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EVALUATION OF THE MEAN IN-PROCESS TIME FOR QUEUEING NETWORKS WITH RANDOM TOPOLOGY*

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A class of queueing networks with topology varying with time at random on each in-process cycle of network is considered. It is also assumed that on each cycle failures can occur with nonzero probability. An upper estimate for the mean in-process time of the networks is proposed based on the idempotent algebra approach.

A class of queueing networks with topology, varying with time at random on each cycle of system, is considered. It is also assumed that on each cycle the network failures may occur with non-zero probability. The upper bound for the mean in-process time of networks is constructed by using the idempotent algebra approach.

Preliminary remarks. The mathematical apparatus of the idempotent $(\max,+)$-algebra is convenient for description of dynamics and analysis of a complex system including queueing networks. In particular, the dynamics of the queueing networks with a customer service synchronization, a topology of which is determined by deterministic acyclic graphs, can be described by means of a generalized linear equation (see, for example, [1–3])

$$x(k) = A(k) \otimes x(k-1) \tag{1}$$

with the matrix

$$A(k) = A(\mathcal{T}_k, G) = (E \oplus \mathcal{T}_k \otimes G^T)^{\otimes r} \otimes \mathcal{T}_k,$$

where $x(k)$ is a time vector of the ending of the k-th service demand at a network node, $\mathcal{T}_k$ is a generalized diagonal random matrix of a service time length, G is a generalized matrix of an adjacency of network graph, r is a length of the largest path in a graph, E is a generalized unit matrix, $\oplus$ and $\otimes$ are matrix operations of a generalized addition and multiplication in the $(\max,+)$-algebra.

Let a system evaluation be regarded as a sequence of service cycles, in which case the k-th cycle is assumed to be completed if at each of the network nodes the k service demands are completed. Under the condition $x(0) = 0$ the time of the ending of the k-th cycles can be determined as

$$\|x(k)\| = \|A_k\|, \qquad A_k = A(k) \otimes \ldots \otimes A(1),$$

where $\|A\| = \bigoplus_{i,j} a_{ij}$ for any matrix $A = (a_{ij})$. Suppose, conditions occur, with non-zero probability, such that they prevent the fulfilling of a subsequent cycle and the ulterior work of system (a probability of

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failure). Assuming that the probability is independent of a cycle number and denoting it by p, we obtain that the mean in-processing time up to the moment of failure is as follows

$$E\|A_\nu\| = (1-p)\sum_{k=1}^{\infty} E\|A_k\|p^k, \tag{2}$$

where the random variable ν is the cycle number of that last cycle, which is normally completed and immediately followed by which a failure occurs.

For this class of systems with deterministic topology, in [3] the upper and lower estimates for the mean in-process time of networks are obtained. Some results of the work [3] are generalized below in the case of networks, a topology of which is described by a random acycle graphs.

Networks with random topology. We assume that on each service cycle k a network topology is chosen at random. Denote by Γ_k a generalized random matrix of an adjacency of the cycle graph, which determines a network topology for the cycle k, and by ρ_k a random variable, describing a length of the largest path in a graph.

Assume that the random matrices $\Gamma_1, \Gamma_2, \dots$ are independent and equally distributed on the space of admissible matrices of adjacency $\mathcal{G} = \{G_1, \dots, G_M\}$. Denote by R the largest length of path for all graphs, corresponding to admissible matrices.

Let the random matrices $\mathcal{T}_1, \mathcal{T}_2, \dots$ be independent and equally distributed, $E\|\mathcal{T}_1\| < \infty$, and $D\|\mathcal{T}_1\| < \infty$. In addition we assume that a sequences of matrices $\mathcal{T} = \{\mathcal{T}_k\}$ and $\Gamma = \{\Gamma_k\}$ are mutually independent.

It is obvious that for the system considered, we can write (1) with the matrix

$$A(k) = A(\mathcal{T}_k, \Gamma_k) = \left(E \oplus \mathcal{T}_k \otimes \Gamma_k^T\right)^{\otimes \rho_k} \otimes \mathcal{T}_k,$$

and introduce also a matrix $A_k = A(\mathcal{T}_k, \Gamma_k) \otimes \dots \otimes A(\mathcal{T}_1, \Gamma_1)$.

The upper estimate of the in-process time of network. It is assumed that for the system under consideration with random topology, on each service cycle the failure may occur with the constant for all cycles probability p. Then, obviously, for the mean in-process time for the network $E\|A_\nu\|$ the presentation (2) is valid.

Lemma 1. *The following estimate holds*

$$E\|A_\nu\| \le \frac{p}{1-p}E\|\mathcal{T}_1\| + \frac{R(M-p)^2}{M(1-p)}\sum_{k=1}^{\infty} E\left[\max_{1\le i\le k}\|\mathcal{T}_i\|\right]\left(\frac{p}{M}\right)^k.$$

Proof. Taking into account a uniform distribution and the independence of matrices $\Gamma_1, \dots, \Gamma_k$, we obtain

$$E\|A_k\| = \frac{1}{M^k}\sum_{G_1,\dots,G_k \in \mathcal{G}} E\|A(\mathcal{T}_k, G_k) \otimes \dots \otimes A(\mathcal{T}_1, G_1)\|.$$

Consider an arbitrary sequence of deterministic matrices $G_1, \dots, G_k$ and introduce the following notion

$$\tilde{A}(i) = A(\mathcal{T}_i, G_i), \quad i = 1, \dots, k, \quad \tilde{A}_k = \tilde{A}(k) \otimes \dots \otimes \tilde{A}(1).$$

It is obvious that in the sequence $G_1, \dots, G_k$ the repeating matrices may occur. Suppose, we have l subsequences of repeating matrices, composed from $m_1, \dots, m_l$ matrices respectively. Introduce the numbers $M_0 = 0,\ M_i = m_1 + \dots + m_i,\ i = 1, \dots, l$. Then we obtain

$$\|\tilde{A}_k\| \le \|\underbrace{\tilde{A}(k) \otimes \dots \otimes \tilde{A}(M_{l-1}+1)}_{m_l \text{ matrices}}\| \otimes \dots \otimes \|\underbrace{\tilde{A}(M_1) \otimes \dots \otimes \tilde{A}(1)}_{m_1 \text{ matrices}}\|.$$

Denote by r_j a length of the largest path in a graph corresponding to the j-th group of matrices. Using the estimates, obtained in [2], we find

$$\|\tilde{A}_k\| \le \bigotimes_{i=1}^{l}\left(\bigotimes_{j=M_{i-1}+1}^{M_i}\|\mathcal{T}_j\|\right) \otimes \left(\bigoplus_{j=M_{j-1}+1}^{M_i}\|\mathcal{T}_j\|\right)^{\otimes r_i} \le \sum_{i=1}^{k}\|\mathcal{T}_i\| + R\sum_{i=1}^{l}\max\{\|\mathcal{T}_{M_{i-1}+1}\|, \dots, \|\mathcal{T}_{M_i}\|\}.$$

Taking into account that the number of sequences of the matrices $G_1, \dots, G_k$, which can be decomposed into l group of repeating matrices, is equal to $M(M-1)^{l-1}$ and that the total number of groups, consisting of m matrices, with respect to all such decompositions of each sequence can be represented in the form

$$N_m(k,l) = \begin{cases} 1, & \text{if } l=1,\ m=k, \\ lC_{k-m-1}^{l-2}, & \text{if } l>1,\ 1\le m\le k-l+1, \\ 0 & \text{otherwise,} \end{cases}$$

we get

$$E\|A_k\| \le E\left[\sum_{i=1}^{k}\|\mathcal{T}_i\|\right] + \frac{R}{M^{k-1}}E\left[\max_{1\le i\le k}\|\mathcal{T}_i\|\right] + \frac{R}{M^{k-1}}\sum_{l=2}^{k}(M-1)^{l-1}\sum_{i=1}^{k-l+1}N_i(k,l)E\left[\max_{1\le j\le i}\|\mathcal{T}_j\|\right]$$

$$= kE\|\mathcal{T}_1\| + \frac{R}{M^{k-1}}E\left[\max_{1\le i\le k}\|\mathcal{T}_i\|\right] + R\frac{M-1}{M}\sum_{i=1}^{k-1}\frac{(M-1)(k-i-1)+2M}{M^i}E\left[\max_{1\le j\le i}\|\mathcal{T}_j\|\right].$$

The inequality obtained can be used for the estimate of sum in (2):

$$\sum_{k=1}^{\infty}E\|A_k\|p^k \le E\|\mathcal{T}_1\|\sum_{k=1}^{\infty}kp^k + RM\sum_{k=1}^{\infty}E\left[\max_{1\le i\le k}\|\mathcal{T}_i\|\right]\left(\frac{p}{M}\right)^k$$

$$+ R\frac{M-1}{M}\sum_{k=1}^{\infty}p^k\sum_{i=1}^{k-1}\frac{(M-1)(k-i-1)+2M}{M^i}E\left[\max_{1\le j\le i}\|\mathcal{T}_j\|\right].$$

Taking into account that a double sum on the right of the above relation can be represented in the following form

$$\sum_{k=1}^{\infty}p^k\sum_{i=1}^{k-1}\frac{(M-1)(k-i-1)+2M}{M^i}E\left[\max_{1\le j\le i}\|\mathcal{T}_j\|\right] = p\frac{2M-p(M+1)}{(1-p)^2}\sum_{k=1}^{\infty}E\left[\max_{1\le i\le k}\|\mathcal{T}_j\|\right]\left(\frac{p}{M}\right)^k,$$

we obtain the required estimate for $E\|A_\nu\|$:

$$E\|A_\nu\| = (1-p)\sum_{k=1}^{\infty}E\|A_k\|p^k \le \frac{p}{1-p}E\|\mathcal{T}_1\| + \frac{R(M-p)^2}{M(1-p)}\sum_{k=1}^{\infty}E\left[\max_{1\le i\le k}\|\mathcal{T}_i\|\right]\left(\frac{p}{M}\right)^k.$$

The following lemma shows how the estimate of the mean in-process time for the network can be calculated under the condition that an average value $E\|\mathcal{T}_1\|$ and dispersion $D\|\mathcal{T}_1\|$ are known.

Lemma 2. *The following inequality holds*

$$E\|A_\nu\| \le \frac{p}{1-p}\left(1+R\frac{M-p}{M}\right)E\|\mathcal{T}_1\| + C\left(\frac{p}{M}\right)\frac{R(M-p)^2}{M(1-p)}\sqrt{D\|\mathcal{T}_1\|}.$$

Here the number $C(x)$ is computed by formula

$$C(x) = \frac{x}{4y}\left(1+\frac{\sqrt{\pi}(1-2y)}{2\sqrt{xy}}(1-\mathrm{erf}\sqrt{y})\right) + H(x),$$

where $y = -\ln(x)/2$,

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-t^2)dt, \qquad H(x) = \max_k\left\{\frac{k-1}{\sqrt{2k-1}}x^k\right\}.$$

Proof. Now we make use of the assertion of the lemma and of the following estimate (see, for example, [2])

$$E\left[\max_{1\le i\le k}\|\mathcal{T}_i\|\right] \le E\|\mathcal{T}_1\| + \frac{k-1}{\sqrt{2k-1}}\sqrt{D\|\mathcal{T}_1\|}.$$

The above implies the inequality

$$E\|A_\nu\| \le \frac{p}{1-p}\left(1+R\frac{M-p}{M}\right)E\|\mathcal{T}_1\| + \frac{R(M-p)^2}{M(1-p)}\sqrt{D\|\mathcal{T}_1\|}\sum_{k=1}^{\infty}\frac{k-1}{\sqrt{2k-1}}\left(\frac{p}{M}\right)^k.$$

For the proof to be completed it is necessary to evaluate the sum on the right of the above relation. In the work [2] it is shown that the following inequality

$$\sum_{k=1}^{\infty}\frac{k-1}{\sqrt{2k-1}}x^k \le C(x)$$

is satisfied, in which case $C(x)$ is computed in the same way as is indicated in the formulation of the lemma.

It is easy to see that for $M = 1$ the estimate obtained takes the form of a corresponding estimate for the system with fixed topology from [3].

References

1. N.K.Krivulin, *Proc. Intern. Workshop on Discrete Event Systems WODES'96*, London, pp. 76–81, 1996.
2. N.K.Krivulin, *Recent Advances in Information Science and Technology*, World Scientific, pp. 147–152, 1998.
3. N.K.Krivulin and D.S.Milov, *Vestnik St.Petersburg Univ. Math.*, Vol. 34, No. 1, pp. 23–30, 2001.

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