymbol{v}^T \otimes \boldsymbol{r}) \otimes \|\boldsymbol{u}\| \otimes \|\boldsymbol{s}\|.$$

Let us suppose that $A(k) = \boldsymbol{u}(k) \otimes \boldsymbol{v}^T(k)$ for any $k = 1, 2, \ldots$. Then for the matrix $A_k = A(1) \otimes \ldots A(k)$, we will have

$$\|A_k\| = \|\boldsymbol{u}(1)\| \otimes \|\boldsymbol{v}(k)\| \otimes \bigotimes_{i=1}^{k-1} \boldsymbol{v}(i)^T \otimes \boldsymbol{u}(i+1) = \|\boldsymbol{u}(1)\| + \|\boldsymbol{v}(k)\| + \sum_{i=1}^{k-1} \boldsymbol{v}(i)^T \otimes \boldsymbol{u}(i+1).$$

In this case the Lyapunov constant will be given as

$$\lambda = \lim_{k\to\infty} \frac{1}{k}\|A_k\| = \mathbb{E}[\boldsymbol{v}(1)^T \otimes \boldsymbol{u}(2)].$$

It is easily verified that for any two matrices $A = (a_{ij})$ and $B = (b_{ij})$,

$$A \otimes B = \bigoplus_{i=1}^{n} \boldsymbol{a}_i \otimes \boldsymbol{b}^i,$$

where $\boldsymbol{a}_i = (a_{1i}, \ldots, a_{ni})^T$ and $\boldsymbol{b}_i = (b_{i1}, \ldots, b_{in})$. Note also that

$$\|A \otimes B\| = \bigoplus_{i=1}^{n} \|\boldsymbol{a}_i\| \otimes \|\boldsymbol{b}^i\|. \tag{5}$$

Let us consider the two successive matrices $A(k)$ and $A(k + 1)$. Obviously, for any $j = 1, \ldots, n$, the following inequality is valid:

$$A(k) \otimes A(k+1) \geq \boldsymbol{a}_j(k) \otimes \boldsymbol{a}^j(k+1),$$

where $\boldsymbol{a}_j(k)$ and $\boldsymbol{a}^j(k + 1)$ are the j-th column of $A(k)$ and the j-th row of $A(k + 1)$, respectively.

Then for $k = 2m$, we will have

$$\|A_k\| \geq \|\boldsymbol{a}_j(1)\| \otimes \|\boldsymbol{a}^j(k)\| \otimes \bigotimes_{i=1}^{m-1} \boldsymbol{a}^j(2i) \otimes \boldsymbol{a}_j(2i+1),$$

whence follows the lower bound

$$\lambda \geq \frac{1}{2} \bigoplus_{j=1}^{n} \mathbb{E}[\boldsymbol{a}^j(1) \otimes \boldsymbol{a}_j(2)]. \tag{6}$$

Example 4. Compute the bound (6) for the system (4). Recalling that

$$\boldsymbol{a}^1(1) = (\alpha_1, \beta_1),\ \boldsymbol{a}^2(1) = (\gamma_1, \delta_1),\ \boldsymbol{a}_1(2) = (\alpha_2, \gamma_2)^T,\ \boldsymbol{a}_2(2) = (\beta_2, \delta_2)^T,$$

we obtain

$$\lambda \geq \frac{1}{2}(\mathbb{E}[\alpha_1 \otimes \alpha_2 \oplus \beta_1 \otimes \gamma_2] \oplus \mathbb{E}[\gamma_1 \otimes \beta_2 \oplus \delta_1 \otimes \delta_2]) = 1{,}375.$$

Let A and B be random matrices. The following inequality holds:

$$\mathbb{E}\|A \otimes B\| \geq \mathbb{E}\left[\bigoplus_{i=1}^{n} \|\boldsymbol{a}_i\| \otimes \mathbb{E}\|\boldsymbol{b}^i\|\right] \geq \mathbb{E}\left[\bigoplus_{i=1}^{n} \|\boldsymbol{a}_i\|\right] \otimes \min_{1\leq i\leq n} \mathbb{E}\|\boldsymbol{b}^i\|.$$

Using the notation

$$\nu(B) = \min_{1\leq i\leq n} \mathbb{E}\|\boldsymbol{b}^i\|,$$

we finally write

$$\mathbb{E}\|A \otimes B\| \geq \mathbb{E}\|A\| \otimes \nu(B).$$

Successive application of the latter inequality to the matrix A_k leads us to conclude that

$$\mathbb{E}\|A_k\| \geq \mathbb{E}\|A(1)\| \otimes \bigotimes_{i=2}^{k} \nu(A(i)),$$

whence it follows

$$\lambda \geq \nu(A(1)).$$

Similarly, it may be shown that $\lambda \geq \nu(A^T(1))$. Combining the two inequalities, we have

$$\lambda \geq \nu(A(1)) \oplus \nu(A^T(1)). \tag{7}$$

Example 5. Application of the bound (7) to the system (4) yields

$$\lambda \geq \min\{\mathbb{E}[\alpha_1 \oplus \beta_1], \mathbb{E}[\gamma_1 \oplus \delta_1]\} \oplus \min\{\mathbb{E}[\alpha_1 \oplus \gamma_1], \mathbb{E}[\beta_1 \oplus \delta_1]\} = 1{,}5.$$

Finally, note that an upper bound for λ may be obtained through use of equality (5). By virtue of the fact that with $k = 2m$, the inequality

$$\|A_k\| \leq \|A(1) \otimes A(2)\| \otimes \ldots \otimes \|A(2m-1) \otimes A(2m)\|,$$

holds, we will have, by (5),

$$\lambda \leq \frac{1}{2}\mathbb{E}\left[\bigoplus_{i=1}^{n} \|\boldsymbol{a}_i(1)\| \otimes \|\boldsymbol{a}^i(2)\|\right]. \tag{8}$$

Example 6. Compute the bound (8) for the system (4):

$$\lambda \leq \frac{1}{2}\mathbb{E}[(\alpha_1 \oplus \gamma_1) \otimes (\alpha_2 \oplus \beta_2) \oplus (\beta_1 \oplus \delta_1) \otimes (\gamma_2 \oplus \delta_2)] = \frac{833}{432} \approx 1{,}9282.$$

5. Isometric matrices

We will say that a matrix A is isometric if for any vector $\boldsymbol{u}$ at least one of the following two equalities holds:

$$\|A \otimes \boldsymbol{u}\| = \|\boldsymbol{u}\|, \ \|A^T \otimes \boldsymbol{u}\| = \|\boldsymbol{u}\|.$$

Obviously, if A is an isometric matrix, for any matrix B at least one of the following two equalities will hold:

$$\|A \otimes B\| = \|B\|, \ \|B \otimes A\| = \|B\|.$$

It is easily seen that a matrix all of whose elements are equal to 0 is isometric. Moreover, the product of such a matrix by any matrix A constitutes an almost isometric matrix to within a constant factor, which is equal to $\|A \otimes 0\| = \|A\|$.

We will say that a matrix A is nonfactorable if the condition $a_{ij} > \varepsilon$ holds for all $i = 1, \ldots, n$ and $j = 1, \ldots, n$.

Let us suppose that the matrix $A(k)$ is nonfactorable with probability 1 for any $k = 1, 2, \ldots$. With $k = 2m$ we will have

$$\mathbb{E}\|A_k\| \geq \mathbb{E}\left\|\bigotimes_{i=1}^{m} A(2i-1) \otimes \mathbb{E}[A(2i)]\right\| \geq \mathbb{E}\left\|\bigotimes_{i=1}^{m} A(2i-1) \otimes 0\right\| \otimes \bigotimes_{i=1}^{m} \mu\left(\mathbb{E}[A(2i)]\right) =$$
$$= m\mathbb{E}\|A(1)\| + m\mu(\mathbb{E}[A(1)]),$$

where $\mu(A) = \min\{a_{ij} | 1 \leq i \leq n, 1 \leq j \leq n\}$ for any matrix A.

From this inequality there follows a bound for the case in which the matrix $A(1)$ is nonfactorable:

$$\lambda \geq \frac{1}{2}\mathbb{E}\|A(1)\| + \frac{1}{2}\mu(\mathbb{E}[A(1)]). \tag{9}$$

Example 7. For the system (4) we have $\mathbb{E}\|A(1)\| \approx 2{,}0833$ and $\mu(\mathbb{E}[A(1)]) = 1$. Then the bound (9) assumes the form

$$\lambda \geq \frac{1}{2}(2.0833 + 1) \approx 1.5416,$$

The next assertion represents sufficient conditions for a matrix to be isometric.

Lemma 1. If a matrix A satisfies at least one of the following conditions:

(1) for every column $\boldsymbol{a}_j$, $j = 1, \ldots, n$, $\|\boldsymbol{a}_j\| = 0$ holds;

(2) for every row $\boldsymbol{a}^i$, $i = 1, \ldots, n$, $\|\boldsymbol{a}^i\| = 0$ holds;

it is isometric.

Proof. Suppose that the matrix A satisfies condition (1) on columns. Then for an arbitrary vector $\boldsymbol{u}$ we will have

$$\|A \otimes \boldsymbol{u}\| = \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{n} a_{ij} \otimes u_j = \bigoplus_{j=1}^{n}\left(u_j \otimes \bigoplus_{i=1}^{n} a_{ij}\right) = \bigoplus_{j=1}^{n}(u_j \otimes \|\boldsymbol{a}_j\|) = \bigoplus_{j=1}^{n} u_j = \|\boldsymbol{u}\|.$$

The condition on rows is considered analogously.

6. Construction of upper bounds

Let us first prove the following assertion.

Theorem 2. Any matrix A may be represented in the form

$$A = \boldsymbol{u} \otimes \boldsymbol{v}^T + S,$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are certain vectors, $\boldsymbol{u} > \varepsilon$, and S is an isometric matrix.

Proof. Obviously, to prove the theorem it is sufficient to specify vectors $\boldsymbol{u} > \varepsilon$ and $\boldsymbol{v}$ for which the matrix $B = A - \boldsymbol{u} \otimes \boldsymbol{v}^T$ will be isometric. For this purpose we require that B satisfy the condition $\|\boldsymbol{b}_j\| = 0$ for all $j = 1, \ldots, n$.

For each j we write

$$0 = \|b_j\| = \bigoplus_{i=1}^{n}(a_{ij} - u_i \otimes v_j) = \bigoplus_{i=1}^{n}(a_{ij} - u_i) - v_j,$$

whence we obtain an expression for determining v_j:

$$v_j = \bigoplus_{i=1}^{n}(a_{ij} - u_i).$$

Thus, it has been proved that any pair of vectors

$$\boldsymbol{u} = (u_1, \ldots, u_n)^T > \varepsilon, \ \boldsymbol{v}^T = \boldsymbol{u}^* \otimes A,$$

where $\boldsymbol{u}^* = -\boldsymbol{u}^T$, yields the required representation of the matrix A.

By Theorem 2, for every matrix $A(k)$, $k = 1, 2, \ldots$, vectors $\boldsymbol{u}(k)$ and $\boldsymbol{v}(k)$ as well as an isometric matrix $S(k)$ may be found such that the following equality holds:

$$A(k) = \boldsymbol{u}(k) \otimes \boldsymbol{v}(k)^T + S(k).$$

Then, in view of (1), we will have

$$\|A_k\| = \|A(1) \otimes \cdots \otimes A(k)\| \le \left\| \bigotimes_{i=1}^{k} \boldsymbol{u}(i) \otimes \boldsymbol{v}(i)^T + \bigotimes_{i=1}^{k} S(i) \right\| \le$$
$$\le \left\| \bigotimes_{i=1}^{k} \boldsymbol{u}(i) \otimes \boldsymbol{v}(i)^T \right\| + \left\| \bigotimes_{i=1}^{k} S(i) \right\| = \|\boldsymbol{u}(i)\| \otimes \|\boldsymbol{v}(k)\| \otimes \bigotimes_{i=1}^{k-1} \boldsymbol{v}^T(i) \otimes \boldsymbol{u}(i+1).$$

Now the upper bound may be computed:

$$\lambda \le \mathbb{E}[\boldsymbol{v}^T(1) \otimes \boldsymbol{u}(2)].$$

We may also express the vector $\boldsymbol{v}(1)$ in terms of $\boldsymbol{u}(1)$ as in Theorem 2. We write the resulting bound in the form

$$\lambda \le \mathbb{E}[\boldsymbol{u}^*(1) \otimes A(1) \otimes \boldsymbol{u}(2)].$$

It is clear that the inequality that has been obtained determines a certain family of bounds. Selection of the optimal bound from the family is not an obvious procedure and requires further analysis. Below, an example is presented that illustrates the construction and computation of one of the bounds in the case of (2×2)-matrices.

Example 8. Consider the system (4) and select the components of the vector $\boldsymbol{u}(1)$ as follows:

$$u_1(1) = 0,\ u_2(1) = \frac{1}{2}(\alpha_1^{-1} \otimes \beta_1^{-1} \otimes \gamma_1 \otimes \delta_1).$$

Computation of the coordinates of the vector $\boldsymbol{v}(1)$ yields

$$v_1(1) = \frac{1}{2}(\alpha_1 \otimes \delta_1^{-1} \otimes (\alpha_1 \otimes \delta_1 \oplus \beta_1 \otimes \gamma_1)),\ v_2(1) = \frac{1}{2}(\beta_1 \otimes \gamma_1^{-1} \otimes (\alpha_1 \otimes \delta_1 \oplus \beta_1 \otimes \gamma_1)).$$

Now we write

$$\boldsymbol{v}^T(1) \otimes \boldsymbol{u}(2) = \frac{1}{2}\Big((\alpha_1 \otimes \delta_1^{-1} \otimes (\alpha_1 \otimes \delta_1 \oplus \beta_1 \otimes \gamma_1))\oplus$$
$$\oplus (\beta_1 \otimes \gamma_1^{-1} \otimes (\alpha_1 \otimes \delta_1 \oplus \beta_1 \otimes \gamma_1) \otimes \alpha_2^{-1} \otimes \beta_2^{-1} \otimes \gamma_2 \otimes \delta_2)\Big).$$

After computation of the mathematical expectation, we finally obtain

$$\lambda \leq \mathbb{E}[\boldsymbol{v}^T(1) \otimes \boldsymbol{u}(2)] = \frac{123}{64} \approx 1{,}9219.$$

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