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Computation of Phase Detector Characteristics in Synchronization Systems

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1 Nonlinear analysis of PLL

For the analysis of PLL it is necessary to consider the models of PLL in signal space and phase space [Viterbi(1966), Gardner(1966), Shakhgil'dyan & Lyakhovkin(1972)]). In this case for constructing of an adequate nonlinear mathematical model of PLL in phase space it is necessary to find the characteristic of phase detector (PD — a nonlinear element, used in PLL for matching tunable signals). The inputs of PD are high-frequency signals of reference and tunable oscillators and the output contains a low-frequency error correction signal, corresponding to a phase difference of input signals. For the suppression of high-frequency component of the output of PD (if such component exists) the low-pass filter can be applied. The characteristic of PD is the dependence of the signal at the output of PD (in the phase space) on the phase difference of signals at the input of PD. This characteristic depends on the realization of PD and the types of signals at the input. Characteristics of the phase detector for standard types of signal are well-known to engineers [Viterbi(1966), Shakhgil'dyan & Lyakhovkin(1972), Abramovitch(2002)].

Further following [Leonov(2008)], on the examples of classical PLL with a phase detector in the form of multiplier, we consider the general principles of computation of phase detector characteristics for different types of signals based on a rigorous mathematical analysis of high-frequency oscillations [Leonov & Seledghi(2005), Kuznetsov *et al.*(2008)Kuznetsov, Leonov & Seledzhi, Kuznetsov *et al.*(2009a)Kuznetsov, Leonov & Seledzhi, Kuznetsov *et al.*(2009b)Kuznetsov, Leonov, Seledzhi & Neittaanmäki, Leonov *et al.*(2010)Leonov, Seledzhi, Kuznetsov & Neittaanmaki].

2 Description of the classical PLL in the signal space

Consider the classical PLL on the level of electronic realization (Fig. 1)

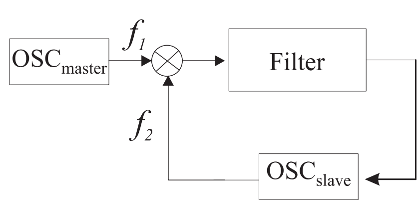

Figure 1: Block diagram of PLL on the level of electronic realization.

Here $\mathrm{OSC}_{\mathrm{master}}$ is a master oscillator, $\mathrm{OSC}_{\mathrm{slave}}$ is a slave (tunable voltage-controlled) oscillator, which generates oscillations $f_j(t)$ with high-frequencies $\omega_j(t)$. Block $\otimes$ is a multiplier (used as PD) of oscillations

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[2] PDF slides http://www.math.spbu.ru/user/nk/PDF/Nonlinear-analysis-of-Phase-locked-loop-PLL.pdf

of $f_1(t)$ and $f_2(t)$ and the signal $f_1(t)f_2(t)$ is its output. The relation between the input $\xi(t)$ and the output $\sigma(t)$ of linear filter has the form

$$\sigma(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau)\xi(\tau)\,\mathrm{d}\tau. \quad (1)$$

Here $\gamma(t)$ is an impulse transient function of filter, $\alpha_0(t)$ is an exponentially damped function, depending on the initial data of filter at the moment $t = 0$.

2.1 High-frequency property of signals

In the simplest ideal case, when two high-frequency sinusoidal signals

$$\begin{aligned} &f_1 = \sin(\omega_1), f_2 = \cos(\omega_2), \\ &f_1 f_2 = [\sin(\omega_1+\omega_2) + \sin(\omega_1-\omega_2)]/2, \end{aligned}$$

are considered, standard engineering assumption is that the low-pass filter has to remove the upper sideband with frequency from the input but leaves the lower sideband without change. Thus it is assumed that the filter output is $\frac{1}{2}\sin(\omega_1-\omega_2)$.

But how to prove this assumption in the general case of signals?

Here to avoid the above non-rigorous arguments we consider mathematical properties of high-frequency oscillations.

Suppose that $f^1(\theta), f^2(\theta)$ — bounded 2π-periodic piecewise differentiable functions. Then, Fourier series, corresponding to the functions $f^1(\theta)$ and $f^2(\theta)$, converge to the function values at points of continuity and to half the sum of left and right limits at the discontinuity points.

Further, since in $L^1_{[-\pi,\pi]}$ functions that differ in a finite number of points are equivalent, we consider $f^1(\theta)$ and $f^2(\theta)$ with the above values at the points of discontinuity, i.e.,

$$\begin{aligned} f^1(\theta) &= c^1 + \sum_{i=1}^{\infty}\big(a_i^1\sin(i\theta) + b_i^1\cos(i\theta)\big), \\ f^2(\theta) &= c^2 + \sum_{i=1}^{\infty}\big(a_i^2\sin(i\theta) + b_i^2\cos(i\theta)\big), \end{aligned} \quad (2)$$

$$\begin{aligned} a_i^p &= \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(x)\sin(ix)dx, \quad b_i^p = \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(x)\cos(ix)dx, \\ c^p &= \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(x)dx, \quad p\in\{1,2\}, \quad i\in\mathbb{N}. \end{aligned}$$

According to the properties of Fourier coefficients for piecewise differentiable functions the following estimates

$$a_i^p = O\left(\frac{1}{i}\right), b_i^p = O\left(\frac{1}{i}\right), \; p\in\{1,2\}. \quad (3)$$

are valid.

A high-frequency property of signals can be reformulated as the following condition. Consider a large fixed time interval $[0, T]$, which can be partitioned into small intervals of the form

$$[\tau, \tau+\delta], \; \tau\in[0,T],$$

where the following relations

$$\begin{aligned}&|\gamma(t)-\gamma(\tau)| \le C\delta, \;\; |\omega_j(t)-\omega_j(\tau)| \le C\delta, \\ &\forall\, t \in [\tau, \tau+\delta], \quad \forall\, \tau \in [0,T],\end{aligned} \tag{4}$$

$$|\omega_1(\tau)-\omega_2(\tau)| \le C_1, \;\; \forall\, \tau \in [0,T], \tag{5}$$

$$\omega_j(t) \ge R, \;\; \forall\, t \in [0,T] \tag{6}$$

are satisfied.

We shall assume that δ is small enough relative to the fixed numbers T, C, C_1 and R is sufficiently large relative to the number δ : $R^{-1} = O(\delta^2)$.

The latter means that on small intervals $[\tau, \tau+\delta]$ the functions $\gamma(t)$ and $\omega_j(t)$ are almost constant and the functions $f_j(t)$ on them are rapidly oscillating. Obviously, such a condition occurs for high-frequency oscillations.

3 Phase-detector characteristic computation

Consider two block diagrams On Fig. 2 $\theta_j(t) = \omega_j(t)t + \psi_j$ are phases of the oscillations $f_j(t)$, PD is a nonlinear

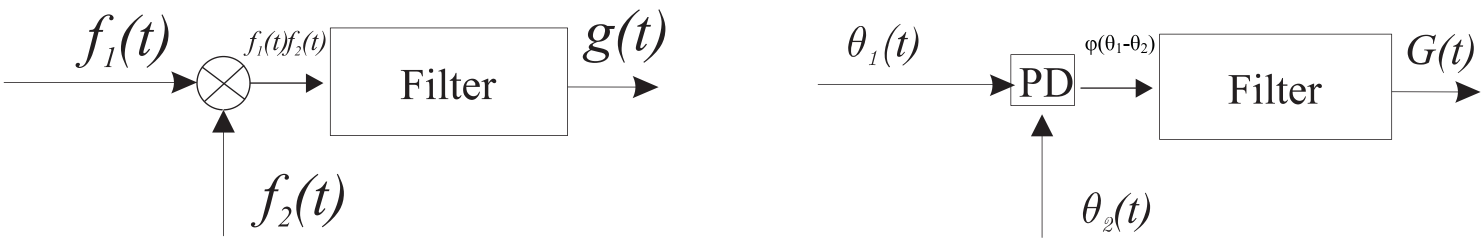

Figure 2: Multiplier and filter. Phase detector and filter.

block with the characteristic $\varphi(\theta)$. The phases $\theta_j(t)$ are the inputs of PD block and the output is the function $\varphi(\theta_1(t) - \theta_2(t))$. The shape of the phase detector characteristic is based on the shape of input signals. The signals $f_1(t)f_2(t)$ and $\varphi(\theta_1(t) - \theta_2(t))$ are inputs of the same filters with the same impulse transient function $\gamma(t)$. The filter outputs are the functions $g(t)$ and $G(t)$, respectively.

Then, using the approaches outlined in [Leonov(2008)] the following result can be proved.

Theorem 1 *If conditions (4)–(6) of high-frequency of signals are satisfied and*

$$\begin{aligned}&\varphi(\theta) = c^1 c^2 + \\ &+\frac{1}{2}\sum_{l=1}^{\infty}\Big((a_l^1 a_l^2 + b_l^1 b_l^2)\cos(l\theta) + (a_l^1 b_l^2 - b_l^1 a_l^2)\sin(l\theta)\Big).\end{aligned} \tag{7}$$

then for the same initial states of filter the following relation

$$|G(t) - g(t)| \le C_2\delta, \quad \forall t \in [0,T] \tag{8}$$

is valid.

Proof.

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Corollary 1. Two sign signals

$$f_k(t) = A_k \text{sign } \sin(\theta_k(t)) =$$
$$= \tfrac{4A_k}{\pi} \sum_{n=0}^{\infty} \tfrac{1}{2n+1} \sin((2n+1)(\omega_k(t)t + \psi_k)), \quad k = 1, 2$$

$$\varphi(\theta_1 - \theta_2) = \tfrac{8A_1A_2}{\pi^2} \sum_{n=0}^{\infty} \tfrac{1}{(2n+1)^2} \cos(\theta_1 - \theta_2)$$

Thus, here phase detector characteristic $\phi(\theta)$ corresponds to 2π-periodic function

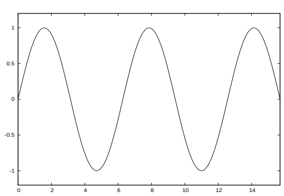
$$A_1A_2\left(1 - \frac{2|\theta|}{\pi}\right), \quad \text{for} \quad \theta \in (-\pi, \pi]. \tag{9}$$

Corollary 2. Sin signal and sign signal

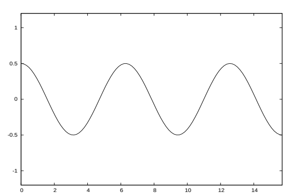
$$f_1(t) = A_1 \sin(\theta_1(t))$$
$$f_2(t) = A_2 \text{sign } \sin(\theta_2(t)) =$$
$$= \tfrac{4A_2}{\pi} \sum_{n=0}^{\infty} \tfrac{1}{2n+1} \sin((2n+1)(\omega_2(t)t + \psi_2))$$
$$\varphi(\theta_1 - \theta_2) = \tfrac{2A_1A_2}{\pi} \cos(\theta_1 - \theta_2)$$

4 Example of phase detector characteristics computation

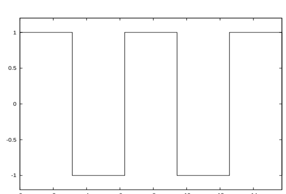
$f^{1,2}(\theta) = \sin(\theta)$



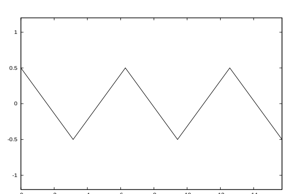
$\varphi(\theta) = \frac{1}{2}\cos(\theta)$

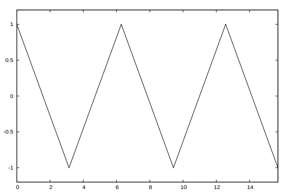
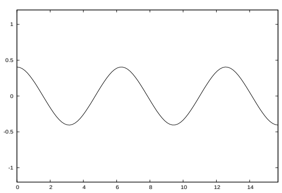
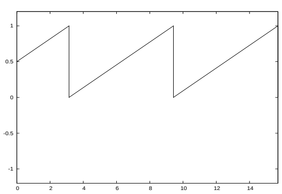
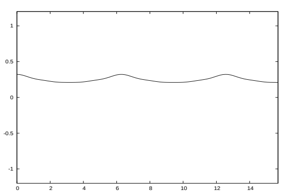


$f^{1,2}(\theta) = \text{sign}\,(\sin(\theta))$



$\varphi(\theta) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\big((2n-1)\theta\big)$



$f^{1,2}(\theta) = \frac{8}{\pi^2} \sum\limits_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\big((2n-1)\theta\big)$ 	$\varphi(\theta) = \frac{32}{\pi^4} \sum\limits_{n=1}^{\infty} \frac{1}{(2n-1)^4} \cos\big((2n-1)\theta\big)$ 
$f^{1,2}(\theta) = \frac{1}{2} + \frac{1}{\pi} \sum\limits_{n=1}^{\infty} \frac{1}{n} \sin\big(n\theta\big)$ 	$\varphi(\theta) = \frac{1}{4} + \frac{1}{2\pi^2} \sum\limits_{n=1}^{\infty} \frac{1}{n^2} \cos\big(n\theta\big)$ 

5 PLL equations in phase-frequency space

From Theorem 1 it follows that block-scheme of PLL in signal space (Fig. 1) can be asymptotically changed (for high-frequency generators) to a block-scheme on the level of frequency and phase relations (Fig. 3).

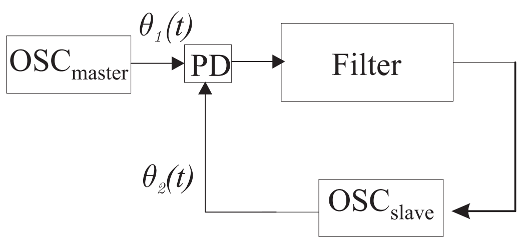


Figure 3: Phase-locked loop with phase detector

Here PD is a phase detector with corresponding characteristics. Thus, here on basis of asymptotical analysis of high-frequency pulse oscillations a characteristics of phase detector can be computed.

Let us make a remark necessary for derivation of differential equations of PLL.

Consider a quantity

$$\dot{\theta}_j(t) = \omega_j(t) + \dot{\omega}_j(t)t.$$

For the well-synthesized PLL such that it possesses the property of global stability, we have exponential damping of the quantity $\dot{\omega}_j(t)$:

$$|\dot{\omega}_j(t)| \leq Ce^{-\alpha t}.$$

Here C and α are certain positive numbers being independent of t. Therefore, the quantity $\dot{\omega}_j(t)t$ is, as a rule, sufficiently small with respect to the number R (see conditions (4)– (6)). From the above we can conclude that the following approximate relation $\dot{\theta}_j(t) \approx \omega_j(t)$ is valid. In deriving the differential equations of this PLL, we make use of a block diagram in Fig. 3 and exact equality

$$\dot{\theta}_j(t) = \omega_j(t). \tag{10}$$

Note that, by assumption, the control law of tunable oscillators is linear:

$$\omega_2(t) = \omega_2(0) + LG(t). \tag{11}$$

Here $\omega_2(0)$ is the initial frequency of tunable oscillator, L is a certain number, and $G(t)$ is a control signal, which is a filter output (Fig. 2). Thus, the equation of PLL is as follows

$$\dot{\theta}_2(t) = \omega_2(0) + L\Big(\alpha_0(t) + \int\limits_0^t \gamma(t-\tau)\varphi\big(\theta_1(\tau) - \theta_2(\tau)\big)\mathrm{d}\tau\Big).$$

Assuming that the master oscillator is such that $\omega_1(t) \equiv \omega_1(0)$, we obtain the following relations for PLL

$$\begin{aligned}&\big(\theta_1(t)-\theta_2(t)\big)' + L\Big(\alpha_0(t) + \int\limits_0^t \gamma(t-\tau)\varphi\big(\theta_1(\tau)-\theta_2(\tau)\big)\mathrm{d}\tau\Big)\\ &= \omega_1(0) - \omega_2(0).\end{aligned} \tag{12}$$

This is an equation of standard PLL.

Characteristic $\varphi(\theta)$, computed in examples 1 and 2, tends to zero if $\theta = (\theta_1 - \theta_2)$ tends to $\pi/2$, so one can proceed to stability analysis [Leonov(2006), Kuznetsov(2008), Leonov *et al.*(2009)Leonov, Kuznetsov & Seledzhi] of differential (or difference) equations depend on the misphasing θ.

In the case when the transfer function of the filter $W(p)$ is non-degenerate (its numerator and denominator do not have common roots) equation (12) is equivalent to the following system of differential equations

$$\dot{z} = Az + b\psi(\sigma), \quad \dot{\sigma} = c^* z. \tag{13}$$

Here $\sigma = \theta_1 - \theta_2$, A is a constant $(n \times n)$-matrix, b and c are constant (n)-vectors, and $\psi(\sigma)$ is 2π-periodic function, satisfying the relations:

$$\begin{aligned}&\rho = -aL, \quad W(p) = L^{-1}c^*(A - pI)^{-1}b,\\ &\psi(\sigma) = \varphi(\sigma) - \frac{\omega_1(0) - \omega_2(0)}{LW(0)}.\end{aligned}$$

The discrete phase-locked loops obey similar equations

$$\begin{aligned}&z(t+1) = Az(t) + b\psi(\sigma(t))\\ &\sigma(t+1) = \sigma(t) + c^* z(t),\end{aligned} \tag{14}$$

where $t \in Z$, Z is a set of integers. Equations (13) and (14) describe the so-called standard PLLs [Shakhgil'dyan & Lyakhovkin(1972)].

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