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EVALUATION OF THE MEAN CYCLE TIME IN FORK-JOIN QUEUEING NETWORKS*

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The problem of evaluating the mean service cycle time in fork-join queueing networks is considered. An approach is proposed based on implementation and further development of the methods of idempotent algebra. It is shown that for acyclic networks under sufficiently general conditions the mean cycle time is determined by mean service time at the network nodes only and it is independent of network topology.

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1. Idempotent algebra. Denote by $\mathbb{R}_\varepsilon$ a set of real numbers, extended by adding the element $\varepsilon = -\infty$. Let on $\mathbb{R}_\varepsilon$ the operations $\oplus$ and $\otimes$ be given as

$$x \oplus y = \max(x, y), \qquad x \otimes y = x + y$$

for any $x, y \in \mathbb{R}_\varepsilon$. Put (in the usual way) $x \otimes \varepsilon = \varepsilon \otimes x = \varepsilon$.

The set $\mathbb{R}_\varepsilon$ with operations $\oplus$ and $\otimes$ is a commutative ring with idempotent addition. The numbers ε and 0 are the zero and unit elements, respectively, relative to the introduced operation. A semiring is usually called an idempotent algebra (see, for example, [1, 2]).

Note that in idempotemt algebra for every $x \neq \varepsilon$ the inverse element x^{-1}, which is to be $-x$ in the usual arithmetic, is defined relative to the operation $\otimes$,

1.1. Matrix algebra. An idempotemt algebra of $(n \times n)$-matrices is introduced in the usual way: for any two matrices $A = (a_{ij})$ and $B = (b_{ij})$ we have

$$\{A \oplus B\}_{ij} = a_{ij} \oplus b_{ij}, \qquad \{A \otimes B\}_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes b_{kj}.$$

Both operations $\oplus$ and $\otimes$ have an associative property but only $\oplus$ is commutative. The matrix $\mathcal{E}$, all the elements of which are equal to ε, and the matrix E, all the elements of which are equal to zero on the principal diagonal and to ε outside, are regarded as the zero and unit matrices, respectively.

Obviously, the operations $\oplus$ and $\otimes$ are monotonic, i. e. from the matrix componentwise inequalities $A \leqslant C$ and $B \leqslant D$ it follows that $A \oplus B \leqslant C \oplus D$ and $A \otimes B \leqslant C \otimes D$.

Let the matrix $A \neq \mathcal{E}$. Then the degrees of the matrices $A^0 = E$ and $A^k = A \otimes A^{k-1} = A^{k-1} \otimes A$ for any integer $k \geqslant 1$ can be found.

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The matrix D is said to be diagonal if all their off-diagonal elements are equal to ε. We can easily check that for any matrix $D = \mathrm{diag}(d_1, \dots, d_n)$ there exists the inverse, with respect to the operation $\otimes$, matrix $D^{-1} = \mathrm{diag}(d_1^{-1}, \dots, d_n^{-1})$ under the condition $d_i > \varepsilon$ for all $i = 1, \dots, n$.

The operation $\otimes$ has an evident distributive property relative to $\oplus$. In other words, the following inequalities

$$A \otimes (B \oplus C) = A \otimes B \oplus A \otimes C, \qquad (A \oplus B) \otimes C = A \otimes C \oplus B \otimes C \tag{1}$$

are valid.

The distributive property can be represented as

$$\bigotimes_{i=1}^{k} \bigoplus_{j=1}^{m} A_{ij} = \bigoplus_{1 \leqslant m_1, \dots, m_k \leq m} \bigotimes_{i=1}^{k} A_{im_i}, \tag{2}$$

where A_{ij} are arbitrary matrices for all $i = 1, \dots, k$, $j = 1, \dots, m$.

1.2. Arithmetic addition. Introduce the usual operation of arithmetic addition so as to be external relative to the algebra under consideration. Since the scalar operations $\otimes$ and $+$ are identical, only the external operation of matrix addition is of interest.

We assume that in algebraic relations for any sequence of operations the arithmetic addition $+$ is performed after the operations $\otimes$ and $\oplus$. For any matrix A, put $A + \mathcal{E} = \mathcal{E} + A = \mathcal{E}$.

We can check that there exists a distributive property of the operation $+$ with respect to $\oplus$:

$$\sum_{i=1}^{k} \bigoplus_{j=1}^{m} A_{ij} = \bigoplus_{1 \leqslant m_1, \dots, m_k \leq m} \sum_{i=1}^{k} A_{im_i},$$

which implies, in particular, that the following inequality holds

$$\sum_{i=1}^{k} \bigoplus_{j=1}^{m} A_{ij} \geqslant \bigoplus_{j=1}^{m} \sum_{i=1}^{k} A_{ij}. \tag{3}$$

In the general case it is difficult to find any simple properties relating the matrix operations $\otimes$ and $+$. However, as will be shown below, some useful relations can be obtained for the special types of matrices involving diagonal support matrices.

1.3. Matrix functions. Consider the arbitrary matrix $A = (a_{ij})$ and introduce the following values

$$\|A\| = \bigoplus_{1 \leqslant i,j \leqslant n} a_{ij}, \qquad \mathrm{tr}(A) = \bigoplus_{i=1}^{n} a_{ii}.$$

Let A and B be any matrices. We can show that the inequality $A \leqslant B$ yields $\|A\| \leqslant \|B\|$ and $\mathrm{tr}(A) \leqslant \mathrm{tr}(B)$. Besides, the following evident relations hold

$$\|A \oplus B\| = \|A\| \oplus \|B\|, \qquad \|A \otimes B\| \leqslant \|A\| \otimes \|B\|,$$
$$\mathrm{tr}(A \oplus B) = \mathrm{tr}(A) \oplus \mathrm{tr}(B), \qquad \|A + B\| \leqslant \|A\| + \|B\|.$$

Finally, for any number $c > 0$ we have $\|cA\| = c\|A\|$ and $\mathrm{tr}(cA) = c\,\mathrm{tr}(A)$ provided the condition $c\varepsilon = \varepsilon$.

1.4. Eigenvalues of matrices. Consider the arbitrary matrix A. The eigenvalue λ and the corresponding to it eigenvector $\boldsymbol{x}$ of the matrix A satisfy the inequality

$$A \otimes \boldsymbol{x} = \lambda \otimes \boldsymbol{x}.$$

The following important result was obtained in [3] (see also [2]).

Theorem 1. *For any $(n \times n)$-matrix A the following relation holds*

$$\lim_{k \to \infty} \frac{1}{k} \|A^k\| = \rho(A),$$

where $\rho(A)$ is a maximum eigenvalue of the matrix A,

$$\rho(A) = \bigoplus_{k=1}^{n} \frac{1}{k} \mathrm{tr}(A^k) = \mathrm{tr}\left(\bigoplus_{k=1}^{n} \frac{1}{k} A^k \right).$$

2. Queueing networks. Let there be given a network with n nodes, the topology of which is described by acyclic graph. At each of the nodes there is a service facility and unlimited storage, intended for the queue to be served. A network node, which does not have entering edges, is regarded as a supplier of the unlimited flow of arriving units. At the initial (zero) moment of time all the facilities are free, the queues at the node-suppliers have infinite length while the queues of all the remaining nodes are empty.

Denote by τ_{ik} an active time and by $x_i(k)$ a time of finish of the k-th service at the node i, $i = 1, \ldots, n$, $k = 1, 2, \ldots$ The values $\tau_{i1}, \tau_{i2}, \ldots$ are assumed to be independent identically distributed random quantities for all $i = 1, \ldots, n$.

In the networks under consideration ordinary and fork-join queueing can be realized [4, 5]. The join operation is performed at the node before a call is connected with a queue and consists of the join of accepted calls, taken by one of each preceding node, and of the replacement of them by a new call, which is connected with a queue end. The fork operation is always performed after a service call at the node is finished. It consists of the replacement of the call by some new calls, one for every subsequent node, after which these new calls are immediately sent into the nodes.

The dynamics of fork-join queueing networks can be described by means of an idempotent algebra technique. As is shown in [5], for acyclic networks the following dynamic equation holds

$$x(k) = \mathcal{A}(k) \otimes x(k-1), \quad \mathcal{A}(k) = \bigoplus_{j=0}^{l} \mathcal{T}_k \otimes (G^T \otimes \mathcal{T}_k)^j, \tag{4}$$

where $x(k) = (x_1(k), \ldots, x_n(k))^T$, $\mathcal{T}_k = \mathrm{diag}(\tau_{1k}, \ldots, \tau_{nk})$, $G = (g_{ij})$ is a matrix of the adjacency graph for a network with the elements

$$g_{ij} = \begin{cases} 0 & \text{if the arc } (i,j) \text{ exists,} \\ \varepsilon & \text{otherwise,} \end{cases}$$

l is a length of maximum path in the graph.

We assume that a work system is a sequence of service cycles. The first cycle is finished after the service of one call at each of network nodes is finished, the second cycle is finished after the same for two calls and so on.

By the property $x(0) = 0$ the time of finish for the k-th cycle can be found as

$$\|x(k)\| = \|\mathcal{A}_k\|, \qquad \mathcal{A}_k = \mathcal{A}(k) \otimes \ldots \otimes \mathcal{A}(1).$$

In many practical problems it is of interest to find the mean cycle time under the condition that the number of cycles increases without limit. In other words, we are interested in the problem of determining the following limit

$$\gamma = \lim_{k\to\infty} \frac{1}{k} \|x(k)\| = \lim_{k\to\infty} \frac{1}{k} \|\mathcal{A}_k\|$$

under the condition that it exists.

It is clear that the existence of the above limit and its value depend on as the structure of the matrix $\mathcal{A}_k$ as the probability properties of its elements. Algebraic properties of the matrices of special type, determining the peculiarities of the structure $\mathcal{A}_k$, and also some auxiliary results, concerning the probabilistic nature of the problem are given below.

3. Diagonal and support matrices. Let D be a diagonal matrix. Then for any matrices A and B we have

$$\begin{aligned} D \otimes (A + B) &= D \otimes A + B = A + D \otimes B, \\ (A + B) \otimes D &= A \otimes D + B = A + B \otimes D. \end{aligned} \tag{5}$$

A square matrix with the elements 0 or ε is a support matrix. Note that any support matrix $G = (g_{ij})$ can be regarded as an adjacency matrix of a certain graph under the condition that $g_{ij} = 0$ implies the existence of the arc (i, j) and $g_{ij} = \varepsilon$ implies that the arc is absent. We can easily check that if the graph is acyclic, then for the corresponding matrix G the relation $G^m = \mathcal{E}$ is valid for all $m > l$, where l is a length of maximum path in the graph.

Let G be a support matrix. For any matrices A and B the following inequality holds

$$G \otimes (A + B) \leqslant G \otimes A + G \otimes B, \qquad (A + B) \otimes G \leqslant A \otimes G + B \otimes G. \tag{6}$$

Further we shall consider the product of alternating diagonal matrices and the certain support matrix G of the form

$$D_0 \otimes G \otimes D_1 \otimes \cdots \otimes G \otimes D_k.$$

Introduce the following notation:

$$\Phi_j(D) = D \otimes (G \otimes D)^j, \qquad \Psi_j^i(D) = G^i \otimes D \otimes G^j.$$

Lemma 1. *For any diagonal matrices $D_0, D_1, \ldots, D_k$ the following inequality*

$$D_0 \otimes \bigotimes_{j=1}^{k} (G \otimes D_j) \leq \sum_{j=0}^{k} G^j \otimes D_j \otimes G^{k-j} \tag{7}$$

is satisfied.

The proof is by induction. For $k = 1$ the assertion of the lemma results from (5) and from the evident relation $D_0 \otimes G = D_0 \otimes G + G$.

Lemma 2. *Let $D_1, \ldots, D_k$ be diagonal matrices, $m_1, \ldots, m_k$ be nonnegative integers and $m_1 + \cdots + m_k = m$.*

The following inequality holds

$$\bigotimes_{i=1}^{k} \Phi_{m_i}(D_i) \leqslant \sum_{j=0}^{m} \Psi_j^{m-j} \left(\bigotimes_{i=r_j}^{s_j} D_i \right), \tag{8}$$

where $r_0 = 1$, $r_j = s_{j-1}$ for all $j = 1, \ldots, m$, and

$$s_j = \begin{cases} k & \text{if } m_1 + \cdots + m_k \leqslant j, \\ \min\{i \mid m_1 + \cdots + m_i > j\} & \text{otherwise} \end{cases}$$

for all $j = 0, 1, \ldots, m$.

Proof. By inequality (7) we obtain

$$\bigotimes_{i=1}^{k} \Phi_{m_i}(D_i) = \bigotimes_{i=1}^{k} (D_i \otimes (G \otimes D_i)^{m_i}) \leqslant \sum_{j=0}^{m} G^j \otimes \widetilde{D}_j \otimes G^{m-j},$$

where $\widetilde{D}_j = D_{r_j} \otimes D_{r_j+1} \otimes \cdots \otimes D_{s_j}$. We can easily check that the indices r_j and s_j are computed due to the lemma. □

Using the induction on k and the properties (5) and (6), we can prove the following

Lemma 3. *Let $D_0^{(1)}, D_1^{(1)}, \ldots, D_k^{(1)}$ and $D_0^{(2)}, D_1^{(2)}, \ldots, D_k^{(2)}$ be diagonal matrices.*

The following inequality holds

$$D_0^{(1)} \otimes \bigotimes_{i=1}^{k} \left(G \otimes D_i^{(1)} \right) + D_0^{(2)} \otimes \bigotimes_{i=1}^{k} \left(G \otimes D_i^{(2)} \right) \geqslant D_0^{(1)} \otimes D_0^{(2)} \otimes \bigotimes_{i=1}^{k} \left(G \otimes D_i^{(1)} \otimes D_i^{(2)} \right).$$

Corollary 1. *Let $D_1^{(1)}, \ldots, D_k^{(1)}$ and $D_1^{(2)}, \ldots, D_k^{(2)}$ be diagonal matrices.*

The following inequality

$$\bigotimes_{i=1}^{k} \Phi_{m_i} \left(D_i^{(1)} \right) + \bigotimes_{i=1}^{k} \Phi_{m_i} \left(D_i^{(2)} \right) \geqslant \bigotimes_{i=1}^{k} \Phi_{m_i} \left(D_i^{(1)} \otimes D_i^{(2)} \right), \tag{9}$$

where $m_1, \ldots, m_k$ are arbitrary nonnegative integers, is satisfied.

4. Random matrices. Consider the matrices, the elements of which are random quantities. Some elements can be degenerated random quantities, which take with probability one the value $\varepsilon = -\infty$.

Let $\mathcal{A}$ be a random matrix. Denote by $\mathbb{E}[\mathcal{A}]$ a matrix obtained from $\mathcal{A}$ by the change of its elements to their mathematical expectations under the condition $\mathbb{E}[\varepsilon] = \varepsilon$.

For any random matrices $\mathcal{A}$ and $\mathcal{B}$ the following inequalities hold

$$\mathbb{E}\|\mathcal{A}\| \geqslant \|\mathbb{E}[\mathcal{A}]\|, \quad \mathbb{E}[\mathcal{A} \oplus \mathcal{B}] \geqslant \mathbb{E}[\mathcal{A}] \oplus \mathbb{E}[\mathcal{B}], \quad \mathbb{E}[\mathcal{A} \otimes \mathcal{B}] \geqslant \mathbb{E}[\mathcal{A}] \otimes \mathbb{E}[\mathcal{B}].$$

Let $\xi_1, \xi_2, \ldots,$ be independent identically distributed random quantities with the mean $\mathbb{E}[\xi_1] = 0$ and the dispersion $\mathbb{D}[\xi_1] < \infty$. Consider a random quantity

$$\zeta_{lk} = \bigoplus_{l \leqslant r \leqslant s \leqslant k} \bigotimes_{i=r}^{s} \xi_i = \max_{l \leqslant r \leqslant s \leqslant k} \{\xi_r + \xi_{r+1} + \cdots + \xi_s\}.$$

It is easy to check that a set of the random quantities $\{\zeta_{lk} \mid l \leqslant k;\ l, k \geqslant 1\}$ satisfies the conditions of the ergodic theorem of Kingman [6]. From this theorem it follows that there exist the limits

$$\lim_{k \to \infty} \frac{1}{k} \zeta_{1k} = a \quad \text{with probability one,} \qquad \lim_{k \to \infty} \frac{1}{k} \mathbb{E}[\zeta_{1k}] = a.$$

On the other hand, we can show (see, for example, [7]) that $\mathbb{E}[\zeta_{1k}] = O(\sqrt{k})$ as $k \to \infty$. Hence

$$a = \lim_{k\to\infty} \frac{1}{k}\mathbb{E}[\zeta_{1k}] = 0.$$

This result can directly be extended to the case of random diagonal matrices. In other words, the following assertion is true.

Lemma 4. *Let $\mathcal{D}_1, \mathcal{D}_2, \ldots,$ be a sequence of random diagonal matrices, the corresponding diagonal elements of which are independent and have identical probability distributions with zero mean and bounded variance.*

Then there exists with probability one the following limit

$$\lim_{k\to\infty} \frac{1}{k} \bigoplus_{1\leqslant r\leqslant s\leqslant k} \bigotimes_{i=r}^{s} \mathcal{D}_i = E.$$

5. Evaluation of the mean cycle time. To solve the problem of evaluating the mean cycle time it is necessary to find the following limit

$$\gamma = \lim_{k\to\infty} \frac{1}{k}\|\mathcal{A}_k\|.$$

Consider a transposed matrix of the system (4):

$$\mathcal{A}^T(k) = \bigoplus_{j=0}^{l} \mathcal{T}_k \otimes (G\otimes \mathcal{T}_k)^j = \bigoplus_{j=0}^{l} \Phi_j(\mathcal{T}_k).$$

Then

$$\mathcal{A}_k^T = \mathcal{A}^T(1)\otimes\cdots\otimes\mathcal{A}^T(k) = \bigotimes_{i=1}^{k}\bigoplus_{j=0}^{l} \mathcal{T}_i \otimes (G\otimes \mathcal{T}_i)^j = \bigotimes_{i=1}^{k}\bigoplus_{j=0}^{l} \Phi_j(\mathcal{T}_i).$$

Consider the following matrices

$$S = \mathbb{E}[\mathcal{T}_1], \qquad B^T = \bigoplus_{j=0}^{l} S\otimes(G\otimes S)^j = \bigoplus_{j=0}^{l}\Phi_j(S), \qquad \mathcal{R}_k = S^{-1}\otimes\mathcal{T}_k,$$

$$\mathcal{C}^T(k) = \bigoplus_{j=0}^{l} \mathcal{R}_k\otimes(G\otimes\mathcal{R}_k)^j = \bigoplus_{j=0}^{l}\Phi_j(\mathcal{R}_k),$$

$$\mathcal{C}_k^T = \mathcal{C}^T(1)\otimes\cdots\otimes\mathcal{C}^T(k) = \bigotimes_{i=1}^{k}\bigoplus_{j=0}^{l} \mathcal{R}_i\otimes(G\otimes\mathcal{R}_i)^j = \bigotimes_{i=1}^{k}\bigoplus_{j=0}^{l}\Phi_j(\mathcal{R}_i).$$

Now we prove two auxiliary assertions.

Lemma 5. *The following inequality holds*

$$\mathcal{A}_k \leqslant B^k + \mathcal{C}_k. \tag{10}$$

Proof. Using relation (2) and property (3), we can write

$$\begin{aligned}(B^T)^k + \mathcal{C}_k^T &= \bigoplus_{0\leqslant m_1,\ldots,m_k\leqslant l}\bigotimes_{i=1}^{k}\Phi_{m_i}(S) + \bigoplus_{0\leqslant m_1,\ldots,m_k\leqslant l}\bigotimes_{i=1}^{k}\Phi_{m_i}(\mathcal{R}_i)\\ &\geqslant \bigoplus_{0\leqslant m_1,\ldots,m_k\leqslant l}\left(\bigotimes_{i=1}^{k}\Phi_{m_i}(S) + \bigotimes_{i=1}^{k}\Phi_{m_i}(\mathcal{R}_i)\right).\end{aligned}$$

By (9) the following inequality

$$\bigotimes_{i=1}^{k}\Phi_{m_i}(S) + \bigotimes_{i=1}^{k}\Phi_{m_i}(\mathcal{R}_i) \geqslant \bigotimes_{i=1}^{k}\Phi_{m_i}(S\otimes S^{-1}\otimes\mathcal{T}_i) = \bigotimes_{i=1}^{k}\Phi_{m_i}(\mathcal{T}_i)$$

is satisfied.

Finally,

$$(B^T)^k + \mathcal{C}_k^T \geqslant \bigoplus_{0\leqslant m_1,\dots,m_k\leqslant l} \bigotimes_{i=1}^{k} \Phi_{m_i}(\mathcal{T}_i) = \bigotimes_{i=1}^{k} \bigoplus_{j=0}^{l} \Phi_j(\mathcal{T}_i) = \mathcal{A}_k^T. \quad \square$$

Lemma 6. *The following inequalities*

$$\mathcal{C}_k^T \leqslant \bigoplus_{m=0}^{l} \sum_{j=0}^{m} \Psi_j^{m-j} \left(\bigoplus_{1\leqslant r\leqslant s\leqslant k} \bigotimes_{i=r}^{s} \mathcal{R}_i \right) \tag{11}$$

are valid.

Proof. Since the network graph is acyclic, the matrix $\mathcal{C}_k^T$ can be represented in the form

$$\mathcal{C}_k^T = \bigoplus_{0\leqslant m_1,\dots,m_k\leqslant l} \bigotimes_{i=1}^{k} \Phi_{m_i}(\mathcal{R}_i) = \bigoplus_{m=0}^{l} \bigoplus_{m_1+\dots+m_k=m} \bigotimes_{i=1}^{k} \Phi_{m_i}(\mathcal{R}_i).$$

Using sequentially (8), (3) and (1), we obtain

$$\bigoplus_{m_1+\dots+m_k=m} \bigotimes_{i=1}^{k} \Phi_{m_i}(\mathcal{R}_i) \leqslant \bigoplus_{m_1+\dots+m_k=m} \sum_{j=0}^{m} \Psi_j^{m-j} \left(\bigotimes_{i=r_j}^{s_j} \mathcal{R}_i \right)$$
$$\leqslant \sum_{j=0}^{m} \bigoplus_{1\leqslant r\leqslant s\leqslant k} \Psi_j^{m-j} \left(\bigotimes_{i=r}^{s} \mathcal{R}_i \right) = \sum_{j=0}^{m} \Psi_j^{m-j} \left(\bigoplus_{1\leqslant r\leqslant s\leqslant k} \bigotimes_{i=r}^{s} \mathcal{R}_i \right),$$

which proves the lemma. $\square$

Theorem 2. *Let $\mathcal{T}_1, \mathcal{T}_2, \dots,$ be a sequence of identically distributed independent random diagonal matrices and $\mathbb{E}\|\mathcal{T}_1\| < \infty$, $\mathbb{D}\|\mathcal{T}_1\| < \infty$.*

Then there exists, with probability one, the limit

$$\lim_{k\to\infty} \frac{1}{k} \|\mathcal{A}_k\| = \rho(B),$$

where $\rho(B)$ is the maximum eigenvalue of the matrix B.

Proof. In [8] there is shown that if the conditions of the theorem are satisfied, then there exist the following limits

$$\lim_{k\to\infty} \frac{1}{k} \mathcal{A}_k = A \quad \text{with probability one}, \qquad \lim_{k\to\infty} \frac{1}{k} \mathbb{E}[\mathcal{A}_k] = A.$$

Hence by the continuity of the function $\|\cdot\|$ the following limits exist

$$\lim_{k\to\infty} \frac{1}{k} \|\mathcal{A}_k\| = \|A\| \quad \text{with probability one}, \qquad \lim_{k\to\infty} \frac{1}{k} \mathbb{E}\|\mathcal{A}_k\| = \|A\|.$$

We show first that $\|A\| \geqslant \rho(B)$. Consider a mathematical expectation

$$\mathbb{E}\|\mathcal{A}_k^T\| = \mathbb{E} \left\| \bigotimes_{i=1}^{k} \bigoplus_{j=0}^{l} \Phi_j(\mathcal{T}_i) \right\| \geqslant \left\| \bigotimes_{i=1}^{k} \bigoplus_{j=0}^{l} \Phi_j(S) \right\| = \left\| \bigotimes_{i=1}^{k} B^T \right\| = \|(B^T)^k\|.$$

Proceeding to limit, by Theorem 1 we obtain

$$\|A\| = \lim_{k\to\infty} \frac{1}{k} \mathbb{E}\|\mathcal{A}_k\| \geqslant \lim_{k\to\infty} \frac{1}{k} \|B^k\| = \rho(B).$$

We now check that the converse inequality is also valid. From (10) it is follows that

$$\frac{1}{k} \|\mathcal{A}_k\| \leqslant \frac{1}{k} \|B^k\| + \|\frac{1}{k} \mathcal{C}_k\|.$$

Taking into account (11), we have

$$\frac{1}{k} \mathcal{C}_k^T \leqslant \frac{1}{k} \bigoplus_{m=0}^{l} \sum_{j=0}^{m} \Psi_j^{m-j} \left(\bigoplus_{1\leqslant r\leqslant s\leqslant k} \bigotimes_{i=r}^{s} \mathcal{R}_i \right) = \bigoplus_{m=0}^{l} \sum_{j=0}^{m} \Psi_j^{m-j} \left(\frac{1}{k} \bigoplus_{1\leqslant r\leqslant s\leqslant k} \bigotimes_{i=r}^{s} \mathcal{R}_i \right).$$

By Lemma 4 as $k \to \infty$ we have, with probability one, the following relation

$$\frac{1}{k} \bigoplus_{1\leqslant r\leqslant s\leqslant k} \bigotimes_{i=r}^{s} \mathcal{R}_i \longrightarrow E.$$

Then

$$\lim_{k\to\infty}\frac{1}{k}\mathcal{C}_k^T \leqslant \bigoplus_{m=0}^{l} G^m = G^* \quad \text{with probability one.}$$

Evidently, $\|G^*\| = 0$ and we finally obtain

$$\|A\| = \lim_{k\to\infty}\frac{1}{k}\|\mathcal{A}_k\| \leqslant \lim_{k\to\infty}\frac{1}{k}\|B^k\| = \rho(B) \quad \text{with probability one.} \quad \square$$

Corollary 2. *The mean time of the cycle γ is determined by the relation*

$$\gamma = \|S\| = \|\mathbb{E}[\mathcal{T}_1]\|.$$

Proof. Taking into account that the network graph is acyclic, the matrix B can be represented in an upper triangle form. Since in this case all the elements of the matrix B, laying under the principal diagonal, are equal to ε, we have

$$\rho(B) = \operatorname{tr}(B) = \operatorname{tr}\left(\bigoplus_{j=0}^{l}\Phi_j(S)\right) = \bigoplus_{j=0}^{l}\operatorname{tr}(\Phi_j(S)).$$

The fact that the matrix S is diagonal implies that $\operatorname{tr}(\Phi_j(S)) = \varepsilon$ for all $j \geqslant 1$. Since $\Phi_0(S) = S$, we finally obtain

$$\gamma = \rho(B) = \operatorname{tr}(S) = \|S\| = \|\mathbb{E}[\mathcal{T}_1]\|. \quad \square$$

It remains to note that the mean cycle time is independent of network topology.

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