Monotonicity Properties and Simple Bounds on the Mean Cycle Time in Acyclic Fork-Join Queueing Networks

Nikolai K. Krivulin*
Faculty of Mathematics and Mechanics, St.Petersburg State University
Bibliotechnaya Sq.2, Petrodvorets, St.Petersburg, 198904 Russia,
E-mail: nikolai@knk.usr.pu.ru
URL: http://www.math.spbu.ru/~krivulin

Abstract

The (max,+)-algebra approach is applied to establish some monotonicity properties and to get algebraic bounds on the service cycle completion times in acyclic fork-join queueing networks. The obtained results are extended to derive simple lower and upper bounds on the mean cycle time in stochastic networks.

1 Introduction

We consider acyclic fork-join queueing networks which consist of single-server nodes with infinite buffers [1, 4]. A network includes both source nodes representing external arrival streams of customers, and internal nodes. At the initial time, the buffers in the source nodes are assumed to have infinite number of customers, whereas the buffers in the internal nodes may have any fixed numbers of customers.

One of the problems of interest in the analysis of stochastic queueing networks is to evaluate the mean cycle time of a network. Both the mean cycle time and its inverse which can be regarded as a throughput, present performance measures commonly used to describe efficiency of the network operation.

It is frequently rather difficult to evaluate the mean cycle time exactly, even though the network under study is quite simple. To get information about the performance measure in this case, one can apply computer simulation to produce reasonable estimates. Another approach is to derive bounds on the mean cycle time (see [2, 4] for related examples).

The paper is concerned with the derivation of a lower and upper bounds on the mean cycle time for stochastic acyclic fork-join queueing networks. A useful way to represent dynamics of the networks is based on the (max, +)-algebra approach [6, 4]. We apply the (max, +)-algebra dynamic representation proposed in [10, 11] to establish some monotonicity properties with respect to the initial numbers of customers in the internal nodes and to get algebraic bounds on the cycle completion times. The obtained results are then extended to derive bounds on the mean cycle time in the stochastic case.

*This work was supported in part by the Russian Federation Ministry of Education Programme "The Universities of Russia" under Grant #4233

2 Algebraic Preliminaries

The (max, +)-algebra is an idempotent commutative semiring (idempotent semifield) which is defined as $\mathbb{R}_{\max} = (\underline{\mathbb{R}}, \oplus, \otimes)$ with $\underline{\mathbb{R}} = \mathbb{R} \cup \{\varepsilon\}$, $\varepsilon = -\infty$, and binary operations $\oplus$ and $\otimes$ defined as

$$x \oplus y = \max(x, y), \quad x \otimes y = x + y \quad \forall x, y \in \underline{\mathbb{R}}.$$

As it is easy to see, the operations $\oplus$ and $\otimes$ retain most of the properties of the ordinary addition and multiplication, including associativity, commutativity, and distributivity of multiplication over addition. However, the operation $\oplus$ is idempotent; that is, for any $x \in \underline{\mathbb{R}}$, one has $x \oplus x = x$.

There are the null and identity elements, namely ε and 0, to satisfy the conditions $x \oplus \varepsilon = \varepsilon \oplus x = x$, and $x \otimes 0 = 0 \otimes x = x$, for any $x \in \underline{\mathbb{R}}$. The null element ε and the operation $\otimes$ are related by the usual absorption rule involving $x \otimes \varepsilon = \varepsilon \otimes x = \varepsilon$.

Non-negative power of any $x \in \mathbb{R}$ is defined as

$$x^0 = 0, \quad x^q = \underbrace{x \otimes \cdots \otimes x}_{q \text{ times}}, \quad q \geq 1.$$

Clearly, the new power x^q corresponds to qx in conventional algebra. In this paper, we will use the power notations only in the (max, +)-algebra sense.

The (max, +)-algebra of matrices is readily introduced in the regular way. Specifically, for any $(n \times n)$-matrices $X = (x_{ij})$ and $Y = (y_{ij})$, the entries of $U = X \oplus Y$ and $V = X \otimes Y$ are calculated respectively as

$$u_{ij} = x_{ij} \oplus y_{ij}, \quad \text{and} \quad v_{ij} = \sum_{k=1}^{n}{}_{\oplus}\, x_{ik} \otimes y_{kj},$$

where $\sum_{\oplus}$ stands for the iterated operation $\oplus$. As the null element, the matrix $\mathcal{E}$ with all entries equal to ε is taken in the algebra, whereas the diagonal matrix $E = \mathrm{diag}(0, \ldots, 0)$ with the off-diagonal entries set to ε presents the identity.

For any square matrix $X \neq \mathcal{E}$, one can define

$$X^0 = E, \quad X^q = \underbrace{X \otimes \cdots \otimes X}_{q \text{ times}}, \quad q \geq 1.$$

However, idempotency in this algebra leads, in particular, to the matrix identity

$$(X \oplus Y)^q = X^q \oplus X^{q-1} \otimes Y \oplus \cdots \oplus Y^q.$$

As direct consequences of the above identity, one has

$$\begin{aligned}(X \oplus Y)^q &\geq X^p \otimes Y^{q-p}, \\ (E \oplus X)^q &\geq (E \oplus X)^p \geq X^p,\end{aligned}$$

for all $p = 0, 1, \ldots, q$.

For any matrix X, its norm is defined as

$$\|X\|_\oplus = \sum\nolimits_{i,j}^{\oplus} x_{ij} = \max_{i,j} x_{ij}.$$

The matrix norm possesses the usual properties. Specifically, for any matrix X, it holds $\|X\|_\oplus \geq \varepsilon$, and $\|X\|_\oplus = \varepsilon$ if and only if $X = \mathcal{E}$. Furthermore, $\|c \otimes X\|_\oplus = c \otimes \|X\|_\oplus$ for any $c \in \mathbb{R}$, and

$$\begin{aligned}\|X \oplus Y\|_\oplus &= \|X\|_\oplus \oplus \|Y\|_\oplus, \\ \|X \otimes Y\|_\oplus &\leq \|X\|_\oplus \otimes \|Y\|_\oplus\end{aligned}$$

for any two conforming matrices X and Y. Note that for any $c > 0$, we also have $\|cX\|_\oplus = c\|X\|_\oplus$.

Consider an $(n \times n)$-matrix X with its entries $x_{ij} \in \mathbb{R}$. It can be treated as an adjacency matrix of an oriented graph with n nodes, provided each entry $x_{ij} \neq \varepsilon$ implies the existence of the arc (i,j) in the graph, while $x_{ij} = \varepsilon$ does the lack of the arc.

It is easy to verify that for any integer $q \geq 1$, the matrix X^q has its the entry $x_{ij}^{(q)} \neq \varepsilon$ if and only if there exists a path from node i to node j in the graph, which consists of q arcs. Furthermore, if the graph associated with the matrix X is acyclic, we have $X^q = \mathcal{E}$ for all $q > p$, where p is the length of the longest path in the graph. Otherwise, provided that the graph is not acyclic, one can construct a path of any length, lying along circuits, and then it holds that $X^q \neq \mathcal{E}$ for all $q \geq 0$.

3 Further Algebraic Results

For any graph, its adjacency matrix G with the elements equal either to 0 or ε is said to be standard. It is easy to verify the next statement.

Proposition 1 *For any matrix X, it holds*

$$X \leq \|X\|_\oplus \otimes G,$$

where G is the standard adjacency matrix of the graph associated with X.

In particular, for a matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$, we have $D \leq \|D\|_\oplus \otimes E$.

Proposition 2 *Let matrices $X_1, \ldots, X_k$ have a common associated acyclic graph, p be the length of the longest path in the graph, and*

$$X = X_1^{m_1} \otimes \cdots \otimes X_k^{m_k},$$

where $m_1, \ldots, m_k$ are nonnegative integers.

If it holds that $m_1 + \cdots + m_k > p$, then $X = \mathcal{E}$.

Proof: It follows from Proposition 1 that

$$\begin{aligned}X &= X_1^{m_1} \otimes \cdots \otimes X_k^{m_k} \\ &\leq \|X_1\|_\oplus^{m_1} \otimes \cdots \otimes \|X_k\|_\oplus^{m_k} \otimes G^m,\end{aligned}$$

where $m = m_1 + \cdots + m_k$.

Since the associated graph is acyclic, it holds that $G^q = \mathcal{E}$ for all $q > p$. Therefore, if $m > p$, then $G^m = \mathcal{E}$, and we arrive at the inequality $X \leq \mathcal{E}$ which leads us to the desired result. ∎

Lemma 1 *Let matrices $X_1, \ldots, X_k$ have a common associated acyclic graph, and p be the length of the longest path in the graph.*

If $\|X_i\|_\oplus \geq 0$ for all $i = 1, \ldots, k$, then for any nonnegative integers $m_1, \ldots, m_k$, it holds

$$\left\| \prod_{i=1}^{k}{}_{\otimes} (E \oplus X_i)^{m_i} \right\|_\oplus \leq \left(\sum_{i=1}^{k}{}_{\oplus} \|X_i\|_\oplus \right)^p,$$

where $\prod_\otimes$ denotes the iterated operation $\otimes$.

Proof: Consider the matrix

$$X = \prod_{i=1}^{k}{}_{\otimes} (E \oplus X_i)^{m_i},$$

and represent it in the form

$$\begin{aligned}X &= (E \oplus X_1)^{m_1} \otimes \cdots \otimes (E \oplus X_k)^{m_k} \\ &= \sum_{i_1=0}^{m_1}{}_{\oplus} X_1^{i_1} \otimes \cdots \otimes \sum_{i_k=0}^{m_k}{}_{\oplus} X_k^{i_k} \\ &= \sum_{i_1=0}^{m_1}{}_{\oplus} \cdots \sum_{i_k=0}^{m_k}{}_{\oplus} X_1^{i_1} \otimes \cdots \otimes X_k^{i_k} \\ &\leq \sum_{0 \leq i_1 + \cdots + i_k \leq m}{}_{\oplus} X_1^{i_1} \otimes \cdots \otimes X_k^{i_k},\end{aligned}$$

where $m = m_1 + \cdots + m_k$. From Proposition 2 we may replace the last term to get

$$X \leq \sum_{0 \leq i_1 + \cdots + i_k \leq p}{}_{\oplus} X_1^{i_1} \otimes \cdots \otimes X_k^{i_k}.$$

Proceeding to the norm, with its additive and multiplicative properties, we arrive at the inequality

$$\|X\|_\oplus \leq \sum_{0 \leq i_1 + \cdots + i_k \leq p}{}_{\oplus} \|X_1\|_\oplus^{i_1} \otimes \cdots \otimes \|X_k\|_\oplus^{i_k}.$$

Since $0 \leq \|X_i\|_\oplus \leq \|X_1\|_\oplus \oplus \cdots \oplus \|X_k\|_\oplus$ for all $i = 1, \ldots, k$, we finally have

$$\begin{aligned}\|X\|_\oplus &\leq \sum_{i=0}^{p}{}_{\oplus} (\|X_1\|_\oplus \oplus \cdots \oplus \|X_k\|_\oplus)^p \\ &= (\|X_1\|_\oplus \oplus \cdots \oplus \|X_k\|_\oplus)^p.\end{aligned}$$

∎

4 Elements of Probability

In this section we present some probabilistic results associated with (max, +)-algebra concepts introduced above. We start with a lemma which states properties of the expectation with respect to the operations $\oplus$ and $\otimes$.

Lemma 2 *Let $\xi_1, \dots, \xi_k$ be random variables taking their values in $\underline{\mathbb{R}}$, and such that their expected values $\mathbb{E}[\xi_i]$, $i = 1, \dots, k$, exist. Then it holds*

1. $\mathbb{E}[\xi_1 \oplus \cdots \oplus \xi_k] \geq \mathbb{E}[\xi_1] \oplus \cdots \oplus \mathbb{E}[\xi_k]$,
2. $\mathbb{E}[\xi_1 \otimes \cdots \otimes \xi_k] = \mathbb{E}[\xi_1] \otimes \cdots \otimes \mathbb{E}[\xi_k]$.

Proof: Clearly, it will suffice to prove the lemma only for $k = 2$, and then extend the proof by induction to the case of arbitrary k.

To verify the first inequality, first assume that one of the expectation on its right side, say $\mathbb{E}[\xi_2]$, is infinite; that is $\mathbb{E}[\xi_2] = \varepsilon$. Since it holds that $\xi_1 \oplus \xi_2 \geq \xi_1$, we have

$$\mathbb{E}[\xi_1 \oplus \xi_2] \geq \mathbb{E}[\xi_1] = \mathbb{E}[\xi_1] \oplus \xi_2 = \mathbb{E}[\xi_1] \oplus \mathbb{E}[\xi_2].$$

Suppose now that $\mathbb{E}[\xi_1], \mathbb{E}[\xi_2] > \varepsilon$, and consider the obvious identity

$$x \oplus y = \frac{1}{2}(x + y + |x - y|) \quad \forall x, y \in \mathbb{R}.$$

With ordinary properties of expectation, we get

$$\begin{aligned} \mathbb{E}[\xi_1 \oplus \xi_2] &\geq \frac{1}{2}(\mathbb{E}[\xi_1] + \mathbb{E}[\xi_2] + |\mathbb{E}[\xi_1] - \mathbb{E}[\xi_2]|) \\ &= \mathbb{E}[\xi_1] \oplus \mathbb{E}[\xi_2]. \end{aligned}$$

The second assertion of the lemma is trivial. ■

The next result [7, 8] provides an upper bound for the expected value of the maximum of independent and identically distributed (i.i.d.) random variables.

Lemma 3 *Let $\xi_1, \dots, \xi_k$ be i.i.d. continuous random variables, and there exist $\mathbb{E}[\xi_1] < \infty$ and $\mathbb{D}[\xi_1] < \infty$. Then it holds*

$$\mathbb{E}\left[\sum_{i=1}^{k}{}_{\oplus}\, \xi_i\right] \leq \mathbb{E}[\xi_1] + \frac{k-1}{\sqrt{2k-1}}\sqrt{\mathbb{D}[\xi_1]}.$$

Consider a random matrix X with its entries x_{ij} taking values in $\underline{\mathbb{R}}$. We denote by $\mathbb{E}[X]$ the matrix obtained from X by replacing each entry x_{ij} by its expected value $\mathbb{E}[x_{ij}]$. With Lemma 2, it is easy to verify the next statement.

Lemma 4 *For any random matrix X, it holds*

$$\mathbb{E}\|X\|_{\oplus} \geq \|\mathbb{E}[X]\|_{\oplus}.$$

Proof: It follows from Lemma 2 that

$$\begin{aligned} \mathbb{E}\|X\|_{\oplus} &= \mathbb{E}\left[\sum_{i,j}{}_{\oplus}\, x_{ij}\right] \\ &\geq \sum_{i,j}{}_{\oplus}\, \mathbb{E}[x_{ij}] = \|\mathbb{E}[X]\|_{\oplus}. \end{aligned}$$

■

5 Acyclic Fork-Join Networks

We consider a network with n single-server nodes and customers of a single class. The topology of the network is described by an oriented acyclic graph $\mathcal{G} = (\mathbf{N}, \mathbf{A})$, where $\mathbf{N} = \{1, \dots, n\}$ represents the set of nodes, and $\mathbf{A} = \{(i,j)\} \subset \mathbf{N} \times \mathbf{N}$ is the set of arcs which determine the transition routes of customers. It is assumed that there are nodes in the graph which have no either incoming or outgoing arcs. Each node with no predecessors is assumed to represent an infinite external arrival stream of customers; provided that a node has no successors, it is considered as an output node intended to release customers from the network.

Each node of the network includes a server and infinite buffer which together present a single-server queue operating under the first-come, first-served queueing discipline. At the initial time, the server at each node is assumed to be free of customers, the buffers in nodes with no predecessors have infinite number of customers, whereas the buffers in the other nodes may have finite numbers of customers.

In addition to the usual service procedure, special join and fork operations [1] may be performed in a node respectively before and after service of a customer. The join operation is actually thought to cause each customer which comes into a node not to enter the buffer at the server but to wait until at least one customer from every preceding node arrives. As soon as these customers arrive, they, taken one from each preceding node, are united to be treated as being one customer which then enters the buffer to become a new member of the queue.

The fork operation at a node is initiated every time the service of a customer is completed; it consists in giving rise to several new customers one for each succeeding nodes. These customers simultaneously depart the node, each being passed to separate node related to the first one. We assume that the execution of fork-join operations when appropriate customers are available, as well as the transition of customers within and between nodes require no time.

For the queue at node i, we denote the kth departure epoch by $x_i(k)$, and the service time of the kth customer by τ_{ik}. We assume that $\tau_{ik} \geq 0$ are given parameters for all $i = 1, \dots, n$, and $k = 1, 2, \dots$, while $x_i(k)$ are considered as unknown state variables. With the condition that the network starts operating at time zero, it is convenient to set $x_i(0) = 0$ and $x_i(k) = \varepsilon$ for all $k < 0$, $i = 1, \dots, n$. Finally, we denote the number of customers in the buffer at node i at the initial time by r_i with $0 \leq r_i \leq \infty$, and introduce $M = \max\{r_i |\ r_i < \infty;\ i = 1, \dots, n\}$.

An example of an acyclic fork-join network with $n = 5$ is shown in Fig. 1.

It has been shown in [10, 11] that the dynamics of the network can be described by the state equation

$$x(k) = \sum_{m=1}^{M}{}_{\oplus}\, A_m(k) \otimes x(k-m), \qquad (1)$$

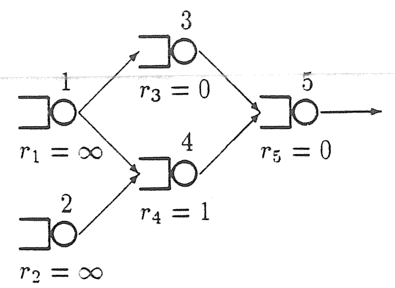

Figure 1: An acyclic fork-join network.

with the state transition matrices

$$\begin{aligned} A_1(k) &= (E \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k \otimes (E \oplus G_1^T), \\ A_m(k) &= (E \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k \otimes G_m^T, \\ & \quad m = 2, \ldots, M, \end{aligned}$$

where $\mathcal{T}_k = \mathrm{diag}(\tau_{1k}, \ldots, \tau_{nk})$ with ε as the off-diagonal entries, $G_m = (g_{ij}^m)$, $m = 0, 1, \ldots, M$, are matrices with their entries defined by the condition

$$g_{ij}^m = \begin{cases} 0, & \text{if } (i,j) \in \mathbf{A} \text{ and } m = r_j, \\ \varepsilon, & \text{otherwise,} \end{cases}$$

and p is the length of the longest path in the graph associated with the matrix G_0.

In fact, G_m presents the standard adjacency matrix of the partial graph $\mathcal{G}_m = (\mathbf{N}, \mathbf{A}_m)$ with $\mathbf{A}_m = \{(i,j)|\ (i,j) \in \mathbf{A};\ r_j = m\}$. Since the graph of the entire network is acyclic, all its partial graphs $\mathcal{G}_m$, $m = 0, 1, \ldots, M$, possess the same property.

6 Monotonicity Properties

In this section, a useful property of monotonicity is established which relates the system state vector $x(k)$ to the initial numbers of customers r_i. It is actually shown that the entries of $x(k)$ for all $k = 1, 2, \ldots$, do not decrease when the numbers r_i with $0 < r_i < \infty$, $i = 1, \ldots, n$, are reduced to 0.

We prove this assertion in two steps: first we consider the effect of reducing the numbers $r_i > 1$ to 1, and then replace $r_i = 1$ with $r_i = 0$. Note that the change in the initial numbers of customers results only in modifications to partial graphs $\mathcal{G}_m$ and so to their adjacency matrices G_m. Specifically, reducing these numbers to 1 leads us to new matrices $\widetilde{G}_1 = G_1 \oplus \cdots \oplus G_M$, and $\widetilde{G}_m = \mathcal{E}$ for all $m = 2, \ldots, M$, whereas G_0 remains unchanged.

Lemma 5 *Let $x(k)$ be determined by (1). Suppose that the vector $\widetilde{x}(k)$ satisfies the dynamic equation*

$$\widetilde{x}(k) = \widetilde{A}_1(k) \otimes \widetilde{x}(k-1), \quad \widetilde{x}(0) = 0,$$

with $\widetilde{A}_1(k) = (E \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k \otimes (E \oplus \widetilde{G}_1^T)$, where $\widetilde{G}_1 = G_1 \oplus \cdots \oplus G_M$.

Then for all $k = 1, 2, \ldots$, it holds

$$x(k) \leq \widetilde{x}(k).$$

Proof: Since $x(k_1) \leq x(k_2)$ for any $k_1 < k_2$, we have from (1)

$$x(k) \leq \left(\sum_{i=1}^{M}{}_{\oplus} A_m(k) \right) \otimes x(k-1).$$

Now we can define the matrix $\widetilde{A}_1(k)$ as

$$\begin{aligned} \widetilde{A}_1(k) &= \sum_{m=1}^{M}{}_{\oplus} A_m(k) \\ &= (E \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k \otimes \left(E \oplus \sum_{m=1}^{M}{}_{\oplus} G_m^T \right). \end{aligned}$$

Starting with the condition $x(0) = \widetilde{x}(0) = 0$, we successively verify that the relations

$$\begin{aligned} x(k) &\leq \widetilde{A}_1(k) \otimes x(k-1) \\ &\leq \widetilde{A}_1(k) \otimes \widetilde{x}(k-1) = \widetilde{x}(k) \end{aligned}$$

are valid for each $k = 1, 2, \ldots$. ∎

The next lemma shows that in the system with finite initial numbers r_i equal either to 0 or 1, the entries of its transition matrix $A_1(k)$ do not decrease as all these numbers are set to 0. Clearly, this also involves nondecrease in the entries of the system state vector $x(k)$.

Lemma 6 *For all $k = 1, 2, \ldots$, it holds*

$$A_1(k) \leq \widetilde{A}(k)$$

with $\widetilde{A}(k) = (E \oplus \mathcal{T}_k \otimes \widetilde{G}^T)^q \otimes \mathcal{T}_k$, where $\widetilde{G} = G_0 \oplus G_1$, and q is the length of the longest path in the graph associated with the matrix $\widetilde{G}$.

Proof: Consider the matrix

$$A_1(k) = (E \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k \otimes (E \oplus G_1^T).$$

As one can see, to prove the lemma, it will suffice to verify both inequalities

$$\widetilde{A}(k) \geq (E \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k, \tag{2}$$

$$\widetilde{A}(k) \geq (E \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k \otimes G_1^T. \tag{3}$$

Let us write the obvious representation

$$(E \oplus \mathcal{T}_k \otimes \widetilde{G}^T)^q = \sum_{i=1}^{q}{}_{\oplus} (E \oplus \mathcal{T}_k \otimes G_0^T)^i \otimes (\mathcal{T}_k \otimes G_1^T)^{q-i}.$$

Since $q \geq p$, we get from the representation

$$\begin{aligned} (E \oplus \mathcal{T}_k \otimes \widetilde{G}^T)^q &\geq (E \oplus \mathcal{T}_k \otimes \widetilde{G}^T)^p \\ &\geq (E \oplus \mathcal{T}_k \otimes G_0^T)^p. \end{aligned}$$

It remains to multiply both sides of the above inequality by $\mathcal{T}_k$ on the right so as to arrive at (2).

To verify (3), let us first assume that $q > p$. In this case, we obtain

$$\begin{aligned} (E \oplus \mathcal{T}_k \otimes \widetilde{G}^T)^q &\geq (E \oplus \mathcal{T}_k \otimes \widetilde{G}^T)^{p+1} \\ &\geq (E \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k \otimes G_1^T. \end{aligned}$$

Suppose now that $q = p$. Then it is necessary that $G_1 \otimes G_0^p = \mathcal{E}$. If this were not the case, there would be a path in the graph associated with the matrix $\widetilde{G} = G_0 \oplus G_1$, which has its length greater than p, and we therefore would have $q > p$.

Clearly, the condition $G_1 \otimes G_0^p = \mathcal{E}$ results in the equality $(\mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k \otimes G_1^T = \mathcal{E}$, and thus we get

$$\begin{aligned}(E \oplus \mathcal{T}_k \otimes \widetilde{G}^T)^p &\geq (E \oplus \mathcal{T}_k \otimes G_0^T)^{p-1} \otimes \mathcal{T}_k \otimes G_1^T \\ &= (E \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k \otimes G_1^T.\end{aligned}$$

Since it holds $(E \oplus \mathcal{T}_k \otimes \widetilde{G}^T)^p \otimes \mathcal{T}_k \geq (E \oplus \mathcal{T}_k \otimes \widetilde{G}^T)^p$, one can conclude that inequality (3) is also valid. ∎

Lemmas 5 and 6 can be summarized as follows.

Theorem 1 *In the acyclic fork-join queueing network, reducing the initial numbers of customers from any fixed values to 0 does not decrease the entries of the system state vector $\boldsymbol{x}(k)$ for all $k = 1, 2, \ldots$.*

7 Service Cycle Time

We consider the evolution of the system (1) as a sequence of service cycles: the 1st cycle starts at the initial time, and it is terminated as soon as all the servers in the network complete their 1st service, the 2nd cycle is terminated as soon as the servers complete their 2nd service, and so on. Clearly, the completion time of the kth cycle can be represented as

$$\max_i x_i(k) = \|\boldsymbol{x}(k)\|_\oplus .$$

The next lemma provides simple lower and upper bounds on the kth cycle completion time.

Lemma 7 *For all $k = 1, 2, \ldots$, it holds*

$$\left\| \sum_{i=1}^k \mathcal{T}_i \right\|_\oplus \leq \|\boldsymbol{x}(k)\|_\oplus \leq \sum_{i=1}^k \|\mathcal{T}_i\|_\oplus + p \left(\sum_{i=1}^k{}_\oplus \|\mathcal{T}_i\|_\oplus \right).$$

Proof: To verify the left inequality, first note that

$$A_1(k) = (E \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k \otimes (E \oplus G_1^T) \geq \mathcal{T}_k.$$

With this condition, we have from equation (1)

$$\begin{aligned}\boldsymbol{x}(k) &\geq \mathcal{T}_k \otimes \boldsymbol{x}(k-1) \\ &\geq \mathcal{T}_k \otimes \cdots \otimes \mathcal{T}_1 \otimes \boldsymbol{x}(0)\end{aligned}$$

with $\boldsymbol{x}(0) = \mathbf{0}$. Taking the norm, and considering that $\mathcal{T}_i$, $i = 1, \ldots, k$, are diagonal matrices, we get

$$\begin{aligned}\|\boldsymbol{x}(k)\|_\oplus &\geq \|\mathcal{T}_k \otimes \cdots \otimes \mathcal{T}_1\|_\oplus \\ &= \|\mathcal{T}_1 + \cdots + \mathcal{T}_k\|_\oplus .\end{aligned}$$

To obtain an upper bound, let us replace the general system (1) with that governed by the equation

$$\boldsymbol{x}(k) = A(k) \otimes \boldsymbol{x}(k-1), \quad \boldsymbol{x}(0) = \mathbf{0}, \qquad (4)$$

where $A(k) = (E \oplus \mathcal{T}_k \otimes G^T)^p \otimes \mathcal{T}_k$ with the matrix $G = G_0 \oplus G_1 \oplus \cdots \oplus G_M$, and p is the length of the longest path in the graph associated with G. As it follows from Theorem 1, such a replacement does not decrease the vector $\boldsymbol{x}(k)$ for all $k = 1, 2, \ldots$.

Let us denote $A_k = A(k) \otimes \cdots \otimes A(1)$. With the condition that $\boldsymbol{x}(0) = \mathbf{0}$, we can write from (4)

$$\|\boldsymbol{x}(k)\|_\oplus = \|A(k) \otimes \cdots \otimes A(1)\|_\oplus = \|A_k\|_\oplus .$$

From Proposition 1 we get for the matrix A_k

$$A_k \leq \prod_{i=1}^k{}_\otimes \|\mathcal{T}_i\|_\oplus \otimes \prod_{i=1}^k{}_\otimes \left(E \oplus \mathcal{T}_{k-i+1} \otimes G^T\right)^p .$$

By applying Lemma 1, we obtain the inequality

$$\begin{aligned}\|A_k\|_\oplus &\leq \prod_{i=1}^k{}_\otimes \|\mathcal{T}_i\|_\oplus \otimes \left(\sum_{i=1}^k{}_\oplus \|\mathcal{T}_i\|_\oplus \right)^p \\ &= \sum_{i=1}^k \|\mathcal{T}_i\|_\oplus + p \left(\sum_{i=1}^k{}_\oplus \|\mathcal{T}_i\|_\oplus \right),\end{aligned}$$

which provides us with the desired result. ∎

Let us define the average cycle time of the system:

$$y(k) = \frac{1}{k} \|\boldsymbol{x}(k)\|_\oplus .$$

In many applications, one is normally interested in investigating the steady-state mean cycle time; that is, the limit of $y(k)$ as k tends to ∞. We will consider this problem with relation to the stochastic network models in the next section.

8 Stochastic Networks

Suppose that for each node $i = 1, \ldots, n$, the service times $\tau_{i1}, \tau_{i2}, \ldots$, form a sequence of i.i.d. continuous non-negative random variables with $\mathbb{E}[\tau_{ik}] < \infty$ and $\mathbb{D}[\tau_{ik}] < \infty$ for all $k = 1, 2, \ldots$. With these conditions, $\mathcal{T}_k$ are i.i.d. random matrices, whereas $\|\mathcal{T}_k\|_\oplus$ present i.i.d. random variables with $\mathbb{E}\|\mathcal{T}_k\|_\oplus < \infty$ and $\mathbb{D}\|\mathcal{T}_k\|_\oplus < \infty$ for all $k = 1, 2, \ldots$.

In the analysis of the mean cycle time of the system, one first has to convince himself that the limit

$$\lim_{k \to \infty} \frac{1}{k} \|\boldsymbol{x}(k)\|_\oplus = \gamma \qquad (5)$$

exists. A standard technique to verify the existence of the above limit is based on the Subadditive Ergodic Theorem proposed in [9]. One can find examples of the implementation of the theorem in the (max, +)-algebra framework in [5, 3, 4, 12].

With the above probabilistic conditions, it is not difficult to apply the theorem in the same way as in [5, 3, 4, 12] to prove existence of the limit in (5) for the system (1). In more exact terms, the theorem allows one to prove that this limit exists with the probability one, and, at the same time, it holds

$$\lim_{k \to \infty} \frac{1}{k} \mathbb{E}\|\boldsymbol{x}(k)\|_\oplus = \gamma. \qquad (6)$$

Now we present our main result which offers simple bounds on the mean cycle time γ.

Theorem 2 *In the stochastic dynamical system (1) the mean cycle time γ satisfies the double inequality*

$$\|\mathbb{E}[\mathcal{T}_1]\|_\oplus \leq \gamma \leq \mathbb{E}\|\mathcal{T}_1\|_\oplus \,. \tag{7}$$

Proof: Note that according to the Subadditive Ergodic Theorem, we may represent the mean cycle time in the form (6).

Let us first prove the left inequality in (7). From Lemmas 7 and 4, we have

$$\begin{aligned} \frac{1}{k}\mathbb{E}\|x(k)\|_\oplus &\geq \frac{1}{k}\mathbb{E}\left\|\sum_{i=1}^{k}\mathcal{T}_i\right\|_\oplus \\ &\geq \left\|\frac{1}{k}\sum_{i=1}^{k}\mathbb{E}[\mathcal{T}_i]\right\|_\oplus = \|\mathbb{E}[\mathcal{T}_1]\|_\oplus, \end{aligned}$$

independently of k.

With the upper bound offered by Lemma 7, we get

$$\frac{1}{k}\mathbb{E}\|x(k)\|_\oplus \leq \mathbb{E}\|\mathcal{T}_1\|_\oplus + \frac{p}{k}\mathbb{E}\left[\sum_{i=1}^{k}{}_\oplus \|\mathcal{T}_i\|_\oplus\right].$$

From Lemma 3, the second term on the right-hand side may be replaced by that of the form

$$\frac{p}{k}\left(\mathbb{E}\|\mathcal{T}_1\|_\oplus + \frac{k-1}{\sqrt{2k-1}}\sqrt{\mathbb{D}\|\mathcal{T}_1\|_\oplus}\right),$$

which tends to 0 as $k \to \infty$. ∎

9 Concluding Remarks

In conclusion, we briefly discuss the behaviour of the bounds (7) under various assumptions concerning the service times in the network. First note that the derivation of the bounds does not require the kth service times τ_{ik} to be independent for all $i = 1, \ldots, n$. Moreover, if $\tau_{ik} = \tau_k$ for all i, we have $\|\mathbb{E}[\mathcal{T}_1]\|_\oplus = \mathbb{E}\|\mathcal{T}_1\|_\oplus$, and so the lower and upper bound coincide. Therefore, one can expect that the accuracy of the bounds increases with strengthening the dependency.

Furthermore, let us consider the effect of decreasing the variance $\mathbb{D}[\tau_{i1}]$ on the bounds on γ. Note that if τ_{i1} were degenerate random variables with zero variance, the lower and upper bounds in (7) would coincide. This leads us to the conclusion that with decreasing the variance of τ_{i1}, the bounds become more accurate.

Finally, let us choose a node of the network, say node $i = 1$. Suppose that, with the random variables τ_{i1}, $i \neq 1$, being fixed, the random service time τ_{11} is changed in such a way so that the probability

$$\mathbb{P}\left\{\tau_{11} > \sum_{i\neq 1}{}_\oplus \tau_{i1}\right\}$$

increases. It is easy to see that as the above probability tends to one, both the lower and upper bounds tend to the common value $\mathbb{E}[\tau_{11}]$.

Since, in particular, this probability increases as $\mathbb{E}[\tau_{11}]$ increases, we conclude that in this case, the greater difference between $\mathbb{E}[\tau_{11}]$ and $\mathbb{E}[\tau_{i1}]$ for all $i \neq 1$, the smaller difference between the lower and upper bounds in (7).

References

[1] F. Baccelli and A.M. Makowski, Queueing Models for Systems with Synchronization Constraints, *Proc. IEEE* 77 (1989), no.1, 138-160.

[2] F. Baccelli and P. Konstantopoulos, Estimates of Cycle Times in Stochastic Petri Nets, *Applied Stochastic Analysis*, Lecture Notes in Control and Information Sciences, 177, Springer-Verlag: New York, 1991, 1-20.

[3] F. Baccelli, Ergodic Theory of Stochastic Petri Networks, *Ann. Probab.* 20 (1992), no.1, 375-396.

[4] F. Baccelli, G. Cohen, G.J. Olsder, and J.-P. Quadrat, *Synchronization and Linearity*, Wiley: Chichester, 1992.

[5] J.E. Cohen, Subadditivity, Generalized Products of Random Matrices and Operations Research, *SIAM Review* 30 (1988), no.1, 69-86.

[6] R.A. Cuninghame-Green, *Minimax Algebra*, Lecture Notes in Economics and Mathematical Systems, 166, Springer-Verlag: Berlin, 1979.

[7] E.J. Gumbel, The Maxima of the Mean Largest Value and on the Range, *Ann. Math. Statist.* 25 (1954), 76-84.

[8] H.O. Hartly and H.A. David, Universal Bounds for Mean Range and Extreme Observations, *Ann. Math. Statist.* 25 (1954), 85-99.

[9] J.F.C. Kingman, Subadditive Ergodic Theory, *Ann. Probab.* 1 (1973), 883-909.

[10] N.K. Krivulin, An Algebraic Approach in Modelling and Simulation of Queueing Networks, in *Proc. Circuits, Systems and Computers '96, Hellenic Naval Academy, Piraeus, Greece, July 15-17, 1996* 2, 668-672.

[11] N.K. Krivulin, Max-Plus Algebra Models of Queueing Networks, in *Proc. International Workshop on Discrete Event Systems (WODES'96), 19-21 August 1996, University of Edinburgh, UK*, IEE: London, 1996, 76-81.

[12] J. Mairesse, Stochastic Linear Systems in the (max, +) Algebra, in *Proceedings of the 11th International Conference on Analysis and Optimization of Systems, Sophia-Antipolis, June 15-17, 1994 (G. Cohen and J.-P. Quadrat, Eds.)*, Springer-Verlag: London, 1994, 257-265.