UNBIASED GRADIENT ESTIMATION IN QUEUEING NETWORKS WITH PARAMETER-DEPENDENT ROUTING

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Abstract

A stochastic queueing network model with parameter-dependent service times and routing mechanism, and its related performance measures are considered. An estimate of performance measure gradient is proposed, and rather general sufficient conditions for the estimate to be unbiased are given. A gradient estimation algorithm is also presented, and its validity is briefly discussed.

Keywords: queueing networks, parameter-dependent routing, performance measure gradient, unbiased estimate.

1 Introduction

The evaluation of performance measure gradient presents one of the main issues of analysis of queueing network performance. Except in a few particular models, there are generally no closed-form representations as functions of network parameters available for performance measures and their gradients. In this situation, one normally applies the Monte Carlo approach to estimate gradient of network performance measures.

In the last decade, infinitesimal perturbation analysis (IPA) [6, 5, 14] has received wide acceptance in queueing system performance evaluation as an efficient technique underlying the calculation of gradient estimates as well as the examination of their unbiasedness. Specifically, this technique was employed in [2] to calculate gradient estimates in closed networks with an ordinary probabilistic routing mechanism and general service time distributions. An extension of IPA, smoothed perturbation analysis, has been applied in [4] to the development of asymptotically unbiased gradient estimates in queueing networks with parameter-dependent routing.

Another approach based on the analysis of algebraic representation of queueing system dynamics and their performance has been implemented in [8, 9, 10]. This approach offers a convenient and unified way of analytical study of gradient estimates, and it leads to computational procedures closely similar to those of IPA. In this paper, based on this approach, a rather general queueing network model with parameter-dependent service times and routing mechanism is presented. For the performance measures which one normally chooses in analysis of network performance, we propose a gradient estimate, and give sufficient conditions for the estimate to be unbiased. These conditions are rather general and normally met in analysis of queueing network performance. Finally, an algorithm of estimating gradient of a particular performance measure is presented, and its validity is briefly discussed.

2 The Underlying Network Model

We consider a generalized model of a queueing network consisting of N nodes, with customers of a single class. As is customary in queueing network models, customers are assumed to circulate through the network to receive service at appropriate nodes. We do not restrict ourselves to a particular type of nodes, it is suggested that any node may have a single server as well as several servers operating either in parallel or in tandem.

Furthermore, there is a buffer with infinite capacity in each node, in which customers are placed at their arrival to wait for service if it cannot be initiated immediately. We assume the queue discipline underlying the operation of any node to be first-come, first-served. Upon his service completion at one node, each customer goes to another node chosen according to some routing procedure described below. We suppose that the transition of any customer between nodes requires no time, and he therefore arrives immediately into the next node. Finally, we assume that the network starts operating at time zero; at the initial time, the server at any node n is free, whereas its buffer contains K_n customers, $0 \leq K_n \leq \infty,\ n = 1, \ldots, N$.

We now turn to the formal description of the network dynamics from an algebraic viewpoint, and then introduce randomness into the network model.

2.1 Algebraic Description of Node Dynamics

In a general sense, each node can be regarded as a processor which produces an output sequence of departure times of customers from another two, an input and control sequences formed respectively by the arrival times and the service times of customers. Let us denote for every node n, $1 \leq n \leq N$, the kth arrival epoch to the node by A_n^k, and the kth departure epoch from the node by D_n^k. Notice, because the transition of customers from one node to another is immediate, each A_n^k coincides with some D_i^j with the exception of $A_n^k = 0$ for all $k \leq K_n$. Finally, we denote the service time associated with the kth service initiation in node n, by τ_n^k. The set of all service times $\boldsymbol{T} = \{\tau_n^k | \, n = 1, \ldots, N; k = 1, 2, \ldots\}$ is assumed to be given.

The usual way to represent the operation of a node is based on recursive equations describing evolution of D_n^k as a state variable [3, 7, 12]. Note that these recursive equations are often rather difficult to resolve. Below are given two equations which describe dynamics of nodes operating as the $G/G/1$ and $G/G/2$ queueing systems. Other examples may be found in [3, 7, 11, 12].

2.1.1 The $G/G/1$ queue.

Suppose first that node n is represented as the $G/G/1$ queue. Its associated recursive equation may be written as [3]

$$D_n^k = (A_n^k \vee D_n^{k-1}) + \tau_n^k,$$

where $\vee$ denotes the maximum operator, and $D_n^k \equiv 0$ for all $k < 0$. It is easy to see that the solution of the equation in terms of arrival and service times has the form

$$D_n^k = \bigvee_{i=1}^{k} \left(A_n^i + \sum_{j=i}^{k} \tau_n^j \right).$$

2.1.2 The $G/G/2$ queue.

The equation which describes the dynamics of a node operating as the $G/G/2$ queue may be considered as rather difficult to handle. For node n, it is written as [11]

$$D_n^k = \bigvee_{i=1}^{k} \left((A_n^i \vee D_n^{i-2}) + \tau_n^i \right) \wedge \left((A_n^{k+1} \vee D_n^{k-1}) + \tau_n^{k+1} \right),$$

where $\wedge$ stands for the minimum operator. Although there are no closed-form solutions of the equation, known to the author, it is clear that it exists.

2.2 Routing Mechanism and Interaction of Nodes

The routing mechanism inherent in the network is defined by the sequences $\boldsymbol{R}_n = \{\rho_n^1, \rho_n^2, \ldots\}$ given for each node n, where ρ_n^k represents the next node to be visited by the customer who is the kth to depart from node n, $\rho_n^k \in \{1, \ldots, N\}$, $k = 1, 2, \ldots$. The matrix

$$\boldsymbol{R} = \begin{pmatrix} \rho_1^1 & \rho_1^2 & \ldots & \rho_1^k & \ldots \\ \rho_2^1 & \rho_2^2 & \ldots & \rho_2^k & \ldots \\ \vdots & \vdots & & \vdots & \\ \rho_N^1 & \rho_N^2 & \ldots & \rho_N^k & \ldots \end{pmatrix}$$

is referred to as the routing table of the network.

In order to describe the dynamics of the network completely, it remains to define formally interactions between nodes. In fact, a relationship between arrival and departure times of distinct nodes is to be established. To this end, for each node n, let us introduce the set

$$\boldsymbol{D}_n = \{D_i^j | \, \rho_i^j = n; \, i = 1, \ldots, N; \, j = 1, 2, \ldots\} \qquad (1)$$

which is constituted by the departure times of the customers who have to go to node n. Furthermore, we denote by $\mathcal{A}_n^k$ the arrival time of the customer which is the kth to arrive into node n after his service at any node of the network. In other words, the symbol $\mathcal{A}_n^k$ differs from A_n^k in that it refers only to the customers really arriving into node n, and does not to those occurring in this node at the initial time.

It has been shown in [9, 10] that it holds

$$\mathcal{A}_n^k = \bigwedge_{\{D_1, \ldots, D_k\} \subset \boldsymbol{D}_n} (D_1 \vee \cdots \vee D_k), \qquad (2)$$

where minimum is taken over all k-subsets of the set $\boldsymbol{D}_n$. The times A_n^k and $\mathcal{A}_n^k$ are related by the equality

$$A_n^k = \begin{cases} 0, & \text{if } k \leq K_n \\ \mathcal{A}_n^{k-K_n}, & \text{otherwise} \end{cases}. \qquad (3)$$

Clearly, if deterministic routing with an integer matrix R as the routing table is adopted in the model, each set $\boldsymbol{D}_n$, $n = 1, \ldots, N$, is determined uniquely from (1), and then a straightforward algebraic representation of A_n^k may be obtained using (2-3). In this case, starting from the above representations of node dynamics, one may eventually arrive at algebraic expressions for any arrival time A_n^k and departure time D_n^k, which are written in terms of service times $\tau \in \boldsymbol{T}$, and involve only the operations of maximum, minimum, and addition.

2.3 Representation of Network Performance

One of the features of the formal network model described above is that it offers the potential for representing network performance criteria in a rather simple and convenient way. Suppose that we observe the network until the Kth service completion at node n, $1 \leq n \leq N$. As performance criteria for node n in the observation period, one normally chooses the following average quantities [3, 8, 9, 10, 12]

system time of one customer:	$S_n^K = \sum_{k=1}^K (D_n^k - A_n^k)/K$,
waiting time of one customer:	$W_n^K = \sum_{k=1}^K (D_n^k - A_n^k - \tau_n^k)/K$,
throughput rate of the node:	$T_n^K = K/D_n^K$,
utilization of the server:	$U_n^K = \sum_{k=1}^K \tau_n^k/D_n^K$,
number of customers:	$J_n^K = \sum_{k=1}^K (D_n^k - A_n^k)/D_n^K$,
queue length at the node:	$Q_n^K = \sum_{k=1}^K (D_n^k - A_n^k - \tau_n^k)/D_n^K$.

It is easy to see that with the routing mechanism determined by an integer matrix, all these criteria may be represented only in terms of service times in closed form.

2.4 Stochastic Aspect and Performance Evaluation

Let us suppose that for all $n = 1,\ldots,N$, and $k = 1, 2, \ldots$, the service times are defined as random variables $\tau_n^k = \tau_n^k(\theta,\omega)$, where $\theta \in \Theta \subset \mathbb{R}$ is a decision parameter, ω is a random vector which represents the random effects involved in network behaviour. First we assume the routing table $\boldsymbol{R} = R$ to be an integer matrix. Since deterministic routing leads to algebraic expressions in terms of the random variables $\tau \in \boldsymbol{T}$ for the performance criteria introduced above, one can conclude that these criteria also present random variables.

Let $F = F(\theta,\omega)$ be a random performance criterion of the network. As is customary, we define the performance measure associated with F as the expected value

$$\boldsymbol{F}(\theta) = E_\omega[F(\theta,\omega)]. \quad (4)$$

Although we may express F in closed form, it is often very difficult or impossible to obtain analytically the performance measure $\boldsymbol{F}$. In this situation, one generally applies a simulation technique which allows of obtaining values of $F(\theta,\omega)$, and then estimate the network performance by using the Monte Carlo approach.

We now turn to the networks with parameter-dependent probabilistic routing. We assume $\rho_n^k = \rho_n^k(\theta,\omega)$ to be a discrete random variable ranging over the set $\{1,\ldots,N\}$. The routing mechanism of the network is now defined by the random matrix $\boldsymbol{R} = \boldsymbol{R}(\theta,\omega)$ with particular routing tables as its values. We denote the set of all possible routing tables R by $\mathcal{R}$.

Obviously, the expression of A_n^k defined by (2-3) may change from one shape into another, depending on particular routing tables. To take this into account, we now define the random performance criteria in (4) as [9, 10]

$$F(\theta,\omega,\boldsymbol{R}(\theta,\omega)) = \sum_{R\in\mathcal{R}} \mathbf{1}_{\{\boldsymbol{R}(\theta,\omega)=R\}} F_R(\theta,\omega), \quad (5)$$

where $\mathbf{1}_{\{\boldsymbol{R}(\theta,\omega)=R\}}$ is the indicator function of the event $\{\boldsymbol{R}(\theta,\omega) = R\}$, and $F_R(\theta,\omega) = F(\theta,\omega,R)$ is the performance criterion evaluated under the condition that the network operates according to the deterministic routing mechanism defined by the routing table $R \in \mathcal{R}$.

3 Performance Measure Gradient Estimation

Since there are generally no explicit representations as functions of system parameters θ, available for the performance measure $\boldsymbol{F}$, one may evaluate its gradient $\partial\boldsymbol{F}(\theta)/\partial\theta$ by no way other than through the use of either finite difference estimates [1, 5] or the estimate

$$g(\theta,\omega_1,\ldots,\omega_M) = \frac{1}{M}\sum_{i=1}^M \frac{\partial}{\partial\theta}F(\theta,\omega_i), \quad (6)$$

where ω_i, $i = 1,\ldots,M$, are independent realization of ω, provided that the derivative $\partial F(\theta,\omega)/\partial\theta$ exists.

Very efficient procedures of obtaining gradient estimates may be designed using the IPA technique [6, 5, 14]. Such a procedure can yield the exact values of the derivative $\partial F(\theta,\omega)/\partial\theta$ by performing only one simulation run. Furthermore, in the case of a vector parameter $\theta \in \mathbb{R}^d$, the IPA procedures provide all partial derivatives $\partial F(\theta,\omega)/\partial\theta_i$, $i = 1,\ldots,d$, simultaneously, and take an additional computational cost which is normally very small compared with that required for the simulation run alone. Finally, it can be easily shown [1, 14] that if the IPA estimate of the derivative in (6) is unbiased, the mean square error of g has the order which is significantly less than those of any finite difference estimates based on the same number of simulation runs.

A sufficient condition for the estimate (6) to be unbiased at some $\theta \in \Theta$ requires [1, 5, 14]

$$\frac{\partial}{\partial\theta}E[F(\theta,\omega)] = E\left[\frac{\partial}{\partial\theta}F(\theta,\omega)\right]. \quad (7)$$

A usual way of examining the interchange in (7) involves the application of the Lebesgue dominated convergence theorem [13]

Theorem 1 *Let $(\Omega,\mathcal{F},P)$ be a probability space, $\Theta \subset \mathbb{R}^d$, and $F : \Theta\times\Omega \to \mathbb{R}$ be a $\mathcal{F}$-measurable function for any $\theta \in \Theta$ and such that the following conditions hold:*

(i) *for every $\theta \in \Theta$, there exists $\partial F(\theta,\omega)/\partial\theta$ at θ w.p. 1,*

(ii) *for all $\theta_1,\theta_2 \in \Theta$, there is a random variable $\lambda(\omega)$ with $E\lambda < \infty$ and such that*

$$|F(\theta_1,\omega) - F(\theta_2,\omega)| \le \lambda(\omega) \parallel \theta_1 - \theta_2 \parallel \quad \textit{w.p. 1.} \quad (8)$$

Then equation (7) holds on Θ.

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In [8, 9, 10] the approach based on the implementation of Theorem 1 has been applied to analyze estimates of performance gradient in the networks models with the parameter-independent probabilistic routing mechanism determined by a random routing table $\boldsymbol{R} = \boldsymbol{R}(\omega)$. Specifically, starting from the representations of network dynamics, discussed in Section 2, it has been shown that

(i) if each service time $\tau \in \boldsymbol{T}$ satisfies the conditions of Theorem 1, and for every $\theta \in \Theta$, all $\tau \in \boldsymbol{T}$ present continuous and independent random variables, then the average total time S_n^K and waiting time W_n^K satisfy the conditions of Theorem 1;

(ii) if in addition to previous assumptions, there exist random variables $\mu, \nu > 0$ such that for all $\tau \in \boldsymbol{T}$ it holds $\nu \leq |\tau| \leq \mu$ w.p. 1 for all $\theta \in \Theta$, and the condition $E[\mu\lambda/\nu^2] < \infty$ is fulfilled, where λ is the random variable providing τ with (8), then the average throughput rate T_n^K, utilization U_n^K, number of customers J_n^K, and queue length Q_n^K satisfy the conditions of Theorem 1.

Note that the above conditions do not involve independence at each θ between the random variables $\tau(\theta, \omega)$ and $\rho(\omega)$ in the probabilistic sense.

4 An Unbiased Gradient Estimate for Networks with Parameter-Dependent Routing

We start the section with an example which exhibits difficulties arising in gradient estimation when there is a parameter dependence involved in the routing mechanism of the network, and then present our main result offering an unbiased estimate of performance measure gradient in networks with parameter-dependent routing.

4.1 Preliminary Analysis

Suppose that there is a parameter dependence of the routing mechanism in the network, that is $\boldsymbol{R} = \boldsymbol{R}(\theta, \omega)$. In this case, the random performance criterion F generally violates condition (8). As an illustration, one can consider the following example.

Let $\Theta = [0, 1]$, $\Omega_1 = \Omega_2 = [0, 1]$, and $(\Omega, \mathcal{F}, P)$ be a probability space, where $\mathcal{F}$ is the σ-field of Borel sets of $\Omega = \Omega_1 \times \Omega_2$, P is the Lebesgue measure on Ω. Denote $\omega = (\omega_1, \omega_2)^T$, and define the function

$$F(\theta, \omega, \rho(\theta, \omega)) = \begin{cases} \tau_1(\theta, \omega), & \text{if } \rho(\theta, \omega) = 1 \\ \tau_2(\theta, \omega), & \text{if } \rho(\theta, \omega) = 2 \end{cases},$$

where

$$\tau_1(\theta, \omega) = \theta + \omega_1 + 1, \quad \tau_2(\theta, \omega) = \theta + \omega_1,$$

and

$$\rho(\theta, \omega) = \begin{cases} 1, & \text{if } \omega_2 \leq \theta \\ 2, & \text{if } \omega_2 > \theta \end{cases}.$$

We may treat τ_1 and τ_2 as the service time of a customer respectively at node 1 and 2. The function F is then assumed to be the service time of the customer which may arrive into either node 1 or 2, according to one of the two possible values of ρ.

Clearly, τ_1 and τ_2 satisfy the conditions of Theorem 1, whereas the function F now represented as

$$F(\theta, \omega) = \begin{cases} \theta + \omega_1 + 1, & \text{if } \omega_2 \leq \theta \\ \theta + \omega_1, & \text{if } \omega_2 > \theta \end{cases},$$

is differentiable w.p. 1 at any $\theta \in \Theta$, and $\partial F(\theta, \omega)/\partial\theta = 1$ w.p. 1. However, for any $\theta_1, \theta_2 \in \Theta$ such that $\theta_1 \geq \omega_2$ and $\theta_2 < \omega_2$, it holds

$$|F(\theta_1, \omega) - F(\theta_2, \omega)| \geq 1,$$

and therefore, condition (8) is violated.

On the other hand, it is easy to verify that

$$E[F(\theta, \omega)] = 2\theta + \tfrac{1}{2}, \qquad \tfrac{\partial}{\partial\theta} E[F(\theta, \omega)] = 2$$

$$\tfrac{\partial}{\partial\theta} F(\theta, \omega) = 1 \text{ w.p. } 1, \qquad E\left[\tfrac{\partial}{\partial\theta} F(\theta, \omega)\right] = 1.$$

In other words, equation (7) proves to be not valid, and we finally conclude that estimate (6) will be biased.

4.2 The Main Result

To suppress the bias in estimates of the gradient

$$\frac{\partial}{\partial\theta}\boldsymbol{F}(\theta) = \frac{\partial}{\partial\theta} E[F(\theta, \omega, \boldsymbol{R}(\theta, \omega))], \tag{9}$$

let us replace (6) by the estimate

$$\widetilde{g}(\theta, \omega_1, \ldots, \omega_M) = \frac{1}{M}\sum_{i=1}^{M} G(\theta, \omega_i), \tag{10}$$

where $G(\theta, \omega)$ will be defined in the next theorem.

Theorem 2 *Suppose that a random performance criterion F is represented in form (5), and for each $R \in \mathcal{R}$ the following conditions hold:*

(i) *F_R satisfies the conditions of Theorem 1;*

(ii) *for any $\theta \in \Theta$, the random variable $F_R(\theta, \omega)$ and the random matrix $\boldsymbol{R}(\theta, \omega)$ are independent;*

(iii) *for any $\theta \in \Theta$, the function $\Phi(\theta, R) = \Pr\{\boldsymbol{R}(\theta, \omega) = R\}$ is continuously differentiable at θ, and $\Phi(\theta, R) > 0$.*

Then for any $\theta_0 \in \Theta$, estimate (10) with

$$\begin{aligned} G(\theta_0, \omega) \quad = \quad & \frac{\partial}{\partial\theta} F(\theta, \omega, \boldsymbol{R}(\theta_0, \omega))\Big|_{\theta=\theta_0} \\ & + F(\theta_0, \omega, \boldsymbol{R}(\theta_0, \omega))\Psi(\theta_0, \boldsymbol{R}(\theta_0, \omega)), \end{aligned}$$

where $\Psi(\theta, R) = \frac{\partial}{\partial\theta} \ln \Phi(\theta, R)$, is unbiased.

Proof. Clearly, it is sufficient to show that the equation

$$\frac{\partial}{\partial\theta}E[F(\theta,\omega,\boldsymbol{R}(\theta,\omega))] = E[G(\theta,\omega)]$$

holds for any $\theta \in \Theta$.

To verify this equation, let us first represent F in form (5), and consider its expected value

$$E[F(\theta,\omega,\boldsymbol{R}(\theta,\omega))] = \sum_{R\in\mathcal{R}} E\left[\mathbf{1}_{\{\boldsymbol{R}(\theta,\omega)=R\}}F(\theta,\omega,R)\right].$$

Since $E[\mathbf{1}_{\{\boldsymbol{R}(\theta,\omega)=R\}}] = \Pr\{\boldsymbol{R}(\theta,\omega) = R\} = \Phi(\theta,R)$, it follows from condition (ii) of the theorem that

$$\begin{aligned} & E[F(\theta,\omega,\boldsymbol{R}(\theta,\omega))] \\ &= \sum_{R\in\mathcal{R}} E[F(\theta,\omega,R)]\Pr\{\boldsymbol{R}(\theta,\omega) = R\} \\ &= \sum_{R\in\mathcal{R}} E[F(\theta,\omega,R)]\Phi(\theta,R). \end{aligned}$$

For any $\theta_0 \in \Theta$, under conditions (i) and (iii), we successively get

$$\begin{aligned} & \frac{\partial}{\partial\theta}E[F(\theta,\omega,\boldsymbol{R}(\theta,\omega))]\bigg|_{\theta=\theta_0} \\ &= \sum_{R\in\mathcal{R}}\Bigg(\frac{\partial}{\partial\theta}E[F(\theta,\omega,R)]\bigg|_{\theta=\theta_0}\Phi(\theta_0,R) \\ &\quad + E[F(\theta_0,\omega,R)]\,\frac{\partial}{\partial\theta}\Phi(\theta,R)\bigg|_{\theta=\theta_0}\Bigg) \\ &= \sum_{R\in\mathcal{R}}\Bigg(E\left[\frac{\partial}{\partial\theta}F(\theta,\omega,R)]\bigg|_{\theta=\theta_0}\right]\Phi(\theta_0,R) \\ &\quad + E[F(\theta_0,\omega,R)]\frac{\frac{\partial}{\partial\theta}\Phi(\theta,R)\big|_{\theta=\theta_0}}{\Phi(\theta_0,R)}\Phi(\theta_0,R)\Bigg) \\ &= \sum_{R\in\mathcal{R}}\Bigg(E\left[\frac{\partial}{\partial\theta}F(\theta,\omega,R)\bigg|_{\theta=\theta_0}\right] \\ &\quad + E[F(\theta_0,\omega,R)]\Psi(\theta_0,R)\Bigg)\Phi(\theta_0,R) \\ &= \sum_{R\in\mathcal{R}} E\Bigg[\frac{\partial}{\partial\theta}(\theta,\omega,R)\bigg|_{\theta=\theta_0} \\ &\quad + F(\theta_0,\omega,R)\Psi(\theta_0,R)\Bigg]\Pr\{\boldsymbol{R}(\theta_0,\omega) = R\} \\ &= E\Bigg[\frac{\partial}{\partial\theta}F(\theta,\omega,\boldsymbol{R}(\theta_0,\omega))\bigg|_{\theta=\theta_0} \\ &\quad + F(\theta_0,\omega,\boldsymbol{R}(\theta_0,\omega))\Psi(\theta_0,\boldsymbol{R}(\theta_0,\omega))\Bigg] \\ &= E[G(\theta_0,\omega)]. \quad \square \end{aligned}$$

It is not difficult to obtain the conditions for estimate (10) to be unbiased for gradient of particular performance measures. They can be stated by combining the conditions in Section 3, related to particular performance criteria in networks with parameter-independent routing, with those of Theorem 2. Note that these conditions are rather general, and normally met in analysis of queueing networks.

Let us now return to the example presented in the previous subsection. First, we have

$$\Phi(\theta,1) = \Pr\{\rho(\theta,\omega) = 1\} = \theta,$$

$$\Phi(\theta,2) = \Pr\{\rho(\theta,\omega) = 2\} = 1-\theta,$$

and then

$$\Psi(\theta,1) = \frac{d}{d\theta}\ln\theta = \frac{1}{\theta},$$

$$\Psi(\theta,2) = \frac{d}{d\theta}\ln(1-\theta) = \frac{1}{1-\theta}.$$

In this case, the function G is defined as

$$G(\theta,\omega) = \begin{cases} 1 + \frac{\theta+\omega_1+1}{\theta}, & \text{if } \omega_2 \le \theta \\ 1 + \frac{\theta+\omega_1}{\theta-1}, & \text{if } \omega_2 > \theta \end{cases}$$

for any $\theta \in (0,1)$. Finally, evaluation of its expected value gives

$$E[G(\theta,\omega)] = \frac{\partial}{\partial\theta}E\left[F(\theta,\omega,\rho(\theta,\omega))\right] = 2.$$

5 Application to Network Simulation

Consider a network with N single-server nodes, and assume that the Kth service completion at a fixed node n, $1 \le n \le N$, comes with probability one after a finite number of service completions in the network. In this case, to observe evolution of the network until the Kth completion at the node, it will suffice to take into consideration only finite routes defined by a right truncated routing table with integer $(N\times L)$-matrices

$$R = \begin{pmatrix} r_1^1 & r_1^2 & \dots & r_1^L \\ r_2^1 & r_2^2 & \dots & r_2^L \\ \vdots & \vdots & & \vdots \\ r_N^1 & r_N^2 & \dots & r_N^L \end{pmatrix}$$

as its values, with some $L \ge K$.

Furthermore, we assume, as is customary in network simulation, that for each $\theta \in \Theta$, the random variables $\rho_n^k(\theta,\omega)$ are independent for all $n = 1,\dots,N$, and $k = 1,\dots,L$. With this condition and the notation $\varphi_n^k(\theta,r) = \Pr\{\rho_n^k(\theta,\omega) = r\}$, we may represent the function Φ introduced in Theorem 2, as

$$\Phi(\theta,R) = \Pr\{\boldsymbol{R}(\theta,\omega) = R\} = \prod_{n=1}^{N}\prod_{k=1}^{L}\varphi_n^k(\theta,r_n^k),$$

and then get the function Ψ in the form

$$\Psi(\theta,R) = \frac{\partial}{\partial\theta}\ln\Phi(\theta,R) = \sum_{n=1}^{N}\sum_{k=1}^{L}\frac{\partial}{\partial\theta}\ln\varphi_n^k(\theta,r_n^k).$$

Suppose now that we have to estimate the gradient of a performance measure, say $\boldsymbol{U}_n^K(\theta) = E[U_n^K(\theta,\omega)]$, the expected value of the average utilization of the server at node n. It results from Theorem 2 that, as an unbiased estimate, the function

$$G(\theta,\omega) = \frac{\partial}{\partial\theta} F(\theta,\omega,R) + F(\theta,\omega,R)\Psi(\theta,R), \qquad (11)$$

may be applied with $R = \boldsymbol{R}(\theta,\omega)$, and

$$F(\theta,\omega,R) = \sum_{k=1}^{K} \tau_n^k(\theta,\omega)/D_n^K(\theta,\omega,R).$$

It is not difficult to construct the next algorithm which produces the value of $G(\theta,\omega)$ for fixed $\theta \in \Theta \subset \mathbb{R}$, and $\omega \in \Omega$, provided that there is a network simulation procedure into which the algorithm may be incorporated. It actually combines an IPA algorithm [6] for obtaining the derivative $\partial F(\theta,\omega,R)/\partial\theta$ with additional computations according to (11).

Algorithm 5.1.

Initialization:
for $i = 1,\ldots,N$ **do** $g_i \longleftarrow 0$;
$s, t, t' \longleftarrow 0$;
$R \longleftarrow \boldsymbol{R}(\theta,\omega)$;

Upon the kth service completion at node i, perform the instructions:
$g_i \longleftarrow g_i + \frac{\partial}{\partial\theta}\tau_i^k(\theta,\omega)$;
if $i = n$ **then** $t \longleftarrow t + \tau_i^k(\theta,\omega)$;
$t' \longleftarrow t' + \frac{\partial}{\partial\theta}\tau_i^k(\theta,\omega)$;
if $k = K$ **then** $d \longleftarrow D_n^K(\theta,\omega,R)$;
stop;
$r \longleftarrow r_i^k$;
$s \longleftarrow s + \frac{\partial}{\partial\theta}\ln\varphi_i^k(\theta,r)$;
if *the server of node* r *is free* **then** $g_r \longleftarrow g_i$.

On completion of the algorithm, it remains to compute $(t'd - tg_n)/d^2 + ts/d$ as the value of G.

Note, in conclusion, that estimate (6) with the function G evaluated using the algorithm will not be unbiased in general. For the estimate to be unbiased, the function $\Psi(\theta,R)$ in (11) must be calculated as the sum with the same number of summands $\partial\ln\varphi_n^k(\theta,r)/\partial\theta$ for any of simulation runs. However, during the simulation runs with distinct realizations of ω, there may be different numbers of service completions encountered at nodes $i \neq n$, and then considered in evaluation of Ψ. This normally involves an insignificant error in estimating the gradient, which becomes inessential as K increases.

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