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STRANGE ATTRACTORS AND CLASSICAL STABILITY THEORY

ST. PETERSBURG UNIVERSITY PRESS

Published by

St. Petersburg University Press
11/21 6th Line, St. Petersburg, 199004, Russia

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ISBN 978-5-288-04500-4

Подписано в печать 18.12.2007. Формат 60×90 $^1/_{16}$.
Бумага офсетная. Печать офсетная. Усл. печ. л. 10.
Тираж 500 экз. Заказ №

Типография Издательства СПбГУ.
199061, С.-Петербург, Средний пр., 41

Printed in Russia by St. Petersburg University Press

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Preface

This book might be more accurately titled, “On the advantages of the classical theory of stability of motion in the study of strange attractors.”

In almost any solid survey or book on chaotic dynamics, one encounters notions from classical stability theory such as Lyapunov exponent and characteristic exponent. But the effect of sign inversion in the characteristic exponent during linearization is seldom mentioned. This effect was discovered by Oscar Perron [110], an outstanding German mathematician, in 1930. The present book sets forth Perron’s results and their further development (see [79, 80, 81]). It is shown that Perron effects may occur on the boundaries of a flow of solutions that is stable by the first approximation. Inside a flow, stability is completely determined by the negativeness of the characteristic exponents of linearized systems.

It is often said that the defining property of strange attractors is the sensitivity of their trajectories with respect to the initial data. But how is this property connected with the classical notions of instability? Many researchers suppose that sensitivity with respect to initial data is adequate for Lyapunov instability [3, 34]. But this holds only for discrete dynamical systems. For continuous systems, it was necessary to remember the almost forgotten notion of Zhukovsky instability. Nikolai Egorovich Zhukovsky, one of the founders of modern aerodynamics and a prominent Russian scientist, introduced his notion of stability of motion in 1882 (see [142, 143]) — ten years before the publication of Lyapunov’s investigations (1892 [94]). The notion of Zhukovsky instability is adequate to the sensitivity of trajectories with respect to the initial data for continuous dynamical systems. In this book we consider the notions of instability according to Zhukovsky [142], Poincaré [117], and Lyapunov [94], along with their adequacy to the sensitivity of trajectories on strange attractors with respect to the initial data.

In order to investigate Zhukovsky stability, a new research tool — a moving Poincaré section — is introduced. With the help of this tool, extensions of the widely-known theorems of Andronov—Witt, Demidovic, Borg, and Poincaré are carried out.

At the present time, the problem of justifying nonstationary linearizations for complicated, nonperiodic motions on strange attractors bears a striking resemblance to the situation that occurred 120 years ago.

J.C. Maxwell (1868 [97, 98]) and I.A. Vyshnegradskii (1876 [98, 135]), the founders of automatic control theory, courageously used linearization in a neighborhood of stationary motions, leaving the justification of such linearization to H. Poincaré (1881–1886 [117]) and A.M. Lyapunov (1892 [94]). Now many specialists in chaotic dynamics believe that the positiveness of the leading characteristic exponent of a linear system of the first approximation implies the instability of solutions of the original system (e.g., [12, 47, 52, 103, 105, 118]). Moreover, there is a great number of computer experiments in which various numerical methods for calculating characteristic exponents and Lyapunov exponents of linear systems of the first approximation are used. As a rule, authors largely ignore the justification of the linearization procedure and use the numerical values of exponents thus obtained to construct various numerical characteristics of attractors of the original nonlinear systems (Lyapunov dimensions, metric entropies, and so on). Sometimes computer experiments serve as arguments for partial justification of the linearization procedure. For example, computer experiments in [119] and [105] show the coincidence of the Lyapunov and Hausdorff dimensions of the attractors of Henon, Kaplan—Yorke, and Zaslavskii. But for B-attractors of Henon and Lorenz, such coincidence does not hold (see [75, 78]).

So linearizations along trajectories on strange attractors require justification. This problem gives great impetus to the development of the nonstationary theory of instability by the first approximation. The present book describes the contemporary state of the art of the problem of justifying nonstationary linearizations.

The method of Lyapunov functions — Lyapunov's so-called direct method — is an efficient research device in classical stability theory. It turns out that even in the dimension theory of strange attractors one can progress by developing analogs of this method. This interesting line of investigation is also discussed in the present book.

When the parameters of a dynamical system are varied, the structure of its minimal global attractor can change as well. Such changes are the subject of bifurcation theory. Here we describe one of these, namely the homoclinic bifurcation.

The first important results concerning homoclinic bifurcation in dissipative systems was obtained in 1933 by the outstanding Italian mathematician Francesco Tricomi [133]. Here we give Tricomi's results along with similar theorems for the Lorenz system.

For the Lorenz system, necessary and sufficient conditions for the existence of homoclinic trajectories are obtained.

Lyapunov's direct method, when combined with the efficient algebraic apparatus of modern control theory, is highly efficient for the study of chaos in discrete dynamical systems. It is usual for discrete systems to have no pictorial representations in phase space, which are characteristic of continuous systems with smooth trajectories that fill this space. However, this drawback can be partially overcome if, instead of observing changes in the distance between two solutions in a discrete time, one considers the line segment joining the initial data at the initial moment and its iterations — a sequence of continuous curves. A change in the length of these curves provides information on the stability or instability and other properties of solutions of discrete systems. The consideration of the lengths of such continuous curves may serve as a basis for definitions of stability and instability, which is of particular importance for noninjective maps determining respective discrete systems (at present, it is precisely such maps that are widely-known generators of chaos). The matter is that in some cases two trajectories may "stick together" for a finite number of iterations. But the iterated segment that joins the initial data may have an increasing length even with regard to such sticking together. It turned out that in a number of cases, the mechanism of stretching the lengths is rather strong, and it can be revealed and estimated with the help of the whole rich arsenal of Lyapunov's direct method, which can be combined with an efficient algebraic apparatus: the Kalman—Szegö and Shepelyavyi lemmas. These yield estimates of the stretching of the lengths of iterated curves in terms of frequency inequalities efficiently verifiable for particular dynamical systems. The methods mentioned above are described in detail in the present book.

For the last fifty years another interesting direction in stability theory has been developed — the method of *a priori* integral estimates, which is based on the application of Fourier transforms as unitary operators in certain functional spaces. In the book it is shown that this method may provide informative estimates for the oscillation periods of discrete dynamical systems. Its application to different one-dimensional maps yields estimates of lack of oscillations with period three. There is a widely-known thesis stating that for one-dimensional discrete dynamical systems, "period three implies chaos". Therefore, nontrivial estimates of periods of oscillations enable one to draw quantitative inferences about scenarios of transition to chaotic dynamics.

The author has tried to make the presentation simple and self-contained. Precise definitions are given for all notions used, and the basic mathematical facts are proved in considerable detail. Many classical results are supplied with new and simpler proofs. The main content of the book is the development of the surveys [78, 80].

The book is primarily addressed to specialists in dynamical systems or applied differential equations. It shows how contemporary problems of chaos theory indicate natural and simple ways for modernizing the classical methods of stability theory. Since the book requires only a standard background in algebra, calculus, and differential equations, it may be useful for both undergraduate and graduate students in mathematics.

My sincere thanks are due to Elmira Gurmuzova for her translating the manuscript from Russian into English and Ludmila Vinogradova for typesetting and copyediting.

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Chapter 1

Definitions of Attractors

The attractor of a dynamical system is an attractive closed invariant set in its phase space.

Consider the dynamical systems generated by the differential equations

$$\frac{dx}{dt} = f(x), \quad t \in \mathbb{R}^1, \ x \in \mathbb{R}^n, \tag{1.1}$$

and by the difference equations

$$x(t+1) = f\left(x(t)\right), \quad t \in \mathbb{Z}, \ x \in \mathbb{R}^n. \tag{1.2}$$

Here $\mathbb{R}^n$ is a Euclidean space, $\mathbb{Z}$ is the set of integers, and $f(x)$ is a vector-function: $\mathbb{R}^n \to \mathbb{R}^n$.

Definition 1.1. We say that (1.1) or (1.2) *generates a dynamical system* if for any initial data $x_0 \in \mathbb{R}^n$ the trajectory $x(t, x_0)$ is uniquely determined for $t \in [0, +\infty)$. Here $x(0, x_0) = x_0$.

It is well known that the solutions of dynamical system (1.1) satisfy the semigroup property

$$x(t+s, x_0) = x\left(t, x(s, x_0)\right) \tag{1.3}$$

for all $t \geq 0$, $s \geq 0$.

For equation (1.1) on $[0, +\infty)$ there are many existence and uniqueness theorems [21, 33, 38, 138] that can be used for determining the corresponding dynamical system with the phase space $\mathbb{R}^n$. The partial differential equations, generating dynamical systems with different infinite-dimensional phase spaces, can be found in [8, 9, 25, 51, 132]. The classical results of the theory of dynamical systems with a metric phase space are given in [106].

For (1.2) it is readily shown that in all cases the trajectory, defined for all $t = 0, 1, 2, \ldots$, satisfying (1.3), and having initial condition x_0, is unique. Thus (1.2) always generates a dynamical system with phase space $\mathbb{R}^n$.

A dynamical system generated by (1.1) is called *continuous*. Equation (1.2) generates a *discrete dynamical system*.

The definitions of attractors are, as a rule, due to [19, 25, 51, 78].

Definition 1.2. We say that K *is invariant* if $x(t, K) = K$, $\forall t \geq 0$. Here

$$x(t, K) = \{x(t, x_0) \mid x_0 \in K\}.$$

Definition 1.3. We say that the invariant set K is *locally attractive* if for a certain ε-neighborhood $K(\varepsilon)$ of K the relation

$$\lim_{t \to +\infty} \rho\left(K, x(t, x_0)\right) = 0, \quad \forall\, x_0 \in K(\varepsilon)$$

is satisfied. Here $\rho(K, x)$ is the distance from the point x to the set K, defined by

$$\rho(K, x) = \inf_{z \in K} |z - x|.$$

Recall that $|\cdot|$ is a Euclidean norm in $\mathbb{R}^n$, and $K(\varepsilon)$ is the set of points x such that $\rho(K, x) < \varepsilon$.

Definition 1.4. We say that the invariant set K is *globally attractive* if

$$\lim_{t \to +\infty} \rho\big(K, x(t, x_0)\big) = 0, \quad \forall\, x_0 \in \mathbb{R}^n.$$

Definition 1.5. We say that the invariant set K is *uniformly locally attractive* if for a certain ε-neighborhood $K(\varepsilon)$ of it and for any $\delta > 0$ and bounded set B there exists $t(\delta, B) > 0$ such that

$$x\big(t, B \cap K(\varepsilon)\big) \subset K(\delta), \quad \forall t \geq t(\delta, B)$$

Here

$$x\big(t, B \cap K(\varepsilon)\big) = \{x(t, x_0) \mid x_0 \in B \cap K(\varepsilon)\}.$$

Definition 1.6. We say that the invariant set K is *uniformly globally attractive* if for any $\delta > 0$ and bounded set $B \subset \mathbb{R}^n$ there exists $t(\delta, B) > 0$ such that

$$x(t, B) \subset K(\delta), \quad \forall t \geq t(\delta, B).$$

Definition 1.7. We say that the invariant set K is *Lyapunov stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x(t, K(\delta)) \subset K(\varepsilon), \quad \forall t \geq 0.$$

Note that if K consists of one trajectory, then the last definition coincides with the classical definitions of the Lyapunov stability of solution. If such K is locally attractive, then we have asymptotic stability in the sense of Lyapunov.

Definition 1.8. We say that K is

(1) an *attractor* if it is an invariant closed and locally attractive set;
(2) a *global attractor* if it is an invariant closed and globally attractive set;
(3) a *B-attractor* if it is an invariant, closed, and uniformly locally attractive set;
(4) a *global B-attractor* if it is an invariant, closed, and uniformly globally attractive set.

A trivial example of an attractor is the whole phase set $\mathbb{R}^n$ if the trajectories are defined for all $t \geq 0$. This shows that it is sensible to introduce the notion of a *minimal attractor*, namely the minimal invariant set possessing the attractive property.

We give the simplest examples of attractors.

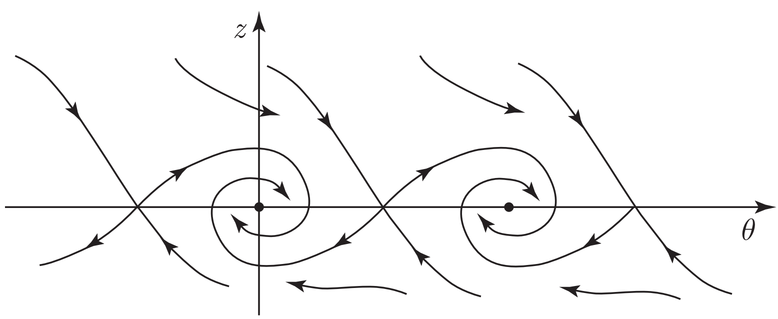

Fig. 1.1 Asymptotic behavior of pendulum equation.

Example 1.1. Consider the equations of pendulum motion:

$$\begin{aligned} \dot{\theta} &= z, \\ \dot{z} &= -\alpha z - \beta \sin\theta, \end{aligned} \tag{1.4}$$

where α and β are positive. The trajectories have a well-known asymptotic behavior (Fig. 1.1).

Any solution of (1.4) tends to a certain equilibrium as $t \to +\infty$. Therefore the minimal global attractor of (1.4) is a stationary set.

Consider now a ball B of small radius centered on the separatrix of the saddle (Fig. 1.2). As $t \to +\infty$ the image $x(t, B)$ of this small ball tends to the set consisting of a saddle equilibrium and of two separatrices, leaving this point and tending to an asymptotically stable equilibrium (Fig. 1.2) as $t \to +\infty$.

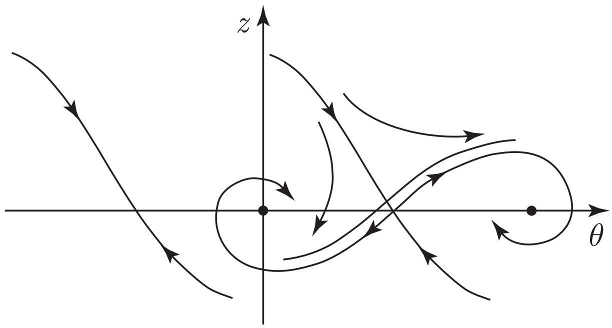

Fig. 1.2 Image of ball centered on the separatrix.

Thus, a global minimal B-attractor is a union of a stationary set and the separatrices, leaving the saddle points (unstable manifolds of the saddle points) (Fig. 1.3).

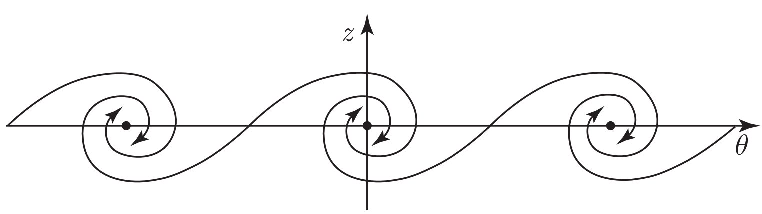

Fig. 1.3 B-attractor of pendulum equation.

In more general situations the B-attractor involves the unstable manifolds of saddle (hyperbolic) points. This fact is often used for estimating the topological dimension of attractors from below [8, 9]. □

Example 1.2. Consider the van der Pol equations

$$\begin{aligned}\dot{y} &= z,\\ \dot{z} &= -\mu(y^2-1)z - y,\end{aligned} \tag{1.5}$$

where $\mu > 0$. It is well known [21] that these have an unstable equilibrium and a limit cycle to which all trajectories (except the equilibrium) are attracted as $t \to +\infty$ (Fig. 1.4).

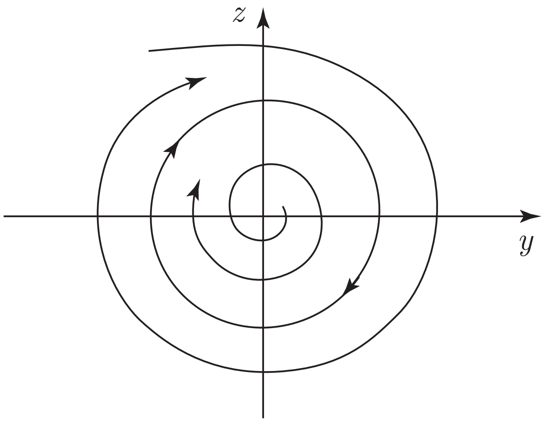

Fig. 1.4 Global minimal attractor of van der Pol equation.

Here a global minimal attractor is a set consisting of a stationary point and a limit cycle. The global minimal B-attractor is a set bounded by the limit cycle (Fig. 1.5). □

Example 1.3. Consider the difference equation (1.2) with $n = 1$ and with a piecewise-linear function $f(x)$:

$$f(x) = \begin{cases} 3x, & x \in [0, 1/2],\\ 3(1-x), & x \in [1/2, 1],\\ 0, & x \notin [0,1]. \end{cases} \tag{1.6}$$

The segments $[0, 1/3]$ and $[2/3, 1]$ are both mapped into $[0, 1]$, while $(1/3, 2/3)$ is mapped outside $[0, 1]$. Hence

$$f(f((1/3, 2/3))) = 0.$$

It is clear that under the mapping $f(f(\cdot))$ the segments $[0, 1/9]$, $[2/9, 3/9]$, $[6/9, 7/9]$, and $[8/9, 1]$ are all mapped into $[0, 1]$, and the solution $x(2)$ with

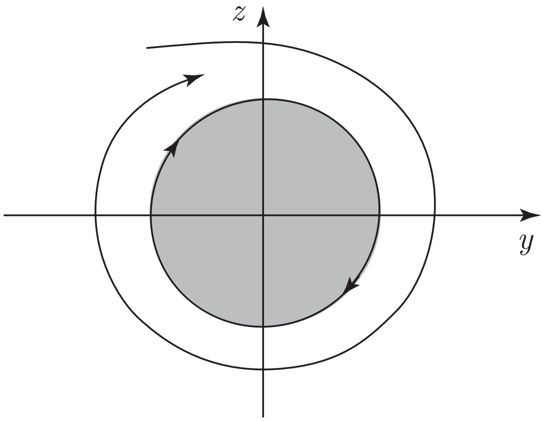

Fig. 1.5 *B*-attractor of van der Pol equation.

the initial data $x(0)$ from the intervals $(1/9, 2/9)$, $(1/3, 2/3)$, and $(7/9, 8/9)$ jumps out of the set $(0, 1)$; therefore, we have $x(3) = 0$.

It is easily seen that at the Nth step under the mapping

$$\underbrace{f(f(\ldots f(\cdot)\ldots))}_{N}$$

into $(0, 1)$ we have the solutions $x(N)$ with the initial data from the sets

$$\left(0, \frac{1}{3^N}\right), \quad \left(\frac{2}{3^N}, \frac{3}{3^N}\right), \quad \left(\frac{6}{3^N}, \frac{7}{3^N}\right), \ldots, \left(\frac{3^N - 1}{3^N}, 1\right)$$

only. The process of ejecting the middles of the remaining intervals is illustrated in Fig. 1.6.

Fig. 1.6 Cantor set.

The part of the segment $[0, 1]$ that remains after infinitely ejecting the open middles of the remaining segments, is called a *Cantor set*. We denote this set by K.

It is clear that K is invariant: $f(K) = K$. From this and the fact that for $x(0) \in \mathbb{R}^1 \setminus K$ the relation

$$\lim_{t \to +\infty} x(t) = 0$$

holds, it follows that K is globally attractive.

Since $\mathbb{R}^1 \setminus K$ is a union of open sets, it is open. Therefore K is closed, and is a global attractor of the system (1.2), (1.6).

Note that for any natural N the point $x_0 \in [0, 1]$ can be found such that $x(N, x_0) = 1/2$. Then $x(N + 1, x_0) = 3/2$. This means that K is not a uniformly globally attractive set. So K is not a global B-attractor. □

The above examples show that we have a substantial distinction between the definitions of attractor and B-attractor. Example 1.3 shows that simple dynamical systems can have attractors of complex structure.

We remark that the natural generalization of the notion of attractor is to weaker requirements of attraction: on sets of positive Lebesgue measure, almost everywhere, and so on [6, 7, 36, 99, 100, 107]. As an illustration of such an approach we give a definition of weak attractor [7, 36].

Definition 1.9. We say that K is a *weak attractor* if K is an invariant closed set for which there exists a set of positive Lebesgue measure $U \subset \mathbb{R}^n$ satisfying the following relation:

$$\lim_{t \to +\infty} \rho(K, x(t, x_0)) = 0, \quad \forall x_0 \in U.$$

Note that for each concrete system it is necessary to detail the set U.

Example 1.4. Consider the Lotka—Volterra system in the degenerate case [88]

$$\begin{aligned} \dot{n}_1 &= \lambda_1 n_1 - \beta n_1 n_2, \\ \dot{n}_2 &= -\lambda_2 n_2. \end{aligned} \tag{1.7}$$

Here n_1 represents the prey population, n_2 the predator population, and λ_j, β are positive parameters. Since the $n_j(t)$ are nonnegative, we have

$$U = \{n_1 \geq 0, \quad n_2 \geq 0\}$$

and

$$K = \{n_1 \geq 0, \quad n_2 = 0\}.$$

□

It turns out that in some important applied problems it is sufficient to consider the special sets U of null Lebesgue measure.

Example 1.5 (limit load of synchronous machine). Consider the equation of a synchronous electrical machine in the simplest idealization [11, 88, 133, 138, 140]

$$\begin{aligned}\dot{\theta} &= z \\ \dot{z} &= -\alpha z - \sin\theta + \gamma.\end{aligned} \tag{1.8}$$

Here $\alpha > 0$ corresponds to the moment of force created by the damping windings of the machine [88], and $0 < \gamma < 1$ corresponds to the load of the machine. If the machine operates in generator mode, then γ corresponds to the circuit load, which is due to the electrical customers. If the machine is used as a motor in the driver of hot rolling mill, then γ is a load, corresponding to the motion of a red-hot bar along the lower rolls only (the process of rolling is lacking, Fig. 1.7).

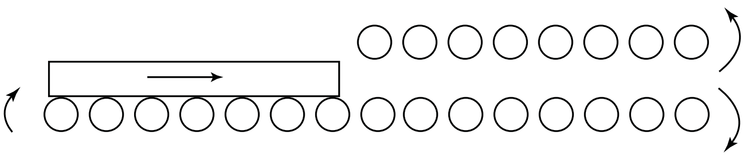

Fig. 1.7 The process of rolling is lacking.

Consider operation in a stationary synchronous mode:

$$\dot{\theta}(t) = z(t) \equiv 0, \quad \theta(t) \equiv \theta_0,$$

where $\theta_0 \in [0, \pi/2]$, $\sin\theta_0 = \gamma$.

A basic issue in the design of synchronous machines is stability with respect to discontinuous loads [88, 138]. Consider a discontinuous load-on at time $\tau > 0$. At this instant a new powerful customer has been connected to a generator. While operating the motor of hot rolling mill, at time τ the red-hot bar enters the space between the lower and upper rolls and the process of the rolling of red-hot bar starts (Fig. 1.8).

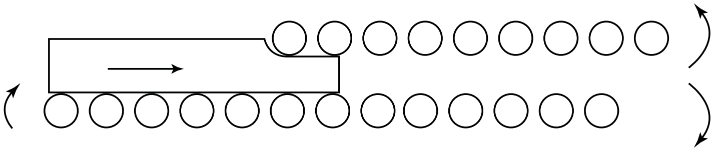

Fig. 1.8 The process of rolling.

Thus, on the segment $[0, \tau]$ we have equations (1.8) and their solutions $\theta(t) = \theta_0$, $z(t) = 0$. For $t \geq \tau$, after the load-on $\Gamma \in (\gamma, 1)$ we have the equations

$$\begin{aligned} \dot{\theta} &= z, \\ \dot{z} &= -\alpha z - \sin\theta + \Gamma, \end{aligned} \tag{1.9}$$

with the initial data $\theta(\tau) = \theta_0$, $z(\tau) = 0$.

It is clear that the solution of (1.9) with the initial data $\theta(\tau) = \theta_0$, $z(\tau) = 0$ is already not an equilibrium. For the synchronous machine to carry on the operation it is necessary that the solution of system (1.9) $\theta(t)$, $z(t)$ with the initial data $\theta(\tau) = \theta_0$, $z(\tau) = 0$ find itself in the domain of attraction of a new stable equilibrium of system (1.9):

$$\lim_{t\to+\infty} \theta(t) = \theta_1, \quad \lim_{t\to+\infty} z(t) = 0. \tag{1.10}$$

Here $\theta_1 \in [0, \pi/2]$, $\sin\theta_1 = \Gamma$. Thus, in this problem we have

$$\begin{aligned} U &= \{\theta \in [\theta_0, \theta_1], \quad z = 0\}, \\ K &= \{\theta = \theta_1, \quad z = 0\}. \end{aligned}$$

(Fig. 1.9).

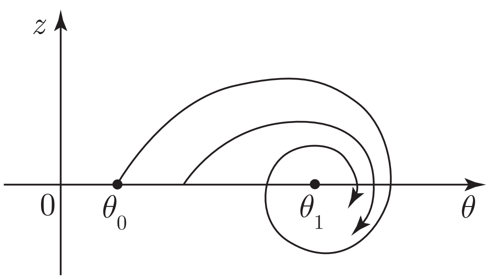

Fig. 1.9 Trajectories after the load-on.

The estimate of θ_0, satisfying (1.10), can be obtained by using the Lyapunov functions

$$V(\theta, z) = \frac{1}{2}z^2 + \int_{\theta_0}^{\theta} (\sin u - \Gamma)\, du.$$

Then it becomes possible to formulate the following

Proposition 1.1. *If*

$$\int_{\theta_0}^{\theta_2} (\sin u - \Gamma)\, du > 0,$$

where $\theta_2 \in [\pi/2, \pi]$, $\sin \theta_2 = \Gamma$, *then relations (1.10) hold.*

Example 1.5 shows that further weakening of the notion of attractor, except for positiveness of the Lebesgue measure for U, is possible. □

Chapter 2

Strange Attractors and the Classical Definitions of Instability

One of the basic characteristics of a strange attractor is the sensitivity of its trajectories to the initial data [5, 12, 13, 28, 29, 35, 37, 90, 102, 108, 120, 121, 123, 128, 129, 131, 136, 137, 141].

We consider the correlation of such "sensitivity" with a classical notion of instability. We recall first the basic definitions of stability.

2.1 Basic Definitions in Classical Stability of Motion

Consider the system

$$\frac{dx}{dt} = F(x,t), \quad t \in \mathbb{R}^1, \quad x \in \mathbb{R}^n, \tag{2.1}$$

where $F(x,t)$ is a continuous vector-function, and

$$x(t+1) = F(x(t),t), \quad t \in \mathbb{Z}, \quad x \in \mathbb{R}^n. \tag{2.2}$$

Denote by $x(t,t_0,x_0)$ the solution of (2.1) or (2.2) with initial data t_0, x_0:

$$x(t_0,t_0,x_0) = x_0.$$

Definition 2.1. The solution $x(t,t_0,x_0)$ is said to be *Lyapunov stable* if for any $\varepsilon > 0$ and $t_0 \geq 0$ there exists $\delta(\varepsilon,t_0)$ such that

(1) all the solutions $x(t,t_0,y_0)$, satisfying the condition

$$|x_0 - y_0| \leq \delta,$$

are defined for $t \geq t_0$,

(2) for these solutions the inequality

$$|x(t,t_0,x_0) - x(t,t_0,y_0)| \le \varepsilon, \quad \forall t \ge t_0$$

is valid.

If $\delta(\varepsilon, t_0)$ is independent of t_0, the Lyapunov stability is called *uniform*.

Definition 2.2. The solution $x(t,t_0,x_0)$ is said to be *asymptotically Lyapunov stable* if it is Lyapunov stable and for any $t_0 \ge 0$ there exists $\Delta(t_0) > 0$ such that the solution $x(t,t_0,y_0)$, satisfying the condition $|x_0 - y_0| \le \Delta$, has the following property:

$$\lim_{t\to+\infty} |x(t,t_0,x_0) - x(t,t_0,y_0)| = 0.$$

Definition 2.3. The solution $x(t,t_0,x_0)$ is said to be *Krasovsky stable* if there exist positive numbers $\delta(t_0)$ and $R(t_0)$ such that for any y_0, satisfying the condition

$$|x_0 - y_0| \le \delta(t_0),$$

the solution $x(t,t_0,y_0)$ is defined for $t \ge t_0$ and satisfies

$$|x(t,t_0,x_0) - x(t,t_0,y_0)| \le R(t_0)|x_0 - y_0|, \quad \forall t \ge t_0.$$

If δ and R are independent of t_0, then Krasovsky stability is called *uniform*.

Definition 2.4. The solution $x(t,t_0,x_0)$ is said to be *exponentially stable* if there exist the positive numbers $\delta(t_0)$, $R(t_0)$, and $\alpha(t_0)$ such that for any y_0, satisfying the condition

$$|x_0 - y_0| \le \delta(t_0),$$

the solution $x(t,t_0,y_0)$ is defined for all $t \ge t_0$ and satisfies

$$|x(t,t_0,x_0) - x(t,t_0,y_0)| \le R(t_0)\exp(-\alpha(t_0)(t-t_0))|x_0 - y_0|, \quad \forall t \ge t_0.$$

If δ, R, and α are independent of t_0, then exponential stability is called *uniform*.

Consider now dynamical systems (1.1) and (1.2). We introduce the following notation:

$$L^+(x_0) = \big\{x(t,x_0) \mid\ t \in [0,+\infty)\big\}.$$

Definition 2.5. The trajectory $x(t, x_0)$ of a dynamical system is said to be *Poincaré stable* (or *orbitally stable*) if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all y_0, satisfying the inequality $|x_0 - y_0| \leq \delta(\varepsilon)$, the relation

$$\rho\big(L^+(x_0),\, x(t, y_0)\big) \leq \varepsilon, \quad \forall\, t \geq 0$$

is satisfied. If, in addition, for a certain number δ_0 and for all y_0, satisfying the inequality $|x_0 - y_0| \leq \delta_0$, the relation

$$\lim_{t \to +\infty} \rho\big(L^+(x_0),\, x(t, y_0)\big) = 0$$

holds, then the trajectory $x(t, x_0)$ is said to be *asymptotically Poincaré stable* (or *asymptotically orbitally stable*).

Note that for continuous dynamical systems we have $t \in \mathbb{R}^1$, and for discrete dynamical systems $t \in \mathbb{Z}$.

We now introduce the definition of Zhukovsky stability for continuous dynamical systems. For this purpose we must consider the following set of homeomorphisms:

$$\text{Hom} = \{\tau(\cdot) \mid \tau : [0, +\infty) \to [0, +\infty), \tau(0) = 0\}.$$

The functions $\tau(t)$ from the set Hom play the role of the reparametrization of time for the trajectories of system (1.1).

Definition 2.6 [60, 80, 84, 142, 143]. The trajectory $x(t, x_0)$ of system (1.1) is said to be *Zhukovsky stable* if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any vector y_0, satisfying the inequality $|x_0 - y_0| \leq \delta(\varepsilon)$, the function $\tau(\cdot) \in \text{Hom}$ can be found such that the inequality

$$|x(t, x_0) - x(\tau(t), y_0)| \leq \varepsilon, \quad \forall t \geq 0$$

is valid. If, in addition, for a certain number $\delta_0 > 0$ and any y_0 from the ball $\{y |\, |x_0 - y| \leq \delta_0\}$ the function $\tau(\cdot) \in \text{Hom}$ can be found such that the relation

$$\lim_{t \to +\infty} |x(t, x_0) - x(\tau(t), y_0)| = 0$$

holds, then the trajectory $x(t, x_0)$ is *asymptotically stable in the sense of Zhukovsky*.

This means that Zhukovsky stability is Lyapunov stability for the suitable reparametrization of each of the perturbed trajectories.

Recall that, by definition, Lyapunov instability is the negation of Lyapunov stability. Analogous statements hold for Krasovsky, Poincaré, and Zhukovsky instability.

2.2 Relationships Between the Basic Stability Notions

The following obvious assertions can be formulated.

Proposition 2.1. *For continuous dynamical systems, Lyapunov stability implies Zhukovsky stability, and Zhukovsky stability implies Poincaré stability.*

Proposition 2.2. *For discrete dynamical systems, Lyapunov stability implies Poincaré stability.*

Proposition 2.3. *For equilibria, all the above definitions due to Lyapunov, Zhukovsky, and Poincaré are equivalent.*

Proposition 2.4. *For periodic trajectories of discrete dynamical systems with continuous $f(x)$, the definitions of Lyapunov and Poincaré stability are equivalent.*

Proof. Let the periodic trajectory of a discrete system be Poincaré stable. Choose the ε-neighborhood $U(\varepsilon, L^+(x_0))$ of the periodic trajectory $L^+(x_0)$ such that

$$U(\varepsilon, u) \cap U(\varepsilon, z) = \emptyset \tag{2.3}$$

for any points $u \in L^+(x_0)$, $z \in L^+(x_0)$, $u \neq z$. Here $\emptyset$ is the empty set.

Consider y_0 from the δ-neighborhood of x_0, and connect x_0 and y_0 by the segment

$$x_0 + \gamma(y_0 - x_0), \quad \gamma \in [0, 1].$$

From the continuity of $f(x)$, the Poincaré stability, and relations (2.3), it follows that

$$x(t, x_0 + \gamma(y_0 - x_0)) \in U(\varepsilon, x(t, x_0)), \quad \forall t \geq 0, \ \forall \gamma \in [0, 1]. \tag{2.4}$$

For $\gamma = 1$, inclusion (2.4) means the Lyapunov stability of the trajectory $x(t, x_0)$. This implies the equivalence of the definitions of Poincaré and Lyapunov stability for the periodic trajectories of discrete systems. □

In a similar way, the equivalence of the asymptotic Poincaré and asymptotic Lyapunov stability can be proved for the periodic trajectories of discrete systems.

Proposition 2.5. *For the periodic trajectories of continuous dynamical systems with differentiable $f(x)$, the definitions of Poincaré and Zhukovsky stability are equivalent.*

Proof. Let a periodic trajectory of a continuous system be Poincaré stable. Choose the ε-neighborhood $U(\varepsilon, L^+(x_0))$ of the periodic trajectory $L^+(x_0)$ in such a way that

$$\begin{aligned} &U(\varepsilon, L^+(x_0), t_1) \cap U(\varepsilon, L^+(x_0), t_2) = \emptyset, \\ &\forall t_1 \neq t_2, \quad t_1 \in [0, T), \quad t_2 \in [0, T), \end{aligned} \tag{2.5}$$

where

$$U(\varepsilon, L^+(x_0), t) = \{y \mid |y - x(t, x_0)| \leq \varepsilon, \quad (y - x(t, x_0))^* f(x(t, x_0)) = 0\}.$$

Choose $\delta = \delta(\varepsilon)$ so that the inequality $|x_0 - y_0| \leq \delta$ implies the relation

$$x(t, y_0) \in U(\varepsilon, L^+(x_0)), \quad \forall t \geq 0.$$

Without loss of generality we can assume that $y_0 \in U(\varepsilon, L^+(x_0), 0)$. Then (2.5) implies that the reparametrization $\tau(t)$ of the trajectory $x(t, y_0)$ can be defined as

$$x(\tau(t), y_0) \in U(\varepsilon, L^+(x_0), t).$$

It is clear that $\tau(\cdot) \in \mathrm{Hom}$ and that

$$|x(\tau(t), y_0) - x(t, x_0)| \leq \varepsilon, \quad t \geq t_0.$$

The latter implies Zhukovsky stability. □

The equivalence of the asymptotic Zhukovsky and Poincaré stability types can be proved similarly. In this case a special reparametrization $\tau(t) = t + c$, called an asymptotic phase, is well known [26, 130].

Also well known are examples of periodic trajectories of continuous systems that happen to be Lyapunov unstable but Poincaré stable. We give one such example.

Example 2.1. Consider the Duffing equation

$$\ddot{x} + x + x^3 = 0$$

or the equivalent two-dimensional system

$$\dot{x} = y, \quad \dot{y} = -x - x^3. \tag{2.6}$$

It is easily seen that all solutions of (2.6) are periodic and satisfy

$$y(t)^2 + x(t)^2 + \frac{1}{2}x(t)^4 \equiv y_0^2 + x_0^2 + \frac{1}{2}x_0^4. \tag{2.7}$$

Here $y_0 = y(0)$, $x_0 = x(0)$. In the sequel we shall assume, without loss of generality, that $y_0 = 0$. This is possible since at a certain time any periodic trajectory of system (2.6) passes through the point having coordinate $y = 0$ (Fig. 2.1).

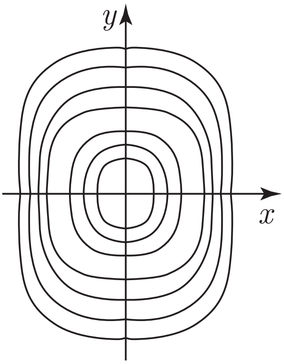

Fig. 2.1 Phase space of Duffing equation.

In Fig. 2.1, the trajectories of (2.6) are graphs of the functions

$$y(x) = \pm\sqrt{c - x^2 - \frac{1}{2}x^4},$$

where the parameter c takes positive values. Denote by $T(x_0)$ the period of the solution passing through the point $y = 0$, $x = x_0$. By (2.6) and (2.7) we find that in the half-plane $\{y \geq 0\}$ the relation

$$\frac{dt}{dx} = \left(x_0^2 + \frac{1}{2}x_0^4 - x^2 - \frac{1}{2}x^4\right)^{-1/2}$$

is valid. From this and from the relation $x(T(x_0)/2) = -x_0$ (see Fig. 2.1) we obtain a formula for the period $T(x_0)$:

$$T(x_0) = 2\int_{-x_0}^{x_0} \left(x_0^2 + \frac{1}{2}x_0^4 - x^2 - \frac{1}{2}x^4\right)^{-1/2} dx. \qquad (2.8)$$

We may contrast this with the linear system

$$\dot{x} = y, \quad \dot{y} = -x, \qquad (2.9)$$

where the period of the solution having initial data $y = 0$, $x = x_0$ is independent of x_0 and equals 2π:

$$T(x_0) = 2\int_{-x_0}^{x_0} \left(x_0^2 - x^2\right)^{-1/2} dx \equiv 2\pi.$$

The period $T(x_0)$ of the solution of system (2.6), computed by (2.8), depends on x_0. Its graph is shown in Fig. 2.2.

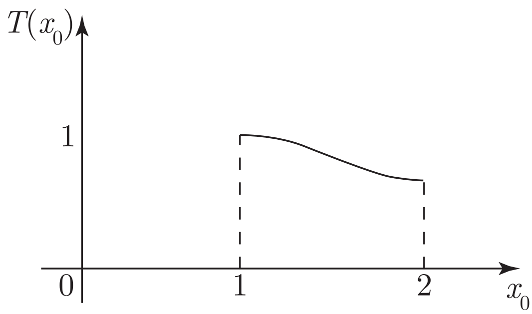

Fig. 2.2 The period $T(x_0)$ of the solution of Duffing equation.

From (2.7) it follows that for small changes in initial data, the trajectories of system (2.6) remain wholly within small neighborhoods of each other; hence they are Poincaré stable. Since all trajectories considered are periodic, they are also Zhukovsky stable. (The equivalence of these two stability types for periodic trajectories was established above.)

However, in this case Lyapunov stability, unlike that for the trajectories of system (2.9), is lacking. This results from the following reasoning. If we consider the solutions of system (2.6) with the initial data $x = x_0$, $y = 0$ and $x = x_1$, $y = 0$, where $x_0 - x_1 = \delta$ and δ is small, then we can choose a natural number $\mathcal{N}$ such that $|\mathcal{N}(T(x_0) - T(x_1))|$ is close to $T(x_1)/2$. It follows that $x(\mathcal{N}T(x_0), x_1)$ is close to the value $-x_1$. Hence the solution $x(t, x_0)$ is not Lyapunov stable: for the small $x_0 - x_1$ the difference $|x(\mathcal{N}T(x_0), x_0) - x(\mathcal{N}T(x_0), x_1)|$ is near $2|x_0|$. □

2.3 Trajectory Sensitivity to Initial Data and the Basic Notions of Instability

Now we proceed to compare the definitions given above with the effect of trajectory sensitivity to the initial data for strange attractors.

Example 2.1 shows that the periodic trajectories of continuous dynamical systems can be Poincaré and Zhukovsky stable and, at the same time, Lyapunov unstable. Thus, Lyapunov instability cannot characterize the "mutual repulsion" of continuous trajectories due to small variations in initial data. Neither can Poincaré instability characterize this repulsion, but for a different reason. In this case, the perturbed solution can leave the ε-neighborhood of a certain segment of the unperturbed trajectory (the effect of repulsion) while simultaneously entering the ε-neighborhood of another segment (the property of Poincaré stability). Thus, mutually repulsive trajectories can be Poincaré stable. Let us consider these effects in more detail.

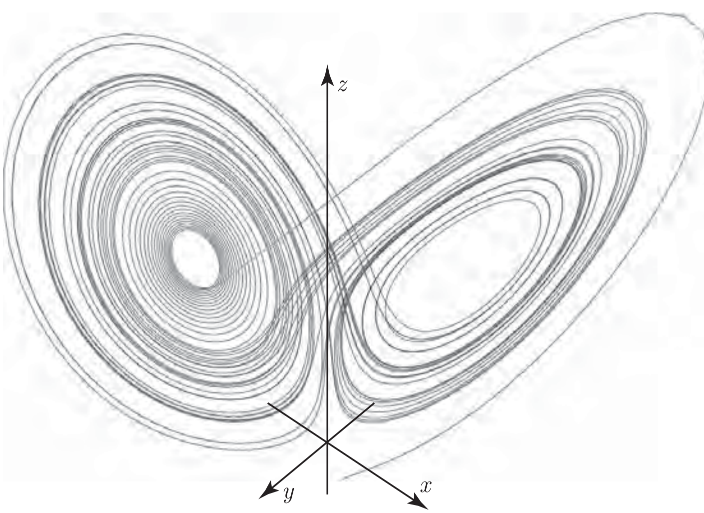

Fig. 2.3 Unstable manifold of the saddle of the Lorenz system. The first fifty turns.

In computer experiments it often happens that the trajectories, situated

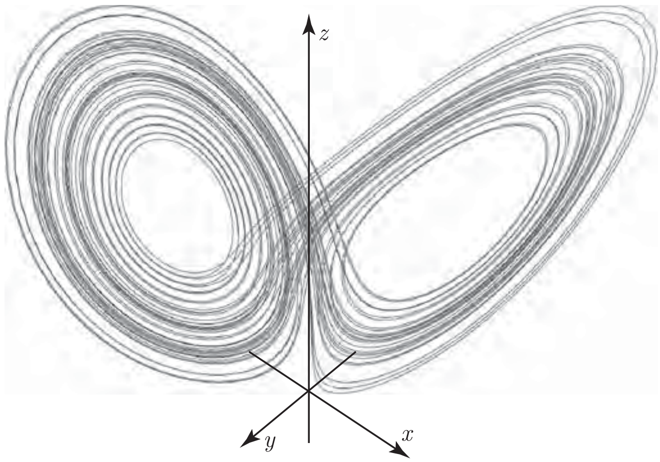

Fig. 2.4 Unstable manifold of the saddle of the Lorenz system. The next fifty turns.

on the unstable manifold of a saddle singular point, everywhere densely fill the B-attractor (or that portion of it consisting of the bounded trajectories). This can be observed on the B-attractor of the Lorenz system [13]

$$\begin{aligned} \dot{x} &= -\sigma(x-y), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy, \end{aligned} \tag{2.10}$$

where $\sigma = 10$, $r = 28$, and $b = 8/3$ (Figs. 2.3 and 2.4). A similar situation occurs for the Henon system [76]

$$\begin{aligned} x(t+1) &= a + by(t) - x(t)^2, \\ y(t+1) &= x(t), \end{aligned} \tag{2.11}$$

where $a = 1.4$, $b = 0.3$ (Fig. 2.5). For these systems we have

$$\rho(L^+(x_0), x(t, y_0)) \equiv 0, \tag{2.12}$$

$\forall x_0 \in M$, $y_0 \in K$, $\forall t \geq 0$. Here M is an unstable manifold of the saddle point, K is the B-attractor mentioned above (or the portion consisting of the bounded trajectories).

From (2.12) it follows that on the attractor K the trajectories $x(t, x_0)$ are asymptotically Poincaré stable. This means that Poincaré instability cannot characterize sensitivity to initial data on strange attractors.

We now give some similar analytic examples.

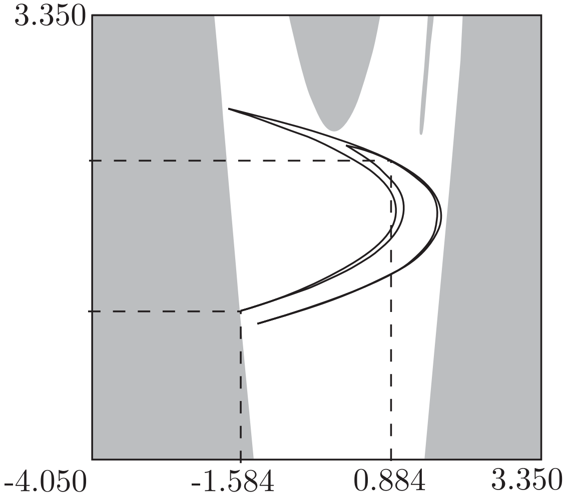

Fig. 2.5 The unstable manifold of the saddle of the Henon system. The shaded part of square is the set of initial data with unbounded solutions.

Example 2.2. Consider the linearized equations of two decoupled pendula:

$$\begin{aligned} \dot{x}_1 &= y_1, \quad \dot{y}_1 = -\omega_1^2 x_1, \\ \dot{x}_2 &= y_2, \quad \dot{y}_2 = -\omega_2^2 x_2. \end{aligned} \tag{2.13}$$

The solutions are

$$\begin{aligned} x_1(t) &= A \sin(\omega_1 t + \varphi_1(0)), \\ y_1(t) &= A\omega_1 \cos(\omega_1 t + \varphi_1(0)), \\ x_2(t) &= B \sin(\omega_2 t + \varphi_2(0)), \\ y_2(t) &= B\omega_2 \cos(\omega_2 t + \varphi_2(0)). \end{aligned}$$

For fixed A and B, the trajectories of system (2.13) are situated on two-dimensional tori

$$\omega_1^2 x_1^2 + y_1^2 = A^2, \qquad \omega_2^2 x_2^2 + y_2^2 = B^2. \tag{2.14}$$

When ω_1/ω_2 is irrational, the trajectories of (2.13) are everywhere densely situated on the tori (2.14) for any initial data $\varphi_1(0)$ and $\varphi_2(0)$.

Thus, for ω_1/ω_2 irrational the relation (2.12) is satisfied for any points $x_0 \in \mathbb{R}^4$, $y_0 \in \mathbb{R}^4$, situated on the tori (2.14). This implies asymptotic Poincaré stability of the trajectories of the dynamical system having phase space (2.14). However, the motion of the points $x(t, x_0)$ and $x(t, y_0)$ along the trajectories occurs in such a way that they do not tend toward each other as $t \to +\infty$. Neither are the trajectories "pressed" toward each other. Hence the intuitive conception of asymptotic stability as a convergence of objects toward each other is in contrast to the formal definition of Poincaré.

It is clear that a similar effect is lacking for the notion of Zhukovsky stability: in the case under consideration, asymptotic Zhukovsky stability does not occur. □

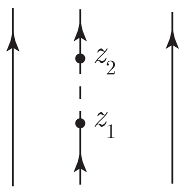

Fig. 2.6 Segment of the trajectory from z_1 to z_2.

Example 2.3. We reconsider the dynamical system (2.13) with ω_1/ω_2 irrational. Change the flow of trajectories on the tori as follows. Cut the toroidal surface along a certain segment of the fixed trajectory from the point z_1 to the point z_2 (Fig. 2.6). Then the surface is stretched diffeomorphically along the torus so that a cut is mapped into the circle with the fixed points z_1 and z_2 (Fig. 2.7). Denote by H the interior of the circle.

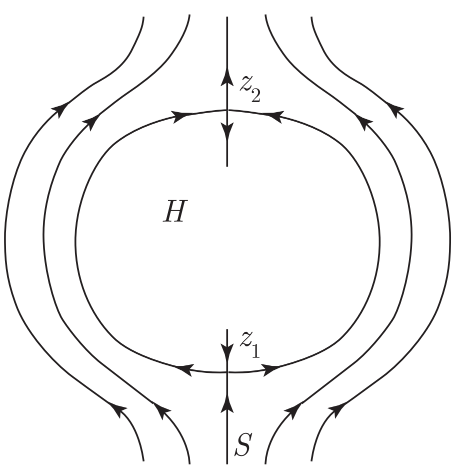

Fig. 2.7 Heteroclinic trajectories with hole H.

Change the dynamical system so that z_1 and z_2 are saddle stationary points and the semicircles connecting z_1 and z_2 are heteroclinic trajectories, tending as $t \to +\infty$ and $t \to -\infty$ to z_2 and z_1, respectively (Fig. 2.7).

Outside the "hole" H, after the diffeomorphic stretching, the disposition of trajectories on the torus is the same.

Consider the behavior of the system trajectories from the Poincaré and Zhukovsky points of view.

Outside the hole H, the trajectories are everywhere dense on torus. They are therefore, as before, asymptotically Poincaré stable.

Now we consider a certain δ-neighborhood of the point z_0, situated on the torus and outside the set H. The trajectory leaving z_0 is either everywhere dense or coincides with the separatrix S of the saddle z_1, tending to z_1 as $t \to +\infty$ (Fig. 2.7). Then there exists a time t such that some trajectories, leaving the δ-neighborhood of z_0, are situated in a small neighborhood of z_1 to the right of the separatrix S. At time t the remaining trajectories, leaving this neighborhood of z_0, are situated to the left of S. It is clear that in this case the trajectories, situated to the right and to the left of S, envelope the hole H on the right and left, respectively. It is also clear that these trajectories are repelled from each other; hence, the trajectory leaving z_0 is Zhukovsky unstable.

Thus, a trajectory can be asymptotically Poincaré stable and Zhukovsky unstable.

This example shows that the trajectories are sensitive to the initial data and can diverge considerably after some time. The notion of Zhukovsky instability is adequate to such a sensitivity.

Note that the set of such sensitive trajectories is situated on the smooth manifold, named "a torus minus the hole H". Thus, the bounded invariant set of trajectories, which are sensitive to the initial data, do not always have a noninteger Hausdorff dimension or the structure of the Cantor set (see Example 1.3 and Chapter 10). □

Hence, from among the classical notions of instability for studying strange attractors, the most adequate ones are Zhukovsky instability (in the continuous case) and Lyapunov instability (in the discrete case).

Recently the following remarkable result [139] was obtained: the ω-limit set of the trajectory, which is asymptotically Zhukovsky stable, consists of a periodic trajectory.

Thus, the limit set of the trajectories, which is asymptotically Zhukovsky stable, has a sufficiently simple structure. On the other hand, as was shown above, Zhukovsky instability can be one of the characteristics of strange attractors.

2.4 Reduction to a Study of the Zero Solution

The first procedure for investigating the stability (or instability) of the solution $x(t, t_0, x_0)$ of system (2.1), as a rule, makes use [94] of the transformation

$$x = y + x(t, t_0, x_0). \tag{2.15}$$

Using this substitution we obtain

$$\frac{dy}{dt} = F(y + x(t, t_0, x_0), t) - F(x(t, t_0, x_0), t), \tag{2.16}$$

which is often called a differential equation of perturbed motion. It is evident that the problem of stability of the solution $x(t, t_0, x_0)$ is reduced to the problem of stability of the trivial solution $y(t) \equiv 0$ to (2.16).

In this case we assume that the right-hand side of (2.16) is known, since $F(x, t)$ and the solution $x(t, t_0, x_0)$ are known. The difficulties of computing $x(t, t_0, x_0)$ are often overcome via numerical methods and computational experiments.

The studies of the Krasovsky and exponential stability types are the same. For investigating the Zhukovsky and Poincaré stability types, another technique is used. It will be described in Chapter 9.

Chapter 3

Characteristic Exponents and Lyapunov Exponents

3.1 Characteristic Exponents

Definition 3.1. The number (or the symbol $+\infty, -\infty$), defined by the formula

$$\lambda = \limsup_{t\to+\infty} \frac{1}{t} \ln |f(t)|,$$

is called a *characteristic exponent* of the vector-function $f(t)$.

Definition 3.2. The characteristic exponent λ of the vector-function $f(t)$ is said to be *sharp* if there exists the following finite limit:

$$\lambda = \lim_{t\to+\infty} \frac{1}{t} \ln |f(t)|.$$

The value

$$\lambda = \liminf_{t\to+\infty} \frac{1}{t} \ln |f(t)|$$

is often called a *lower characteristic exponent* of $f(t)$.

Consider the linear system

$$\frac{dx}{dt} = A(t)x, \quad x \in \mathbb{R}^n, \tag{3.1}$$

where the $n \times n$ matrix $A(t)$ is continuous and bounded on $[0, +\infty)$. Let $X(t) = (x_1(t), \ldots, x_n(t))$ be a fundamental matrix of (3.1) (i.e. $\det X(0) \neq 0$). It is well known [1, 26] (by Corollary 3.1) that under the above conditions the characteristic exponents λ_j of the solutions $x_j(t)$ are numbers.

Definition 3.3. Fundamental matrix $X(t)$ is said to be *normal* if the sum $\sum_{j=1}^{n} \lambda_j$ of the characteristic exponents of the vector-functions $x_j(t)$ is minimal in comparison to other fundamental matrices.

The following substantial and almost obvious results [1, 26] are well-known.

Theorem 3.1 (Lyapunov, on a normal fundamental matrix). *For any fundamental matrix $X(t)$ there exists the constant matrix C ($\det C \neq 0$) such that the matrix*

$$X(t)C \tag{3.2}$$

is a normal fundamental matrix of (3.1).

Theorem 3.2. *For all normal fundamental matrices $(x_1(t), \ldots, x_n(t))$ the number of solutions $x_j(t)$ having the same characteristic exponent is the same.*

We can now introduce the following definitions.

Definition 3.4. The set of characteristic exponents $\lambda_1, \ldots, \lambda_n$ of the solutions $x_1(t), \ldots, x_n(t)$ of certain normal fundamental matrices $X(t)$ is called the *complete spectrum* of linear system (3.1), and the numbers λ_j are called the *characteristic exponents* of (3.1).

Thus, any normal fundamental matrix realizes the complete spectrum of the system (3.1).

In the sequel, by $\Sigma = \sum_{j=1}^{n} \lambda_j$ is denoted the sum of characteristic exponents of system (3.1).

The Lyapunov inequality [1, 26]

$$\Sigma \geq \limsup_{t \to +\infty} \frac{1}{t} \int_0^t \operatorname{Tr} A(\tau)\, d\tau \tag{3.3}$$

is well known. Here Tr is a spur of the matrix A.

Definition 3.5. If the relation

$$\Sigma = \limsup_{t \to +\infty} \frac{1}{t} \int_0^t \operatorname{Tr} A(\tau)\, d\tau$$

is satisfied, then system (3.1) is called *regular*.

It is well-known [1, 26] that each characteristic exponent of a regular system is sharp. Numerous investigations are devoted to criteria for sharp exponents [20, 21, 26, 107].

Definition 3.6. The number

$$\Gamma = \Sigma - \liminf_{t \to +\infty} \frac{1}{t} \int_0^t \operatorname{Tr} A(\tau)\, d\tau$$

is called the *coefficient of irregularity* for (3.1).

We assume further that $\lambda_1 \geq \cdots \geq \lambda_n$. The number λ_1 is called a *higher characteristic exponent*.

3.2 Lyapunov Exponents

Let $X(t)$ be a fundamental matrix of system (3.1). We introduce the singular values $\alpha_1(X(t)) \geq \cdots \geq \alpha_n(X(t)) \geq 0$ of $X(t)$. Recall that the singular values $\alpha_j(X(t))$ of a matrix $X(t)$ are square roots of eigenvalues of the matrix $X(t)^* X(t)$. Geometrically, the $\alpha_j(X(t))$ coincide with the principal axes of the ellipsoid $X(t)B$, where B is the unit ball.

Definition 3.7 [132]. The *Lyapunov exponent* μ_j is the number

$$\mu_j = \limsup_{t \to +\infty} \frac{1}{t} \ln \alpha_j(X(t)).$$

We say that μ_j is *sharp* if there exists the finite limit

$$\lim_{t \to +\infty} \frac{1}{t} \ln \alpha_j(X(t)).$$

Proposition 3.1. *The higher characteristic exponent λ_1 and the Lyapunov exponent μ_1 coincide.*

Proof. Recall that a geometric interpretation of singular values implies the relation $|X(t)| = \alpha_1(X(t))$. Here $|X|$ is a norm of the matrix X, defined by the formula

$$|X| = \max_{|x|=1} |Xx|, \quad x \in \mathbb{R}^n.$$

Then by the relation

$$\limsup_{t \to +\infty} \frac{1}{t} \ln |X(t)| = \lambda_1$$

we obtain $\lambda_1 = \mu_1$. □

Now we show that there exist systems such that $\lambda_2 \neq \mu_2$. Consider system (3.1) with the matrix

$$A(t) = \begin{pmatrix} 0 & \sin(\ln t) + \cos(\ln t) \\ \sin(\ln t) + \cos(\ln t) & 0 \end{pmatrix}, \quad t > 1,$$

and with the fundamental normal matrix

$$X(t) = \begin{pmatrix} e^{\gamma(t)} & e^{-\gamma(t)} \\ e^{\gamma(t)} & -e^{-\gamma(t)} \end{pmatrix},$$

where $\gamma(t) = t\sin(\ln t)$. It is obvious that $\lambda_1 = \lambda_2 = 1$ and

$$\alpha_1(X(t)) = \sqrt{2}\max(e^{\gamma(t)}, e^{-\gamma(t)}),$$
$$\alpha_2(X(t)) = \sqrt{2}\min(e^{\gamma(t)}, e^{-\gamma(t)}).$$

Hence we obtain $\mu_1 = 1$, $\mu_2 = 0$. Thus, $\lambda_2 \neq \mu_2$.

The Lyapunov exponent turned out to be a useful tool for defining the notion of Lyapunov dimension, and for estimating the Hausdorff and fractal dimensions of attractors [75, 78, 132]. This will be considered in detail in Chapter 10.

3.3 Norm Estimates for the Cauchy Matrix

Consider a norm of the fundamental matrix $X(t)$ of system (3.1), and introduce the following notation:

$$\Lambda = \max_j \lambda_j, \quad \lambda = \min_j \lambda_j.$$

Here λ_j is a complete spectrum of (3.1). We call $X(t)X(\tau)^{-1}$ the *Cauchy matrix* of system (3.1). The following result is well-known and often used.

Theorem 3.3. *For any $\varepsilon > 0$ there exists $C > 0$ such that*

$$|X(t)X(\tau)^{-1}| \leq C\exp[(\Lambda + \varepsilon)(t - \tau) + (\Gamma + \varepsilon)\tau], \quad \forall\, t \geq \tau \geq 0, \tag{3.4}$$
$$|X(t)X(\tau)^{-1}| \leq C\exp[\lambda(t - \tau) + (\Gamma + \varepsilon)\tau], \quad \forall\, \tau \geq t \geq 0. \tag{3.5}$$

Proof. We introduce the following notation:

$$X(t) = (x_1(t), \dots, x_n(t)), \quad \widetilde{x}_j(t) = x_j(t)\exp[(-\lambda_j - \varepsilon)t],$$

$$X(t)^{-1} = \begin{pmatrix} u_1(t)^* \\ \vdots \\ u_n(t)^* \end{pmatrix}, \quad \widetilde{u}_j(t) = u_j(t)\exp[(\lambda_j + \varepsilon)t].$$

Here the asterisk denotes transposition. From the definitions of λ_j and the rule of matrix inversion it follows that for some $L > 0$ the inequality

$$|(\widetilde{x}_1(t), \dots, \widetilde{x}_n(t))^{-1} \det(\widetilde{x}_1(t), \dots, \widetilde{x}_n(t))| \le L, \ \forall\, t \ge 0, \tag{3.6}$$

holds. By (3.6) we obtain

$$\begin{aligned} |(\widetilde{x}_1(t), \dots, \widetilde{x}_n(t))^{-1}| &\le L\exp\left[(\Sigma + n\varepsilon)t - \int_0^t \operatorname{Tr} A(\tau)\,d\tau\right] \\ &\le L_1 \exp[(2n\varepsilon + \Gamma)t], \ \forall t \ge 0. \end{aligned} \tag{3.7}$$

Here L_1 is sufficiently large.

We have the following obvious relations:

$$\begin{aligned} |X(t)X(\tau)^{-1}| &= \Big|\sum_j x_j(t)u_j(\tau)^*\Big| \\ &= \Big|\sum_j \widetilde{x}_j(t)\exp[(\lambda_j + \varepsilon)t - (\lambda_j + \varepsilon)\tau]\widetilde{u}_j(\tau)^*\Big|. \end{aligned}$$

Taking into account that for $t \ge 0$ the vector-function $\widetilde{x}_j(t)$ is finite, for sufficiently large L_2 we obtain

$$|X(t)X(\tau)^{-1}| \le L_2 \sum_j \exp[(\lambda_j + \varepsilon)(t - \tau)]|\widetilde{u}_j(\tau)|. \tag{3.8}$$

Since

$$\begin{pmatrix} \widetilde{u}_1(t)^* \\ \vdots \\ \widetilde{u}_n(t)^* \end{pmatrix} = (\widetilde{x}_1(t), \dots, \widetilde{x}_n(t))^{-1},$$

by (3.7) and (3.8) we obtain (3.4) and (3.5). □

Now we prove the inequality of Vazhevsky [1, 26] in a convenient form.

Theorem 3.4. *For the solutions $x(t)$ of system (3.1) the inequalities*

$$|x(\tau)| \exp \int_\tau^t \alpha(s)\, ds \le |x(t)| \le |x(\tau)| \exp \int_\tau^t \beta(s)\, ds, \quad \forall t \ge \tau,$$
$$|x(\tau)| \exp \int_\tau^t \beta(s)\, ds \le |x(t)| \le |x(\tau)| \exp \int_\tau^t \alpha(s)\, ds, \quad \forall t \le \tau,$$

hold, where $\alpha(t)$ and $\beta(t)$ are the smallest and largest eigenvalues of the matrix

$$\frac{1}{2}[A(t) + A(t)^*].$$

Proof. Since the inequalities

$$\alpha(t)|x|^2 \le \frac{1}{2} x^*[A(t) + A(t)^*]x \le \beta(t)|x|^2, \quad \forall x \in \mathbb{R}^n,$$

and the relation

$$\frac{d}{dt}(|x(t)|^2) = x(t)^*[A(t) + A(t)^*]x(t)$$

are satisfied, we have

$$2\alpha(t) \le \frac{(|x(t)|^2)^\bullet}{|x(t)|^2} \le 2\beta(t).$$

Integrating these inequalities between τ and t, for $t \ge \tau$ we obtain the first estimate of the theorem. For $t \le \tau$ we obtain the second estimate. □

Corollary 3.1. *If $|A(t)|$ is bounded on $\mathbb{R}^1$, then there exists ν such that for any t and τ the estimate*

$$|X(t)X(\tau)^{-1}| \le \exp[\nu|t - \tau|] \tag{3.9}$$

is satisfied.

Estimate (3.9) results from Theorem 3.4 and the obvious relation

$$X(t)X(\tau)^{-1}x(\tau) = x(t).$$

In fact, we have

$$\begin{aligned}|X(t)X(\tau)^{-1}| &= \max_{y\neq 0}\frac{|X(t)X(\tau)^{-1}y|}{|y|}\\ &\leq \max\left(\exp\int_\tau^t \beta(s)\,ds, \exp\int_\tau^t \alpha(s)\,ds\right)\\ &\leq \exp[\nu|t-\tau|].\end{aligned}$$

3.4 The Counterexample of Nemytskii—Vinograd [20]

Consider system (3.1) with

$$A(t) = \begin{pmatrix} 1-4(\cos 2t)^2 & 2+2\sin 4t \\ -2+2\sin 4t & 1-4(\sin 2t)^2 \end{pmatrix}.$$

It is obvious that the vector-function

$$x(t) = \begin{pmatrix} e^t \sin 2t \\ e^t \cos 2t \end{pmatrix} \tag{3.10}$$

satisfies (3.1). It follows easily that

$$\det(A(t) - pI) = p^2 + 2p + 1.$$

Therefore the eigenvalues $\nu_1(t)$ and $\nu_2(t)$ of the matrix $A(t)$ are -1:

$$\nu_1(t) = \nu_2(t) = -1.$$

On the other hand, the characteristic exponent λ_j of (3.10) is equal to 1.

This counterexample shows that the matrix $A(t)$ can have all eigenvalues with negative real parts even when the corresponding linear system (3.1) has positive characteristic exponents. It also shows that the formulas obtained in the book [4], namely

$$\lambda_j = \limsup_{t\to+\infty}\frac{1}{t}\int_0^t \operatorname{Re}\nu_j(\tau)\,d\tau,$$

are not true.

Chapter 4

Perron Effects

In 1930, O. Perron [110] showed that the negativity of the higher characteristic exponent of the system of the first approximation does not always imply the stability of the zero solution of the original system. In addition, in an arbitrary small neighborhood of zero the solutions of the original system with positive characteristic exponent can exist. Perron's results impressed the specialists in the theory of motion stability.

The effect of sign reversal for the characteristic exponent of solutions of the system of the first approximation, and of the original system under the same initial data, we shall call the *Perron effect*.

We cite the outstanding result of Perron. Consider a system

$$\begin{aligned} \frac{dx_1}{dt} &= -ax_1, \\ \frac{dx_2}{dt} &= [\sin(\ln(t+1)) + \cos(\ln(t+1)) - 2a]x_2 + x_1^2, \end{aligned} \tag{4.1}$$

where a satisfies

$$1 < 2a < 1 + \frac{1}{2}\exp(-\pi). \tag{4.2}$$

The solution of the equation of the first approximation takes the form

$$\begin{aligned} x_1(t) &= \exp[-at]x_1(0), \\ x_2(t) &= \exp[(t+1)\sin(\ln(t+1)) - 2at]x_2(0). \end{aligned}$$

It is obvious that for the system of the first approximation under condition (4.2) we have $\lambda_1 < 0$.

Now we write the solution of (4.1):

$$\begin{aligned} x_1(t) &= \exp[-at]x_1(0), \\ x_2(t) &= \exp[(t+1)\sin(\ln(t+1)) - 2at]\Big(x_2(0) \\ &\quad + x_1(0)^2 \int_0^t \exp[-(\tau+1)\sin(\ln(\tau+1))]d\tau\Big). \end{aligned} \tag{4.3}$$

Letting $t = \exp[(2k+\frac{1}{2})\pi] - 1$, where k is an integer, we obtain

$$\exp[(t+1)\sin(\ln(t+1)) - 2at] = e(\exp[(1-2a)t]), \quad (1+t)e^{-\pi} - 1 > 0,$$

and

$$\begin{aligned} &\int_0^t \exp[-(\tau+1)\sin(\ln(\tau+1))]d\tau \\ &\quad > \int_{f(k)}^{g(k)} \exp[-(\tau+1)\sin(\ln(\tau+1))]\, d\tau \\ &\quad > \int_{f(k)}^{g(k)} \exp\left[\frac{1}{2}(\tau+1)\right] d\tau \\ &\quad > \int_{f(k)}^{g(k)} \exp\left[\frac{1}{2}(\tau+1)\exp(-\pi)\right] d\tau \\ &\quad = \exp\left[\frac{1}{2}(t+1)\exp(-\pi)\right](t+1)\left(\exp\left(-\frac{2\pi}{3}\right) - \exp(-\pi)\right), \end{aligned}$$

where

$$f(k) = (1+t)\exp[-\pi] - 1, \quad g(k) = (1+t)\exp\left[-\frac{2\pi}{3}\right] - 1.$$

This implies the estimate

$$\begin{aligned} &\exp[(t+1)\sin(\ln(t+1)) - 2at]\int_0^t \exp[-(\tau+1)\sin(\ln(\tau+1))]d\tau \\ &> \exp\left[\frac{1}{2}(2+\exp(-\pi))\right]\left(\exp\left(-\frac{2\pi}{3}\right) - \exp(-\pi)\right) \\ &\times \exp\left[\left(1 - 2a + \frac{1}{2}\exp(-\pi)\right)t\right]. \end{aligned} \tag{4.4}$$

This and condition (4.2) imply that the characteristic exponent λ of the solutions of system (4.1) for $x_1(0) \neq 0$ is positive.

Thus, all characteristic exponents of the system of the first approximation are negative, and almost all solutions of the original system (4.1) tend exponentially to infinity as $k \to +\infty$. □

We consider the similar effect of the sign reversal of characteristic exponents but "on the contrary", namely the solution of the system of the first approximation has a positive characteristic exponent while the solution of the original system with the same initial data has a negative exponent [79, 80, 81]. Consider a system

$$\begin{aligned} \dot{x}_1 &= -ax_1, \\ \dot{x}_2 &= -2ax_2, \\ \dot{x}_3 &= [\sin(\ln(t+1)) + \cos(\ln(t+1)) - 2a]x_3 + x_2 - x_1^2, \end{aligned} \tag{4.5}$$

on the invariant manifold

$$M = \{x_3 \in \mathbb{R}^1,\ x_2 = x_1^2\}.$$

Here a satisfies (4.2). The solutions of (4.5) on the set M are

$$\begin{aligned} x_1(t) &= \exp[-at]x_1(0), \\ x_2(t) &= \exp[-2at]x_2(0), \\ x_3(t) &= \exp[(t+1)\sin(\ln(t+1)) - 2at]x_3(0), \\ x_1(0)^2 &= x_2(0). \end{aligned}$$

Obviously, these have negative characteristic exponents.

Consider now the system of the first approximation in the neighborhood of the zero solution of system (4.5):

$$\begin{aligned} \dot{x}_1 &= -ax_1, \\ \dot{x}_2 &= -2ax_2, \\ \dot{x}_3 &= [\sin(\ln(t+1)) + \cos(\ln(t+1)) - 2a]x_3 + x_2. \end{aligned} \tag{4.6}$$

Its solutions have the form

$$\begin{aligned} x_1(t) &= \exp[-at]x_1(0), \\ x_2(t) &= \exp[-2at]x_2(0), \\ x_3(t) &= \exp[(t+1)\sin(\ln(t+1)) - 2at]\Bigg(x_3(0) \\ &\qquad + x_2(0)\int_0^t \exp[-(\tau+1)\sin(\ln(\tau+1))]\,d\tau\Bigg). \end{aligned} \tag{4.7}$$

Comparing (4.7) with (4.3) and applying (4.4), we find that for $x_2(0) \neq 0$ the relation

$$\limsup_{t\to+\infty} \frac{1}{t} \ln |x_3(t)| > 0$$

holds. It is easily shown that for the solutions of systems (4.5) and (4.6) we have

$$(x_1(t)^2 - x_2(t))^\bullet = -2a(x_1(t)^2 - x_2(t)).$$

Then

$$x_1(t)^2 - x_2(t) = \exp[-2at](x_1(0)^2 - x_2(0)).$$

It follows that M is a global attractor for the solutions of (4.5) and (4.6). This means that the relation $x_1(0)^2 = x_2(0)$ yields $x_1(t)^2 = x_2(t)$ for all $t \in \mathbb{R}^1$ and that for any initial data we have

$$|x_1(t)^2 - x_2(t)| \leq \exp[-2at]|x_1(0)^2 - x_2(0)|.$$

Thus, systems (4.5) and (4.6) have the same global attractor M on which almost all the solutions of the system of the first approximation (4.6) have a positive characteristic exponent and all the solutions of original system (4.5) have negative characteristic exponents.

Here the Perron effect occurs on the two-dimensional manifold, namely

$$\{x_3 \in \mathbb{R}^1,\ x_2 = x_1^2 \neq 0\}.$$

To construct the exponentially stable system for which the first approximation has a positive characteristic exponent, we change (4.5) to

$$\begin{aligned} \dot{x}_1 &= F(x_1, x_2), \\ \dot{x}_2 &= G(x_1, x_2), \\ \dot{x}_3 &= [\sin\ln(t+1) + \cos\ln(t+1) - 2a]x_3 + x_2 - x_1^2. \end{aligned} \tag{4.8}$$

Here the functions $F(x_1, x_2)$ and $G(x_1, x_2)$ have the form

$$F(x_1, x_2) = \pm 2x_2 - ax_1, \quad G(x_1, x_2) = \mp x_1 - \varphi(x_1, x_2),$$

in which case the upper sign is taken for $x_1 > 0$, $x_2 > x_1^2$ and for $x_1 < 0$, $x_2 < x_1^2$, the lower one for $x_1 > 0$, $x_2 < x_1^2$ and for $x_1 < 0$, $x_2 > x_1^2$. The function $\varphi(x_1, x_2)$ is defined as

$$\varphi(x_1, x_2) = \begin{cases} 4ax_2, & |x_2| > 2x_1^2, \\ 2ax_2, & |x_2| < 2x_1^2. \end{cases}$$

The solutions of system (4.8) are credited to A.F. Filippov [33]. Then for the given functions F and G, on the lines of discontinuity $\{x_1 = 0\}$ and $\{x_2 = x_1^2\}$ the system

$$\begin{aligned} \dot{x}_1 &= F(x_1, x_2), \\ \dot{x}_2 &= G(x_1, x_2), \end{aligned} \tag{4.9}$$

has the sliding solutions, which are defined as

$$x_1(t) \equiv 0, \quad \dot{x}_2(t) = -4ax_2(t),$$

and

$$\dot{x}_1(t) = -ax_1(t), \quad \dot{x}_2(t) = -2ax_2(t), \quad x_2(t) \equiv x_1(t)^2.$$

In this case the solutions of system (4.9) with the initial data $x_1(0) \neq 0$, $x_2(0) \in \mathbb{R}^1$ attain the curve $\{x_2 = x_1^2\}$ in a finite time, which does not exceed 2π. The phase picture of such a system is shown in Fig. 4.1.

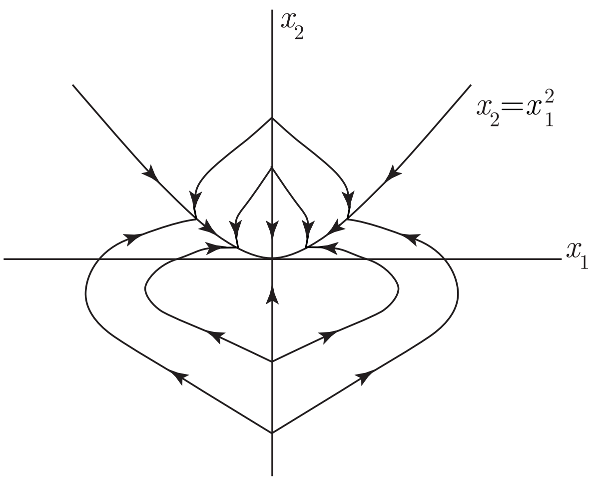

Fig. 4.1 The phase space of system (4.9).

From the above it follows that for the solutions of system (4.8) with the initial data $x_1(0) \neq 0$, $x_2(0) \in \mathbb{R}^1$, $x_3(0) \in \mathbb{R}^1$ for $t \geq 2\pi$ we have the relations $F(x_1(t), x_2(t)) = -ax_1(t)$, $G(x_1(t), x_2(t)) = -2ax_2(t)$. Therefore on these solutions, for $t \geq 2\pi$ system (4.6) is the system of the first approximation.

This system, as we have shown earlier, has a positive characteristic exponent. At the same time all the solutions of system (4.8) tend exponentially to zero. □

The technique considered here permits us to construct the different classes of nonlinear systems for which Perron effects occur.

Chapter 5

Matrix Lyapunov Equation

The *matrix Lyapunov equation* is the equation

$$\dot{H}(t) + P(t)^* H(t) + H(t)P(t) = -G(t) \tag{5.1}$$

with respect to the symmetric differentiable matrix $H(t)$. Here $P(t)$ and $G(t)$ are $n \times n$ matrices, continuous and bounded for $t \geq 0$, and $G^*(t) = G(t)$ $\forall t \geq 0$.

Denote by $X(t)$ a fundamental matrix of the system

$$\frac{dx}{dt} = P(t)x. \tag{5.2}$$

If for certain constants $\alpha > 0$, $C > 0$, $\gamma \geq 0$ the estimate

$$|X(s)X(t)^{-1}| \leq C \exp[-\alpha(s-t) + \gamma t], \quad \forall s \geq t \geq 0, \tag{5.3}$$

is valid, then the solution of (5.1) is the matrix

$$H(t) = \int_t^{+\infty} (X(s)X(t)^{-1})^* G(s)(X(s)X(t)^{-1})\, ds. \tag{5.4}$$

This is verified by substituting (5.4) into (5.1) and using the identity

$$(X(t)^{-1})^\bullet = -X(t)^{-1}P(t).$$

Convergence of the integral (5.4) results from estimate (5.3). In addition, from (5.3) it follows that

$$|H(t)| \leq C^2 \sup_{t \geq 0} |G(t)| \int_t^{+\infty} \exp 2[-\alpha(s-t) + \gamma t]\, ds.$$

Thus, there exists R such that

$$|H(t)| \leq R \exp[2\gamma t], \quad \forall\, t \geq 0. \tag{5.5}$$

Together with estimate (5.5), an important role is played by a lower bound for the quadratic form $z^*H(t)z$. This can be found by using the following

Theorem 5.1. *Suppose the estimates (5.3) and*

$$z^*G(t)z \geq \delta|z|^2, \quad \forall t \geq 0, \quad \forall z \in \mathbb{R}^n, \tag{5.6}$$

hold for some $\delta > 0$. *Then there exists* $\varepsilon > 0$ *such that*

$$z^*H(t)z \geq \varepsilon|z|^2, \quad \forall t \geq 0, \quad \forall z \in \mathbb{R}^1. \tag{5.7}$$

Proof. From (5.6) we obtain the following estimate:

$$\begin{aligned} z^*H(t)z &\geq \delta \int_t^{+\infty} |X(s)X(t)^{-1}z|^2 ds \\ &\geq \delta \int_t^{+\infty} \frac{|z|^2}{|(X(s)X(t)^{-1})^{-1}|^2}\, ds \\ &= \delta|z|^2 \int_t^{+\infty} \frac{1}{|X(t)X(s)^{-1}|^2}\, ds. \end{aligned}$$

From this and Corollary 3.1 we obtain

$$z^*H(t)z \geq \delta|z|^2 \int_t^{+\infty} \exp[2\nu(t-s)]\, ds = \varepsilon|z|^2,$$

where $\varepsilon = \delta(2\nu)^{-1}$. The theorem is proved. □

Consider now the first-order equation

$$\frac{dy}{dt} = Q(t)y$$

with the positive lower characteristic exponent $\rho > 0$:

$$\rho = \liminf_{t\to+\infty} \frac{1}{t} \ln|y(t)|.$$

Since this equation can be represented as

$$\frac{dx}{dt} = -Q(t)x, \quad x(t) = y(t)^{-1},$$

from Theorem 3.3 it follows that for any $\alpha \in (0, \rho)$ and the certain $\gamma > 0$ estimate (5.3) is satisfied. Then for $P(t) = -Q(t)$ the solution $H(t)$ of the Lyapunov equation, with properties (5.5) and (5.7), exists.

Assuming $M(t) = H(t)^{-1}$ and $G(t) \equiv 1$, we obtain the following

Corollary 5.1. *If $\rho > 0$, then there exists a function $M(t)$ — positive, continuously differentiable, and bounded for $t \geq 0$ — such that*

$$\dot{M}(t) + 2Q(t)M(t) = M(t)^2, \quad \forall\, t \geq 0.$$

We remark that for $\gamma = 0$ the existence of $H(t)$ with properties (5.1), (5.5), (5.7) is proved by I.G. Malkin [95].

Consider the equation

$$\dot{x} = A(t)x + f(t, x) \tag{5.8}$$

with $n = 1$. Here $A(t)$ is continuous and bounded on $[0, +\infty)$. The function $f(t, x)$ is continuous and, in a certain neighborhood of the point $x = 0$, satisfies

$$|f(t, x)| \leq \kappa |x|, \quad \forall\, t \geq 0, \tag{5.9}$$

for some κ. We assume that the characteristic exponent λ of the equation

$$\dot{x} = A(t)x \tag{5.10}$$

is negative: $\lambda < 0$. In this case (5.8) can be represented in the form

$$\dot{x} = (P(t) - a)x + f(t, x), \tag{5.11}$$

where $a \in (0, -\lambda)$. Then for equation (5.2) for the certain C and γ estimate (5.3) is valid. Therefore the function $H(t)$, satisfying relations (5.1), (5.5) and (5.7) for $G(t) \equiv 1$, exists. This implies that for $V(t) = x(t)H(t)x(t)$ we have

$$\dot{V}(t) \leq -|x(t)|^2 - 2aV(t) + 2x(t)H(t)f(t, x(t)).$$

From (5.9) it follows that for $\kappa < a$ the inequality

$$V(t) \leq V(0) \exp[-2(a - \kappa)t]$$

holds. Then from (5.7) we obtain

$$|x(t)|^2 \leq \varepsilon^{-1} |H(0)| |x(0)|^2 \exp[-2(a - \kappa)t]. \qquad \square$$

Thus we have

Proposition 5.1. *For the first-order equation, the negativity of the characteristic exponent of the system of the first approximation implies the exponential stability of the zero solution of the original system.*

Consider now equation (5.8), assuming that the lower characteristic exponent $\lambda > 0$. In this case we represent (5.8) in the form (5.11) with $a \in (-\lambda, 0)$. Using Corollary 5.1 and the function $V(t) = x(t)M(t)x(t)$ we obtain

$$\dot{V}(t) \geq -2aV(t) + 2M(t)x(t)f(t, x(t)).$$

From (5.9) it follows that for $\kappa < |a|$ the inequality

$$V(t) \geq V(0)\exp[-2(a+\kappa)t]$$

holds. Then, from the boundedness of $M(t)$ it follows that there exists L such that

$$|x(t)| \geq L|x(0)|\exp[-(a+\kappa)t]. \qquad \square$$

We have obtained

Proposition 5.2. *For the first-order equation, the positiveness of the lower characteristic exponent of the system of the first approximation implies the exponential instability of the zero solution of the original system.*

Consider now equation (5.8) with arbitrary n and with $f(t, x)$ having the special form

$$f(t, x) = \varphi(t, x)x,$$

where $\varphi(t, x)$ is continuous and scalar-valued: $\varphi(t, x)\colon \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^1$. We assume that in a certain neighborhood of $x = 0$ the inequality

$$|\varphi(t, x)| \leq \kappa \quad \forall t \geq 0, \ \forall x \in \mathbb{R}^n \tag{5.12}$$

holds. Suppose the higher characteristic exponent Λ of system (5.10) is negative. Represent (5.8) in the form

$$\dot{x} = (P(t) - aI)x + f(t, x), \tag{5.13}$$

where $a \in (0, -\Lambda)$ and I is a unit matrix. Then by Theorem 3.3 we have estimate (5.3) with $\alpha = -\Lambda - a - \varepsilon$, $\gamma = \Gamma + \varepsilon$. In this case for $G(t) \equiv I$ there exists the matrix $H(t)$ satisfying (5.1), (5.5), and (5.7).

By (5.12) we obtain the estimate for $V(t) = x(t)^* H(t)x(t)$, namely

$$\dot{V}(t) \leq -|x(t)|^2 - 2aV(t) + 2\kappa V(t).$$

Therefore for $a > \kappa$ we have

$$V(t) \leq V(0)\exp[-2(a-\kappa)t].$$

By Theorem 5.1 we obtain the estimate

$$|x(t)|^2 \le \varepsilon^{-1} V(0) \exp[-2(a-\kappa)t]. \qquad \square$$

Thus, we have

Proposition 5.3. *For system (5.8) with $f(t,x)$ satisfying (5.12) to be exponentially stable, it is sufficient that*

$$\Lambda + \kappa < 0.$$

Chapter 6

Stability Criteria by the First Approximation

We now describe the most famous stability criteria by the first approximation for the system

$$\frac{dx}{dt} = A(t)x + f(t,x), \quad t \geq 0, \quad x \in \mathbb{R}^n. \tag{6.1}$$

Here $A(t)$ is a continuous $n \times n$ matrix bounded for $t \geq 0$, and $f(t,x)$ is a continuous vector-function, satisfying in some neighborhood $\Omega(0)$ of the point $x = 0$ the condition

$$|f(t,x)| \leq \kappa |x|^\nu, \quad \forall t \geq 0, \quad \forall x \in \Omega(0). \tag{6.2}$$

Here κ and ν are certain positive numbers, $\nu \geq 1$.

We refer to

$$\frac{dx}{dt} = A(t)x \tag{6.3}$$

as the *system of the first approximation*. Suppose that there exist $C > 0$ and a piecewise continuous function $p(t)$ such that Cauchy matrix $X(t)X(\tau)^{-1}$ of (6.3) satisfies

$$|X(t)X(\tau)^{-1}| \leq C \exp \int_\tau^t p(s)\, ds, \quad \forall t \geq \tau \geq 0.$$

Theorem 6.1. *If condition (6.2) with $\nu = 1$ and the inequality*

$$\limsup_{t \to +\infty} \frac{1}{t} \int_0^t p(s)\, ds + C\kappa < 0$$

hold, then the solution $x(t) \equiv 0$ of (6.1) is asymptotically Lyapunov stable.

Theorem 6.1 shows that for the equation of the first order the negativity of the characteristic exponent of the system of the first approximation implies the asymptotic Lyapunov stability of the zero solution. (Here $\nu > 1$ or $\nu = 1$ and κ is sufficiently small.)

Let us now assume that $X(t)X(\tau)^{-1}$ satisfies

$$|X(t)X(\tau)^{-1}| \leq C \exp[-\alpha(t-\tau) + \gamma\tau], \quad \forall t \geq \tau \geq 0, \tag{6.4}$$

where $\alpha > 0$ and $\gamma \geq 0$. This estimate was mentioned in Chapters 3 and 5.

Theorem 6.2 (Persidskii [95, 109]). *If conditions (6.4) with $\gamma = 0$ and (6.2) with $\nu = 1$ and sufficiently small κ are valid, then the solution $x(t) \equiv 0$ of (6.1) is asymptotically Lyapunov stable.*

Theorem 6.2 results from Theorem 6.1 for $p(t) \equiv -\alpha$.

Theorem 6.3 (Chetaev [22], Malkin [20, 95], Massera [96]). *If conditions (6.4), (6.2), and the inequality*

$$(\nu - 1)\alpha - \gamma > 0 \tag{6.5}$$

hold, then the solution $x(t) \equiv 0$ of (6.1) is asymptotically Lyapunov stable.

Theorem 6.3 strengthens the well-known Lyapunov Theorem on the stability by the first approximation for regular systems [94]. To prove Theorems 6.1 and 6.3 we need the well-known lemmas of Bellman—Gronwall and Bihari [26].

Lemma 6.1 (Bellman—Gronwall). *Suppose $u(t)$ and $v(t)$ are nonnegative and continuous for $t \geq 0$, satisfying for some $C > 0$ the inequality*

$$u(t) \leq C + \int_0^t v(\tau)u(\tau)\,d\tau, \quad \forall t \geq 0. \tag{6.6}$$

Then the estimate

$$u(t) \leq C \exp\left[\int_0^t v(\tau)\,d\tau\right], \quad \forall t \geq 0, \tag{6.7}$$

is valid.

Lemma 6.2 (Bihari). *Suppose $u(t)$ and $v(t)$ are nonnegative and continuous for $t \geq 0$, and that the following hold for some $\nu > 1$ and $C > 0$:*

$$u(t) \leq C + \int_0^t v(\tau)[u(\tau)]^\nu\,d\tau, \quad \forall t \geq 0, \tag{6.8}$$

$$(\nu - 1)C^{\nu-1} \int_0^t v(\tau)\,d\tau < 1, \quad \forall t \geq 0. \tag{6.9}$$

Then

$$u(t) \le C \left[1 - (\nu - 1) C^{\nu - 1} \int_0^t v(\tau)\, d\tau \right]^{-1/(\nu-1)}, \quad \forall t \ge 0. \tag{6.10}$$

Proof of Lemmas 6.1 and 6.2. We introduce the function $\Phi(u) = u$ for Lemma 6.1 and $\Phi(u) = u^\nu$ for Lemma 6.2. Since Φ increases, by (6.6) and (6.8) we have

$$\Phi(u(t)) \le \Phi\left(C + \int_0^t v(\tau) \Phi(u(\tau))\, d\tau \right).$$

This implies the inequality

$$\frac{v(t)\Phi(u(t))}{\Phi\left(C + \int_0^t v(\tau)\Phi(u(\tau))\, d\tau \right)} \le v(t).$$

Denoting $w(t) = C + \int_0^t v(\tau)\Phi(u(\tau))\, d\tau$ and integrating the last inequality between 0 and t, we obtain the estimate

$$\int_0^t \frac{\dot{w}(s)}{\Phi(w(s))}\, ds \le \int_0^t v(s)\, ds.$$

The last inequality can be represented as

$$\frac{w(t)}{w(0)} \le \exp \int_0^t v(s)\, ds$$

for $\Phi(u) = u$ and as

$$\frac{1}{1-\nu}\left(\frac{1}{w(t)^{\nu-1}} - \frac{1}{w(0)^{\nu-1}} \right) \le \int_0^t v(s)\, ds$$

for $\Phi(u) = u^\nu$. Taking into account the condition $u(t) \le w(t)$, $w(0) = C$, and (6.9), we obtain the assertions of Lemmas 6.1 and 6.2. □

Proof of Theorem 6.1. Represent the solution of (6.1) in the form

$$x(t) = X(t)\left(x(0) + \int_0^t X(\tau)^{-1} f(\tau, x(\tau))\, d\tau \right). \tag{6.11}$$

By (6.11) and the hypotheses of theorem we have

$$|x(t)| \le C \exp \int_0^t p(s)\, ds |x(0)| + C \int_0^t \exp\left(\int_\tau^t p(s)\, ds \right) \kappa |x(\tau)|\, d\tau.$$

This estimate can be rewritten as

$$\exp\left(-\int_0^t p(s)\,ds\right)|x(t)| \le C|x(0)| + C\kappa\int_0^t \exp\left(-\int_0^\tau p(s)\,ds\right)|x(\tau)|\,d\tau.$$

By Lemma 6.1 we obtain

$$|x(t)| \le C|x(0)|\exp\left[\int_0^t p(s)\,ds + C\kappa t\right], \quad \forall t \ge 0,$$

which proves the theorem. □

Proof of Theorem 6.3. Relations (6.2), (6.4), and (6.11) yield the estimate

$$|x(t)| \le Ce^{-\alpha t}|x(0)| + C\int_0^t \exp[-\alpha(t-\tau)+\gamma\tau]\kappa|x(\tau)|^\nu\,d\tau.$$

This can be rewritten as

$$e^{\alpha t}|x(t)| \le C|x(0)| + C\kappa\int_0^t \exp[((1-\nu)\alpha+\gamma)\tau](e^{\alpha\tau}|x(\tau)|)^\nu\,d\tau.$$

By Lemma 6.2 we have

$$\begin{aligned}|x(t)| \le C|x(0)|\exp[-\alpha t][1-(\nu-1)(C|x(0)|)^{(\nu-1)}(C\kappa)\\ \times\int_0^t \exp[((1-\nu)\alpha+\gamma)\tau]\,d\tau]^{-1/(\nu-1)}, \quad \forall\, t \ge 0.\end{aligned}$$

From this inequality and condition (6.5), for $|x(0)|$ sufficiently small we obtain the assertion of the theorem. □

Consider a system

$$\frac{dx}{dt} = F(x,t), \quad t \ge 0, \quad x \in \mathbb{R}^n, \tag{6.12}$$

where $F(x,t)$ is a twice continuously differentiable vector-function. Suppose that for the solutions of system (6.12) with the initial data $y = x(0,y)$ from a certain domain Ω, the following condition is satisfied. The maximal singular value $\alpha_1(t,y)$ of the fundamental matrix $X(t,y)$ of the linear system

$$\frac{dz}{dt} = A(t)z \tag{6.13}$$

satisfies the inequality

$$\alpha_1(t,y) \le \alpha(t), \quad \forall t \ge 0, \quad \forall y \in \Omega. \tag{6.14}$$

Here

$$A(t) = \frac{\partial F(x,t)}{\partial x}\bigg|_{x=x(t,y)}$$

is the Jacobian matrix of the vector-function $F(x,t)$ on the solution $x(t,y)$, $X(0,y) = I$.

Theorem 6.4 [67]. *Let the function $\alpha(t)$ be bounded on the interval $(0,+\infty)$. Then the solution $x(t,y)$, $y \in \Omega$, is Lyapunov stable. If, in addition, we have*

$$\lim_{t\to+\infty} \alpha(t) = 0,$$

then the solution $x(t,y)$, $y \in \Omega$, is asymptotically Lyapunov stable.

Proof. It is well known that

$$\frac{\partial x(t,y)}{\partial y} = X(t,y), \quad \forall t \geq 0.$$

It is also well known [143] that for any vectors y, z and numbers $t \geq 0$ there exists a vector w such that

$$|w - y| \leq |y - z|$$

and

$$|x(t,y) - x(t,z)| \leq \left|\frac{\partial x(t,w)}{\partial w}\right| |y - z|.$$

Therefore for any vector z from the ball centered at y and entirely situated in Ω we have

$$|x(t,y) - x(t,z)| \leq |y - z| \sup \alpha_1(t,w) \leq \alpha(t)|y - z|, \quad \forall t \geq 0. \tag{6.15}$$

Here the supremum is taken for all w from the ball $\{w \mid |w - y| \leq |y - z|\}$. Theorem 6.4 is proved. □

Consider now the hypotheses of Theorem 6.4. The theorem establishes the asymptotic Lyapunov stability of solutions with the initial data from Ω if the corresponding equations (6.13) have negative Lyapunov exponents (or negative characteristic exponents). In this case the requirement that the negativity of Lyapunov exponents is uniform by Ω replaces the requirement in Theorem 6.3 that the coefficient of irregularity is small.

Thus, the Perron effects, considered in Chapter 4, are possible on the boundaries of the flow stable by the first approximation only.

Indeed, consider the flow of solutions of system (4.1) with the initial data in some neighborhood of the point $x_1 = x_2 = 0$: $x_1(0, x_{10}, x_{20}) = x_{10}$, $x_2(0, x_{10}, x_{20}) = x_{20}$. It follows easily that

$$x_1(t, x_{10}, x_{20}) = \exp[-at]x_{10}.$$

Therefore in this case the matrix $A(t)$ of system (6.13) takes the form

$$A(t) = \begin{pmatrix} -a & 0 \\ 2\exp[-at]x_{10} & [\sin(\ln(t+1)) + \cos(\ln(t+1)) - 2a] \end{pmatrix}.$$

The solutions of system (6.13) are

$$z_1(t) = \exp[-at]z_1(0),$$

$$z_2(t) = \exp[(t+1)\sin(\ln(t+1)) - 2at]\biggl(z_2(0) + 2z_1(0)x_{10} \times \int_0^t \exp[-(\tau+1)\sin(\ln(\tau+1))]\,d\tau\biggr).$$

In Chapter 4 was shown that if (4.2) and the relation

$$z_1(0)x_{10} \neq 0$$

are satisfied, then the characteristic exponent $z_2(t)$ is positive. Hence, in the arbitrary small neighborhood of the trivial solution $x_1(t) \equiv x_2(t) \equiv 0$ there exists the initial condition x_{10}, x_{20} such that for $x_1(t, x_{10}, x_{20})$, $x_2(t, x_{10}, x_{20})$ the system of the first approximation has a positive higher characteristic exponent (and the Lyapunov exponent μ_1 too).

Thus, for system (4.1) the Perron effect is due to the fact that the uniform by Ω estimate (6.14) is lacking. Here Ω is a certain domain involving the point $x_1 = x_2 = 0$.

Chapter 7

Instability Criteria

Consider a system

$$\frac{dx}{dt} = A(t)x + f(t,x), \quad t \geq 0, \quad x \in \mathbb{R}^n, \tag{7.1}$$

where the $n \times n$ matrix $A(t)$ is continuous and bounded on $[0,\infty)$. We assume that the vector-function $f(t,x)$ is continuous and in some neighborhood $\Omega(0)$ of the point $x = 0$ the inequality

$$|f(t,x)| \leq \kappa |x|^\nu, \quad \forall t \geq 0, \quad \forall x \in \Omega(0), \tag{7.2}$$

holds. Here $\kappa > 0$ and $\nu > 1$.

Consider the normal fundamental matrix

$$Z(t) = (z_1(t), \dots, z_n(t)), \tag{7.3}$$

consisting of the linearly independent solutions $z_j(t)$ of the following linear system of the first approximation:

$$\frac{dz}{dt} = A(t)z. \tag{7.4}$$

A basic approach for instability analysis is the reduction of the linear part of system (7.1) to triangular form. In this case the technique of Perron—Vinograd of the triangulation of the linear system [20, 26] turns out to be the most effective. We describe this method here.

7.1 The Perron—Vinograd Triangulation Method

We apply the Gram—Schmidt orthogonalization procedure to the solutions $z_j(t)$ making up matrix (7.3). Then we obtain

$$\begin{aligned} v_1(t) &= z_1(t), \\ v_2(t) &= z_2(t) - v_1(t)^* z_2(t) \frac{v_1(t)}{|v_1(t)|^2}, \\ &\vdots \\ v_n(t) &= z_n(t) - v_1(t)^* z_n(t) \frac{v_1(t)}{|v_1(t)|^2} - \cdots - v_{n-1}(t)^* z_n(t) \frac{v_{n-1}(t)}{|v_{n-1}(t)|^2}. \end{aligned} \tag{7.5}$$

Relations (7.5) yield at once

$$v_i(t)^* v_j(t) = 0, \quad \forall j \neq i, \tag{7.6}$$

and

$$|v_j(t)|^2 = v_j(t)^* z_j(t). \tag{7.7}$$

Relation (7.7) implies at once the following

Lemma 7.1. *The estimate*

$$|v_j(t)| \leq |z_j(t)|, \quad \forall t \geq 0, \tag{7.8}$$

is valid.

We might ask how far the vector-function $v_j(t)$ can decrease in comparison with the original system of vectors $z_j(t)$. The answer is the following

Lemma 7.2. *If for some C the inequality*

$$\prod_{j=1}^{n} |z_j(t)| \leq C \exp \int_0^t \operatorname{Tr} A(s)\, ds, \quad \forall\, t \geq 0, \tag{7.9}$$

holds, then there exists $r > 0$ such that

$$|z_j(t)| \leq r |v_j(t)|, \quad \forall t \geq 0, \quad j = 1, \ldots, n. \tag{7.10}$$

Proof. By the Ostrogradsky—Liouville formula [26] and inequality (7.9) we have

$$\begin{aligned}&\left|\det\left(\frac{z_1(t)}{|z_1(t)|},\dots,\frac{z_n(t)}{|z_n(t)|}\right)\right|\\ &\quad=(|z_1(t)|,\dots,|z_n(t)|)^{-1}|\det(z_1(0),\dots,z_n(0))|\exp\int_0^t \operatorname{Tr} A(s)\,ds\\ &\quad\geq C^{-1}|\det(z_1(0),\dots,z_n(0))|,\quad \forall\, t\geq 0.\end{aligned}$$

This implies that for the linear subspace $L(t)$ spanned by the vectors $z_1(t),\dots,z_m(t)$ $(m<n)$, the number $\varepsilon\in(0,1)$ can be found such that the estimate

$$\frac{|z_{m+1}(t)^* e(t)|}{|z_{m+1}(t)|}\leq 1-\varepsilon,\quad \forall t\geq 0, \tag{7.11}$$

is valid for all $e(t)\in L(t)$ such that $|e(t)|=1$. Relations (7.5) can be rewritten as

$$\frac{v_j(t)}{|z_j(t)|}=\prod_{i=1}^{j-1}\left(I-\frac{v_i(t)v_i(t)^*}{|v_i(t)|^2}\right)\frac{z_j(t)}{|z_j(t)|}. \tag{7.12}$$

Suppose the lemma is false. Then there exists the sequence $t_k\to+\infty$ such that

$$\lim_{k\to+\infty}\frac{v_j(t_k)}{|z_j(t_k)|}=0.$$

But by (7.12) there exists $l<j$ such that

$$\lim_{k\to+\infty}\left[\frac{z_j(t_k)}{|z_j(t_k)|}-\frac{v_l(t_k)}{|v_l(t_k)|}\right]=0. \tag{7.13}$$

Since $v_l(t)\in L(t)$, relations (7.11) and (7.13) are incompatible. This contradiction proves (7.10). □

It follows easily that condition (7.9) is necessary and sufficient for the existence of $r>0$ such that estimate (7.10) is satisfied.

Note that (7.9) is necessary and sufficient for the nondegeneracy, as $t\to+\infty$, of the normalized fundamental matrix of system of the first approximation (7.4):

$$\liminf_{t\to+\infty}\left|\det\left(\frac{z_1(t)}{|z_1(t)|},\dots,\frac{z_n(t)}{|z_n(t)|}\right)\right|>0.$$

Now we proceed to describe *the triangulation procedure of Perron—Vinograd.* Consider the unitary matrix

$$U(t) = \left(\frac{v_1(t)}{|v_1(t)|}, \ldots, \frac{v_n(t)}{|v_n(t)|}\right),$$

and perform the change of variable $z = U(t)w$ in system (7.4). The unitary nature of $U(t)$ implies that for the columns $w(t)$ of $W(t) = (w_1(t), \ldots, w_n(t)) = U(t)^* Z(t)$ the relations $|w_j(t)| = |z_j(t)|$ are valid. By (7.5)–(7.7) the matrix $W(t)$ has the upper triangular form

$$W(t) = \begin{pmatrix} |v_1(t)| & \cdots & \\ & \ddots & \vdots \\ 0 & & |v_n(t)| \end{pmatrix}. \tag{7.14}$$

The matrix $W(t)$ is fundamental for the system

$$\frac{dw}{dt} = B(t)w, \tag{7.15}$$

where

$$B(t) = U(t)^{-1}A(t)U(t) - U(t)^{-1}\dot{U}(t). \tag{7.16}$$

Since $W(t)$ is upper triangular, so are $W(t)^{-1}$ and $\dot{W}(t)$. Therefore

$$B(t) = \dot{W}(t)W(t)^{-1}$$

is an upper triangular matrix of the type

$$B(t) = \begin{pmatrix} (\ln|v_1(t)|)^\bullet & \cdots & \\ & \ddots & \vdots \\ 0 & & (\ln|v_n(t)|)^\bullet \end{pmatrix}. \tag{7.17}$$

We shall show that if $A(t)$ is bounded for $t \geq 0$, then so are $B(t)$, $U(t)$, and $\dot{U}(t)$. The boundedness of $U(t)$ always occurs and is obvious. Hence $U(t)^{-1}A(t)U(t) = U(t)^*A(t)U(t)$ is also bounded. The unitary nature of $U(t)$ implies that

$$\left(U(t)^{-1}\dot{U}(t)\right)^* = -U(t)^{-1}\dot{U}(t). \tag{7.18}$$

From this, (7.16), and (7.17), we conclude that the modulus of an element of $U(t)^{-1}\dot{U}(t)$ coincides with that of a certain element of $U(t)^{-1}A(t)U(t)$. Thus $U(t)^{-1}\dot{U}(t)$ is bounded for $t \geq 0$. By (7.16) we see that $B(t)$ is bounded. This and the relation $\dot{U}(t) = AU(t) - U(t)B(t)$ imply the boundedness of $\dot{U}(t)$. We have therefore proved

Theorem 7.1 [26] (Perron, on triangulation of linear system). *By the unitary transformation $z = U(t)w$, the system (7.4) can be reduced to system (7.15) with the matrix $B(t)$ of the type (7.17). If the matrix $A(t)$ is bounded for $t \geq 0$, then so are the matrices $B(t)$, $U(t)$, and $\dot{U}(t)$.*

We now obtain one more useful estimate for the vector-function $v_n(t)$.

Lemma 7.3. *The estimate*

$$\frac{|v_n(t)|}{|v_n(\tau)|} \geq \exp\left[\int_\tau^t \operatorname{Tr} A(s)\,ds\right] \prod_{j=1}^{n-1} \frac{|v_j(\tau)|}{|z_j(t)|} \tag{7.19}$$

is valid.

Proof. By (7.14) we have

$$\frac{|v_n(t)|}{|v_n(\tau)|} = \frac{\det W(t) \prod\limits_{j=1}^{n-1} |v_j(\tau)|}{\det W(\tau) \prod\limits_{j=1}^{n-1} |v_j(t)|}.$$

From the Ostrogradsky—Liouville formula, (7.15), (7.16), and (7.18) it follows that

$$\det W(t) = \det W(\tau) \int_\tau^t \operatorname{Tr} B(s)\,ds = \det W(\tau) \exp \int_\tau^t \operatorname{Tr} A(s)\,ds.$$

These and estimate (7.8) imply at once the assertion of the lemma. □

7.2 Theorems on Instability

Theorem 7.2 [77]. *If*

$$\sup_k \liminf_{t\to+\infty} \left[\frac{1}{t}\left(\int_0^t \operatorname{Tr} A(s)ds - \sum_{j\neq k} \ln|z_j(t)|\right)\right] > 0, \tag{7.20}$$

then the solution $x(t) \equiv 0$ of system (7.1) is Krasovsky unstable.

Proof. Without loss of generality we can assume that in (7.20) the supremum over k is attained for $k = n$. Performing the change of variables

$x = U(t)y$ in (7.1) we obtain

$$\frac{dy}{dt} = B(t)y + g(t, y), \tag{7.21}$$

where $B(t)$ is defined by (7.16) and

$$g(t, y) = U(t)^{-1} f(t, U(t)y).$$

Thus, the last equation of system (7.21) takes the form

$$\dot{y}_n = (\ln |v_n(t)|)^{\bullet} y_n + g_n(t, y). \tag{7.22}$$

Here y_n and g_n are the nth components of the vectors y and g, respectively.

Suppose now that the solution $x(t) \equiv 0$ is Krasovsky stable. This means that for a certain neighborhood of $x = 0$ there exists $R > 0$ such that

$$|x(t, x_0)| \le R|x_0|, \quad \forall t \ge 0. \tag{7.23}$$

Here $x(0, x_0) = x_0$. Conditions (7.2) and (7.23) imply that

$$|g(t, y(t))| \le \kappa R^{\nu} |y(0)|^{\nu}. \tag{7.24}$$

By Lemma 7.3 from (7.20) we see that there exists $\mu > 0$ such that for sufficiently large t,

$$\ln |v_n(t)| \ge \mu t. \tag{7.25}$$

Note that the solution $y_n(t)$ of (7.22) can be represented as

$$y_n(t) = \frac{|v_n(t)|}{|v_n(0)|} \left(y_n(0) + \int_0^t \frac{|v_n(0)|}{|v_n(s)|} g(s, y(s)) ds \right). \tag{7.26}$$

Estimate (7.25) implies that there exists $\rho > 0$ such that

$$\int_0^t \frac{|v_n(0)|}{|v_n(s)|} ds \le \rho, \quad \forall t \ge 0. \tag{7.27}$$

Now we take the initial condition $x_0 = U(0)y(0)$ in such a way that $y_n(0) = |y(0)| = \delta$ where

$$\delta > \rho \kappa R^{\nu} \delta^{\nu}.$$

Then from (7.24)–(7.27) we obtain for sufficiently large $t \ge 0$ the estimate

$$y_n(t) \ge \exp(\mu t)(\delta - \rho \kappa R^{\nu} \delta^{\nu}).$$

This implies at once the relation

$$\liminf_{t\to+\infty} y_n(t) = +\infty.$$

The latter contradicts the hypothesis on Krasovsky stability of a trivial solution to the system (7.1). This proves the theorem. □

Remark concerning the technique of proving Theorem 7.2. If we assume Lyapunov stability and attempt to proceed similarly, we must prove that

$$y_n(0) + \int_0^{+\infty} \frac{|v_n(0)|}{|v_n(s)|} g(s, y(s))\, ds \neq 0. \tag{7.28}$$

While this inequality is easily established for the case of Krasovsky stability, it becomes an intractable problem for the case of Lyapunov stability.

A similar scheme of reducing the problem to a single scalar equation of the type (7.22) was used by N.G. Chetaev [22, 23] in order to obtain instability criteria. A similar difficulty arises regarding the proof of (7.28). Therefore, at present, Chetaev's technique permits us to obtain criteria for Krasovsky instability.

We must seek criteria for Lyapunov instability by a different approach. It will be presented in Theorem 7.4 through the use of certain additional restrictions. □

Condition (7.20) of Theorem 7.2 is satisfied if

$$\Lambda - \Gamma > 0. \tag{7.29}$$

Here Λ is a maximal characteristic exponent and Γ is a coefficient of irregularity.

The condition for Krasovsky instability (7.29) was obtained by Chetaev [22, 23] under the additional requirement of the analyticity of $f(t, x)$.

Recall the stability condition (6.5) of Theorem 6.3. By Theorem 3.3 it can be represented as

$$(\nu - 1)\Lambda + \Gamma < 0. \tag{7.30}$$

Since Theorems 6.1–6.3 also yield Krasovsky stability, we can formulate the following

Theorem 7.3. *If*

$$\Lambda < \frac{-\Gamma}{(\nu - 1)},$$

then the solution $x(t) \equiv 0$ *is Krasovsky stable. If*

$$\Lambda > \Gamma,$$

it is Krasovsky unstable.

For regular systems (in the case $\Gamma = 0$) Theorem 7.3 completely solved the problem of Krasovsky stability in the noncritical case ($\Lambda \neq 0$).

Note that for system (4.6) the relation $\Gamma = \Lambda + 2a + 1$ holds. Therefore for system (4.6) condition (7.29) is not satisfied.

Theorem 7.4 [81]. *Assume that for some numbers* $C > 0$, $\beta > 0$, *and* $\alpha_j < \beta$ $(j = 1, \dots, n-1)$ *the following conditions are valid:*

(1) for $n > 2$

$$\prod_{j=1}^{n} |z_j(t)| \leq C \exp \int_0^t \operatorname{Tr} A(s)\, ds, \quad \forall t \geq 0 \tag{7.31}$$

(2)

$$\begin{gathered} |z_j(t)| \leq C \exp(\alpha_j (t-\tau)) |z_j(\tau)|, \\ \forall\, t \geq \tau \geq 0, \quad \forall j = 1, \dots, n-1 \end{gathered} \tag{7.32}$$

(3)

$$\frac{1}{(t-\tau)} \int_\tau^t \operatorname{Tr} A(s)\, ds > \beta + \sum_{j=1}^{n-1} \alpha_j, \quad \forall\, t \geq \tau \geq 0. \tag{7.33}$$

Then the zero solution of system (7.1) is Lyapunov unstable.

Proof. Recall that the fundamental matrix $W(t)$ of system (7.15) has the form (7.14). The vector-columns of this matrix $w_j(t)$ are

$$w_1(t) = \begin{pmatrix} w_{11}(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad w_{n-1}(t) = \begin{pmatrix} w_{n-1,1}(t) \\ \vdots \\ w_{n-1,n-1}(t) \\ 0 \end{pmatrix}.$$

Therefore

$$\widetilde{W}(t) = \begin{pmatrix} w_{11}(t) & \cdots & w_{n-1,1}(t) \\ 0 & \cdots & \\ \vdots & & \vdots \\ 0 & \cdots & w_{n-1,n-1}(t) \end{pmatrix}$$

is the fundamental matrix of the system

$$\dot{\widetilde{w}} = \widetilde{B}(t)\widetilde{w}, \quad \widetilde{w} \in \mathbb{R}^{n-1}$$

having

$$\widetilde{B}(t) = \begin{pmatrix} (\ln|v_1(t)|)^\bullet & \cdots & \\ & \ddots & \vdots \\ 0 & & (\ln|v_{n-1}(t)|)^\bullet \end{pmatrix}.$$

From (7.31) and the identities $|w_j(t)| \equiv |z_j(t)|$, which resulted from the unitary nature of $U(t)$, we have the estimate

$$|\widetilde{w}_j(t)| \le C\exp(\alpha_j(t-\tau))|\widetilde{w}_j(\tau)|, \\ \forall t \ge \tau \ge 0, \quad \forall j = 1, \dots, n-1. \tag{7.34}$$

Besides, by Lemma 7.2 from condition (7.31) we have estimate (7.10) and by Lemma 7.3 from conditions (7.32) and (7.33) we have the estimate

$$\frac{|v_n(t)|}{|v_n(\tau)|} \ge \exp(\beta(t-\tau))C^{1-n}\prod_{j=1}^{n-1}\frac{|v_j(\tau)|}{|z_j(\tau)|} \qquad \forall\, t \ge \tau \ge 0. \tag{7.35}$$

Relations (7.10) and (7.35) yield the inequality

$$\frac{|v_n(t)|}{|v_n(\tau)|} \ge \exp(\beta(t-\tau))(Cr)^{1-n}, \quad \forall t \ge \tau \ge 0. \tag{7.36}$$

Since for $n = 2$, $v_1(t) = z_1(t)$, from (7.35) we obtain the estimate

$$\frac{|v_2(t)|}{|v_2(\tau)|} \ge C^{-1}\exp(\beta(t-\tau)), \quad \forall t \ge \tau \ge 0,$$

without assumption (7.31).

Now we have performed the following change of variables

$$x = e^{dt}U(t)y. \tag{7.37}$$

Here $d > 0$ is chosen so that

$$\alpha < d < \beta,$$

where $\alpha = \max \alpha_j$, $j = 1, \dots, n-1$. As a result we obtain

$$\frac{dy}{dt} = (B(t) - dI)y + g(t,y), \tag{7.38}$$

where

$$g(t,y) = e^{-dt}U(t)^{-1}f(t, e^{dt}U(t)y).$$

From (7.2) it follows that for any $\rho > 0$ there exists a neighborhood $\Phi(0)$ of the point $y = 0$ such that

$$|g(t,y)| \le \rho|y|, \quad \forall t \ge 0, \quad \forall y \in \Phi(0). \tag{7.39}$$

Note that by relation (7.34) for the system

$$\dot{\widetilde{y}} = (\widetilde{B}(t) - dI)\widetilde{y}, \quad \widetilde{y} \in \mathbb{R}^{n-1}, \tag{7.40}$$

we have the estimate

$$|\widetilde{y}(t)| \le C \exp[(\alpha - d)(t-\tau)]|\widetilde{y}(\tau)|, \quad \forall t \ge \tau.$$

Then by Theorem 5.1 there exists a matrix $H(t)$, continuously differentiable and bounded on $[0, +\infty)$, along with positive numbers ρ_1 and ρ_2 such that

$$\widetilde{y}^*(\dot{H}(t) + 2H(\widetilde{B}(t) - dI))\widetilde{y} \le -\rho_1|\widetilde{y}|^2, \quad \forall \widetilde{y} \in \mathbb{R}^{n-1}, \quad \forall t \ge 0 \tag{7.41}$$

$$\widetilde{y}^* H(t)\widetilde{y} \ge \rho_2|\widetilde{y}|^2, \quad \forall \widetilde{y} \in \mathbb{R}^{n-1}, \quad \forall t \ge 0. \tag{7.42}$$

By (7.36) for the scalar equation

$$\dot{y}_n = [\ln|v_n(t)|)^\bullet - d]y_n, \quad y_n \in \mathbb{R}^1,$$

in the case $n \ne 2$ we have the estimate

$$|y_n(t)| \ge (Cr)^{1-n} \exp[(\beta - d)(t-\tau)]|y_n(\tau)|, \quad \forall t \ge \tau \ge 0.$$

For $n = 2$ a similar estimate takes the form

$$|y_2(t)| \ge C^{-1} \exp[(\beta - d)(t-\tau)]|y_2(\tau)|, \quad \forall t \ge \tau \ge 0.$$

Therefore by Corollary 5.1 there exists a function $h(t)$, continuously differentiable and bounded on $[0, +\infty)$, along with positive numbers ρ_3 and ρ_4 such that

$$\begin{gathered} \dot{h}(t) + 2h(t)\left[(\ln|v_n(t)|)^\bullet - d\right] \le -\rho_3, \\ h(t) \le -\rho_4, \quad \forall t \ge 0. \end{gathered} \tag{7.43}$$

We now show that for sufficiently large ω the function

$$V(t,y) = \widetilde{y}^* H(t)\widetilde{y} + \omega h(t) y_n^2$$

is the Lyapunov function, satisfying for system (7.38) all conditions of Lyapunov's classical theorem on instability. Indeed, (7.38) can be represented as

$$\begin{aligned} \dot{\widetilde{y}} &= (\widetilde{B}(t) - dI)\widetilde{y} + q(t)y_n + \widetilde{g}(t, \widetilde{y}, y_n), \\ \dot{y}_n &= ((\ln|v_n(t)|)^\bullet - d)y_n + g_n(t, \widetilde{y}, y_n), \end{aligned} \tag{7.44}$$

where vector-function $q(t)$ is bounded on $[0, +\infty)$ and $\widetilde{g}$ and g_n are such that

$$g(t, y) = \begin{pmatrix} \widetilde{g}(t, y) \\ g_n(t, y) \end{pmatrix}.$$

Therefore estimates (7.41), (7.43) imply the inequalities

$$\begin{aligned} \dot{V}(t, y) \le &-\rho_1|\widetilde{y}|^2 - \omega\rho_3 y_n^2 + 2\widetilde{y}^* H(t)q(t)y_n \\ &+ 2\widetilde{y}^* H(t)\widetilde{g}(t, \widetilde{y}, y_n) + 2\omega h(t)y_n g_n(t, \widetilde{y}, y_n) \\ \le &-\rho_1|\widetilde{y}|^2 - \omega\rho_3 y_n^2 + 2[(|y_n||\widetilde{y}| \sup_t |H(t)|q(t)| \\ &+ |\widetilde{y}| \sup_t |H(t)|\rho(|\widetilde{y}| + |y_n|) + \omega|y_n| \sup_t |h(t)|\rho|(|\widetilde{y}| + |y_n|)]. \end{aligned}$$

From this and the boundedness of the matrix-function $H(t)$, the vector-function $q(t)$, and the function $h(t)$, it follows that for sufficiently large ω and sufficiently small ρ a positive number θ can be found such that

$$\dot{V}(t, y) \le -\theta|y|^2. \tag{7.45}$$

The boundedness of $H(t)$ and $h(t)$ implies that there exists a such that

$$|y|^2 \ge -aV(t, y), \quad \forall t \ge 0, \quad \forall y \in \mathbb{R}^n.$$

By (7.45) we have

$$\dot{V}(t, y) \le a\theta V(t, y), \quad \forall t \ge 0, \quad \forall y \in \mathbb{R}^n. \tag{7.46}$$

Now choose the initial condition $y(0)$ so that $V(0, y(0)) < 0$. Then by (7.45) we have

$$V(t, y(t)) < 0, \quad \forall\, t \ge 0,$$

and by (7.46) obtain

$$-V(t, y(t)) \ge e^{a\theta t}(-V(0, y(0))).$$

From this and inequalities (7.42) and (7.43) we have the estimate

$$-\omega h(t) y_n(t)^2 \geq e^{a\theta t}(-V(0, y(0))), \quad \forall t \geq 0.$$

Thus,

$$y_n(t)^2 \geq \frac{e^{a\theta t}}{\omega \sup\limits_t (-h(t))}(-V(0, y(0))). \tag{7.47}$$

This inequality results in Lyapunov instability of the solution $y(t) \equiv 0$. In addition, (7.47) implies that in the neighborhood of $y = 0$ the increase of the solution $y(t)$ with the initial data $V(0, y(0)) < 0$ has an exponential form.

Since $d > 0$ and $U(t)$ is unitary, the zero solution of system (7.1) is also Lyapunov unstable. □

The natural question arises whether the instability conditions due to Theorems 7.2 and 7.4 can be weakened. The answer is that the Perron effects impose restrictions on such weakening.

Let us reconsider the ensemble of solutions $x(t, t_0, x_0)$ of the system

$$\frac{dx}{dt} = F(x, t), \quad t \geq 0, \quad x \in \mathbb{R}^n, \tag{7.48}$$

where $F(x, t)$ is a continuously differentiable function (cf., the end of Chapter 6). Here $x_0 \in \Omega$, where Ω is a certain bounded open set in $\mathbb{R}^n$, and t_0 is a certain fixed nonnegative number.

Assume that for the fundamental matrix $X(t, t_0, x_0)$ of the system

$$\frac{dz}{dt} = \left(\left. \frac{\partial F(x, t)}{\partial x} \right|_{x = x(t, t_0, x_0)} \right) z \tag{7.49}$$

with the initial data $X(t_0, t_0, x_0) = I$ and a certain vector-function $\xi(t)$ the relations

$$|\xi(t)| = 1, \quad \inf_{\Omega} |X(t, t_0, x_0)\xi(t)| \geq \alpha(t), \quad \forall t \geq t_0, \tag{7.50}$$

are valid.

Theorem 7.5 [67]. *Suppose that the function $\alpha(t)$ satisfies*

$$\limsup_{t \to +\infty} \alpha(t) = +\infty. \tag{7.51}$$

Then any solution $x(t, t_0, x_0)$ with the initial data $x_0 \in \Omega$ is Lyapunov unstable.

Proof. Holding a certain pair $x_0 \in \Omega$ and $t \geq t_0$ fixed, we choose in any δ-neighborhood of the point x_0 the vector y_0 in such a way that

$$x_0 - y_0 = \delta\xi(t). \tag{7.52}$$

Let δ be so small that the ball of radius δ centered in x_0 is wholly situated in Ω.

It is well known [144] that for any fixed numbers t, j and the vectors x_0, y_0 there is a vector $w_j \in \mathbb{R}^n$ such that

$$|x_0 - w_j| \leq |x_0 - y_0|,$$

$$x_j(t, t_0, x_0) - x_j(t, t_0, y_0) = X_j(t, t_0, w_j)(x_0 - y_0). \tag{7.53}$$

Here $x_j(t, t_0, x_0)$ is the jth component of the vector-function $x(t, t_0, x_0)$, and $X_j(t, t_0, w)$ is the jth row of the matrix $X(t, t_0, w)$. By (7.53) we obtain the estimate

$$\begin{aligned}
|x(t, t_0, x_0) - x(t, t_0, y_0)| &= \sqrt{\sum_j |X_j(t, t_0, w_j)(x_0 - y_0)|^2} \\
&\geq \delta \max\{|X_1(t, t_0, w_1)\xi(t)|, \ldots, |X_n(t, t_0, w_n)\xi(t)|\} \\
&\geq \delta \max_j \inf_\Omega |X_j(t, t_0, x_0)\xi(t)| \\
&= \delta \inf_\Omega \max_j |X_j(t, t_0, x_0)\xi(t)| \\
&\geq \frac{\delta}{\sqrt{n}} \inf_\Omega |X(t, t_0, x_0)\xi(t)| \geq \frac{\alpha(t)\delta}{\sqrt{n}}.
\end{aligned}$$

This and condition (7.51) imply that for any positive ε and δ there exists $t \geq t_0$ and a vector y_0 such that

$$|x_0 - y_0| = \delta, \quad |x(t, t_0, x_0) - x(t, t_0, y_0| > \varepsilon.$$

Hence the solution $x(t, t_0, x_0)$ is Lyapunov unstable. □

The hypotheses of Theorem 7.5 require that at least one Lyapunov exponent of the linearization of the flow of solutions with the initial data from Ω is positive; moreover, the "unstable directions $\xi(t)$" (or the unstable manifolds) of these solutions must depend continuously on the initial data x_0. Actually if this property occurs, then regarding (if necessary) the domain Ω as the union of the domains Ω_i of arbitrary small diameter on which condition (7.50) and (7.51) are valid, we obtain the Lyapunov instability of the whole flow of solutions with the initial data from Ω.

Next, we apply Theorem 7.5 to the system (4.5). For the solutions $x(t, t_0, x_0)$ of (4.5) with the initial data $t_0 = 0$,

$$\begin{aligned} x_1(0, x_{10}, x_{20}, x_{30}) &= x_{10}, \\ x_2(0, x_{10}, x_{20}, x_{30}) &= x_{20}, \\ x_3(0, x_{10}, x_{20}, x_{30}) &= x_{30}, \end{aligned}$$

we have

$$x_1(t, x_{10}, x_{20}, x_{30}) = \exp[-at]x_{10},$$

$$\frac{\partial F(x,t)}{\partial x}\bigg|_{x=x(t,0,x_0)} = \begin{pmatrix} -a & 0 & 0 \\ 0 & -2a & 0 \\ -2\exp[-at]x_0 & 1 & r(t) \end{pmatrix}.$$

Here

$$r(t) = \sin(\ln(t+1)) + \cos(\ln(t+1)) - 2a.$$

The solutions of (7.49) are

$$\begin{aligned} z_1(t) &= \exp[-at]z_1(0), \\ z_2(t) &= \exp[-2at]z_2(0), \\ z_3(t) &= p(t)(z_3(0) + (z_2(0) - 2x_{10}z_1(0))q(t)), \end{aligned} \tag{7.54}$$

where

$$\begin{aligned} p(t) &= \exp[(t+1)\sin(\ln(t+1)) - 2at], \\ q(t) &= \int_0^t \exp[-(\tau+1)\sin(\ln(\tau+1))]\, d\tau. \end{aligned}$$

By (7.54)

$$X(t, 0, x_0) = \begin{pmatrix} \exp[-at] & 0 & 0 \\ 0 & \exp[-2at] & 0 \\ -2x_{10}p(t)q(t) & p(t)q(t) & p(t) \end{pmatrix}.$$

If we put

$$\xi(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

then for $\Omega = \mathbb{R}^n$ and

$$\alpha(t) = \sqrt{\exp[-4at] + (p(t)q(t))^2}$$

relations (7.50) and (7.51) are satisfied (see estimate (4.4)). Thus, by Theorem 7.5 any solution of system (4.5) is Lyapunov unstable.

More narrowly, consider the manifold

$$M = \{x_3 \in \mathbb{R}^1, \quad x_2 = x_1^2\}.$$

In this case the initial data of the unperturbed solution x_0 and the perturbed solution y_0 belong to the manifold M:

$$x_0 \in M, \quad y_0 \in M. \tag{7.55}$$

From the proof of Theorem 7.5 (see (7.52)) we can conclude that here on the vector-function $\xi(t)$ the following additional requirement is imposed. If (7.52) and (7.55) are satisfied, then from the inequality $\xi_2(t) \neq 0$ we have the relation $\xi_1(t) \neq 0$. In this case (7.50) and (7.51) are not satisfied since for either $2x_{10}\xi_1(t) = \xi_2(t) \neq 0$ or $\xi_2(t) = 0$ the value

$$|X(t, 0, x_0)\xi(t)|$$

is bounded on $[0, +\infty)$. Thus, for system (4.5) on the set M the Perron effects are possible because of the lack of uniformity with respect to x_0 in conditions (7.50) and (7.51) under certain additional restrictions on the vector-function $\xi(t)$. □

7.3 Conclusion

Let us summarize the investigations of stability by the first approximation, considered in Chapters 4–7.

Theorems 6.4 and 7.5 give a complete solution to the problem for the flows of solutions in the noncritical case when for small variations of the initial data of the original system, the system of the first approximation preserves its stability (or the instability in the certain "direction" $\xi(t)$).

Thus, here the classical problem on the stability by the first approximation of time-varying motions is completely proved in the generic case ([95], p. 370).

The Perron effects, described in Chapter 4, are possible on the boundaries of flows that are either stable or unstable by the first approximation only. From this point of view here we have a nongeneric case.

Progress in the generic case became possible since the theorem on finite increments permits us to reduce the estimate of the difference between perturbed and unperturbed solutions to the analysis of the system of the first approximation, linearized along a certain "third" solution of the original system. Such an approach renders the proof of the theorem "almost obvious".

Thus, the difficulties in studying the individual solutions are due to the fact that they can situate on the boundaries of flows that are stable (or unstable) by the first approximation. In this case a nongeneric situation results; here it is necessary to develop finer (and, naturally, more complicated) tools for investigation. Such techniques for investigating the individual solutions are stated in Chapters 5–7.

Note that all results obtained in Chapters 3–7 can be rewritten for discrete systems. The main distinction is that the fundamental matrix $X(t)$ of the linear system

$$X(t+1) = A(t)X(t),$$

which is represented as

$$X(t) = \prod_{j=0}^{t-1} A(j)X(0),$$

can be degenerate.

Therefore in the discrete case one usually assumes that for some $\varepsilon > 0$ the inequality

$$\left| \prod_{j=0}^{t} \det A(j) \right| \geq \varepsilon, \quad \forall\, t = 0, 1, \dots,$$

holds. Extensions of the results of Chapters 3–7 on discrete systems can be found in [48–50].

Chapter 8

Stability in the Large

Here the methodology developed for proving Theorem 6.4 is applied to the global analysis of nonlinear nonautonomous systems.

Consider the systems

$$\frac{dx}{dt} = F(x,t), \qquad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^1, \tag{8.1}$$

and

$$x(t+1) = F(x(t),t), \qquad x \in \mathbb{R}^n, \quad t \in \mathcal{Z}, \tag{8.2}$$

where the vector-function $F(x,t)$ is continuously differentiable in x. When considering (8.1), we shall assume that $F(x,t)$ is continuous in t. Denote by

$$A(x,t) = \frac{\partial F(x,t)}{\partial x}$$

the Jacobian matrix of $F(x,t)$ at the point $x \in \mathbb{R}^n$.

Consider the solutions of (8.1) and (8.2) with the initial data $x(t,t_0,x_0) = x_0$. Suppose they are defined for all $t \geq t_0$. Take t_0 to be some fixed number.

Introduce the following linearization of systems (8.1) and (8.2) along the solutions $x(t,t_0,x_0)$:

$$\dot{z} = A(x(t,t_0,x_0),t)z, \qquad z \in \mathbb{R}^n, \quad t \in \mathbb{R}^1, \quad t \geq t_0, \tag{8.3}$$

$$z(t+1) = A(x(t,t_0,x_0),t)z(t), \qquad z \in \mathbb{R}^n, \quad t \in \mathcal{Z}, \quad t \geq t_0. \tag{8.4}$$

Denote by $Z(t,x_0)$ the fundamental matrix of system (8.3), satisfying the condition $Z(t_0,x_0) = I$ where I is the $n \times n$ unit matrix. The fundamental matrix $Z(t, x_0)$ for (8.4) is introduced in the following way: $Z(t_0, x_0) = I$,

$$Z(t, x_0) = \prod_{j=t_0}^{t-1} A(x(j, t_0, x_0), j), \qquad t > t_0 + 1.$$

Lemma 8.1. *For the solutions $x(t, t_0, x_0)$ and $x(t, t_0, y_0)$ of (8.1) (or (8.2)) the estimate*

$$|x(t, t_0, x_0) - x(t, t_0, y_0)| \le \sup_{v \in B} |Z(t, v)||x_0 - y_0|, \quad \forall t \ge t_0, \tag{8.5}$$

holds, where $B = \{v \mid |x_0 - v| \le |x_0 - y_0|\}$.

Proof. First consider (8.1). The assumptions imposed on $F(x, t)$ imply [38] differentiability of the solutions $x(t, t_0, x_0)$ with respect to the initial data x_0. Differentiating in x_0 the left- and right-hand sides of the equation

$$\frac{dx(t, t_0, x_0)}{dt} = F(x(t, t_0, x_0), t),$$

we obtain

$$\frac{d}{dt} \frac{\partial x(t, t_0, x_0)}{\partial x_0} = A(x(t, t_0, x_0), t) \frac{\partial x(t, t_0, x_0)}{\partial x_0}.$$

Hence

$$\frac{\partial x(t, t_0, x_0)}{\partial x_0} = Z(t, x_0). \tag{8.6}$$

By the theorem on finite increments [144] for the fixed $t \ge t_0$ we know that there exists $v = v(t) \in B$ such that

$$|x(t, t_0, x_0) - x(t, t_0, y_0)| \le \left| \frac{\partial x(t, t_0, v)}{\partial v} \right| |x_0 - y_0|. \tag{8.7}$$

Relations (8.6) and (8.7) yield estimate (8.5).

Consider system (8.2) and its solution

$$x(t, t_0, x_0) = F(F(\ldots F(x_0, t_0) \ldots), t - 1).$$

Differentiating this relation in x_0 we obtain

$$\frac{\partial x(t, t_0, x_0)}{\partial x_0} = Z(t, x_0), \qquad t \ge t_0, \qquad t \in \mathcal{Z}. \tag{8.8}$$

Estimate (8.7) for $t \ge t_0$, $t \in \mathcal{Z}$ can be found in exactly the same way as for (8.1) in the continuous case. Relations (8.8) and (8.7) yield (8.5). $\square$

Consider the open arcwise connected set Ω. Suppose any two points Ω can be connected by a path $\gamma(s) \subset \Omega$ of class C^1. Denote by $l(\gamma, x_0, y_0)$ the length of $\gamma(s) \subset \Omega$, $s \in [0,1]$, connecting the points x_0 and y_0: $\gamma(0) = x_0$, $\gamma(1) = y_0$.

Lemma 8.2. *The estimate*

$$|x(t, t_0, x_0) - x(t, t_0, y_0)| \leq \sup_{v \in \Omega} |Z(t, v)| l(\gamma, x_0, y_0) \tag{8.9}$$

is valid.

Proof. For any points $\gamma(s)$, $s \in [0,1]$, there is a ball $B(s)$ centered at $\gamma(s)$ such that $B(s) \subset \Omega$. By the Heine—Borel theorem we can choose a finite subcovering $B(s_k)$ $(k = 1, \ldots, \mathcal{N})$ of the compact $\{\gamma(s), s \in [0,1]\}$. Without loss of generality we can assume that $s_1 = 0$, $s_{\mathcal{N}} = 1$. On the path γ choose points v_k so that $v_k \in B(s_k) \cap B(s_{k+1})$ $(k = 1, \ldots, \mathcal{N} - 1)$. By Lemma 8.1,

$$\begin{aligned}
&|x(t, t_0, x_0) - x(t, t_0, y_0)| \leq |x(t, t_0, \gamma(s_1)) - x(t, t_0, v_1)| \\
&\quad + |x(t, t_0, v_1) - x(t, t_0, \gamma(s_2))| + |x(t, t_0, \gamma(s_2)) - x(t, t_0, v_2)| + \cdots \\
&\quad + |x(t, t_0, v_{\mathcal{N}-1}) - x(t, t_0, \gamma(s_{\mathcal{N}}))| \leq \sup_{v \in B(s_1)} |Z(t, v)||\gamma(s_1) - v_1| \\
&\quad + \sup_{v \in B(s_2)} |Z(t, v)||\gamma(s_2) - v_1| + \sup_{v \in B(s_2)} |Z(t, v)||\gamma(s_2) - v_2| + \cdots \\
&\quad + \sup_{v \in B(s_{\mathcal{N}})} |Z(t, v)||v_{\mathcal{N}-1} - \gamma(s_{\mathcal{N}})| \\
&\quad \leq \sup_{v \in \Omega} |Z(t, v)| l(\gamma, x_0, y_0).
\end{aligned}$$

The lemma is proved. □

From estimate (8.9) at once follows

Theorem 8.1. *If*

$$\lim_{t \to +\infty} \sup_{v \in R^n} |Z(t, v)| = 0, \tag{8.10}$$

then system (8.1) (or (8.2)) is stable in the large.

Recall that a system is said to be *stable in the large* if for the Lyapunov stable solution $x(t, t_0, x_0)$ the relation

$$\lim_{t \to +\infty} |x(t, t_0, x_0) - x(t, t_0, y_0)| = 0$$

is also satisfied for all $y_0 \in \mathbb{R}^n$.

By Theorem 8.1, Vazhevsky's inequality (Theorem 3.4), or other estimates for solutions of linear systems [1], we can obtain various extensions of the well-known conditions of stability in the large for nonautonomous systems [26, 115]. We show, for example, how from Theorem 8.1 and Vazhevsky's inequality the well-known assertions of B.P. Demidovich [26] and W. Lohmiller and J.J.E. Slotine [93] can be extended.

Theorem 8.2. *Suppose that for a certain symmetric, uniformly positively defined, and continuously differentiable matrix $H(t) = H^*(t) \geq \varepsilon > 0$, $\forall\, t \geq t_0$, and a continuous function $\Lambda(t)$ the relations*

$$\dot{H}(t) + H(t)A(x,t) + A(x,t)^* H(t) \leq \Lambda(t)H(t),\ \forall x \in \mathbb{R}^n,\ \forall t \geq t_0 \tag{8.11}$$

$$\lim_{t\to+\infty} \int_{t_0}^{t} \Lambda(\tau)\, d\tau = -\infty \tag{8.12}$$

are valid. Then system (8.1) is stable in the large.

Proof. It is sufficient to note that Vazhevsky's inequality and relations (8.11) and (8.12) yield (8.10). By Theorem 8.1, system (8.1) is stable in the large. □

For $\Lambda(t) \equiv -\alpha < 0$, Theorem 8.2 coincides with the well-known theorem of Lohmiller—Slotine [93] and for $H(t) \equiv I$, $\Lambda(t) \equiv -\alpha < 0$ with the result of Demidovich [26].

Chapter 9

Zhukovsky Stability

Zhukovsky stability is simply the Lyapunov stability of reparametrized trajectories. To study it, we may apply the arsenal of methods and devices that were developed for the study of Lyapunov stability.

The reparametrization of trajectories permits us to introduce another tool for investigation, the *moving Poincaré section*. The classical Poincaré section [3, 12, 34, 103] is the transversal $(n-1)$-dimensional surface S in the phase space $\mathbb{R}^n$, which possesses a recurring property. The latter means that for the trajectory of a dynamical system $x(t, x_0)$ with the initial data $x_0 \in S$, there exists a time instant $t = T > 0$ such that $x(T, x_0) \in S$. The transversal property means that

$$n(x)^* f(x) \neq 0, \quad \forall x \in S.$$

Here $n(x)$ is a normal vector of the surface S at the point x, and $f(x)$ is the right-hand side of the differential equation

$$\frac{dx}{dt} = f(x), \quad t \in \mathbb{R}^1, \quad x \in \mathbb{R}^n, \tag{9.1}$$

generating a dynamical system.

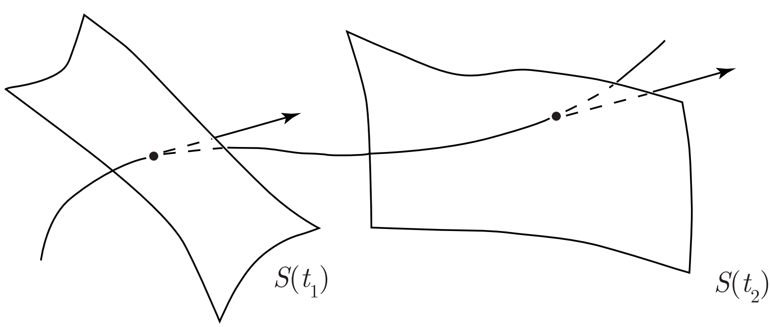

Fig. 9.1 Moving Poincaré section.

We now "force" the Poincaré section to move along the trajectory $x(t, x_0)$ (Fig. 9.1). We assume further that the vector-function $f(x)$ is twice continuously differentiable and that the trajectory $x(t, x_0)$, whose the Zhukovsky stability (or instability) will be considered, is wholly situated in a certain bounded domain $\Omega \subset \mathbb{R}^n$ for $t \geq 0$. Suppose also that $f(x) \neq 0$, $\forall x \in \overline{\Omega}$. Here $\overline{\Omega}$ is a closure of the domain Ω. Under these assumptions there exist positive numbers δ and ε such that

$$f(y)^* f(x) \geq \delta, \quad \forall y \in S(x, \varepsilon), \quad \forall x \in \overline{\Omega}.$$

Here

$$S(x, \varepsilon) = \{y \mid (y - x)^* f(x) = 0, \quad |x - y| < \varepsilon\}.$$

Definition 9.1. The set $S(x(t, x_0), \varepsilon)$ is called a *moving Poincaré section.*

Note that for small ε it is natural to restrict oneself to the family of segments of the surfaces $S(x(t, x_0), \varepsilon)$ rather than arbitrary surfaces. From this point of view a more general consideration does not give new results. It is possible to consider the moving Poincaré section more generally by introducing the set

$$S(x, q(x), \varepsilon) = \{y \mid (y - x)^* q(x) = 0, \quad |x - y| < \varepsilon\},$$

where the vector-function $q(x)$ satisfies the condition $q(x)^* f(x) \neq 0$. Such a consideration can be found in [60]. We treat the most interesting and descriptive case $q(x) \equiv f(x)$.

The classical Poincaré section allows us to clarify the behavior of trajectories using the information at their discrete times of crossing the section. Reparametrization makes it possible to organize the motion of trajectories so that at time t all trajectories are situated on the same moving Poincaré section $S(x(t, x_0), \varepsilon)$:

$$x(\varphi(t), y_0) \in S(x(t, x_0), \varepsilon). \tag{9.2}$$

Here $\varphi(t)$ is a reparametrization of the trajectory $x(t, y_0)$, $y_0 \in S(x_0, \varepsilon)$. This consideration has, of course, a local property and is only possible for t satisfying

$$|x(\varphi(t), y_0) - x(t, x_0)| < \varepsilon. \tag{9.3}$$

We formulate these facts more precisely as follows.

Lemma 9.1. (on parametrization). *For any $y_0 \in S(x_0, \varepsilon)$ there exists a differentiable function $\varphi(t) = \varphi(t, y_0)$ such that either relations (9.2)–(9.3) are valid for all $t \geq 0$, or there exists $T > 0$ such that (9.2)–(9.3) are valid for $t \in [0, T)$ and*

$$|x(\varphi(T), y_0) - x(T, x_0)| = \varepsilon. \tag{9.4}$$

In this case we have

$$\frac{d\varphi}{dt} = \frac{|f(x(t, x_0))|^2 - (x(\varphi(t), y_0) - x(t, x_0))\frac{\partial f}{\partial x}(x(t, x_0))f(x(t, x_0))}{f(x(\varphi(t), y_0))^* f(x(t, x_0))}. \tag{9.5}$$

Here we denote by

$$\frac{\partial f}{\partial x}(x(t, x_0))$$

the Jacobian matrix of the vector-function f at the point $x(t, x_0)$.

Proof. Consider the following function of two variables:

$$F(t, \tau) = (x(\tau, y_0) - x(t, x_0))^* f(x(t, x_0)),$$

for which $F(t, \tau) = 0$. From this and the inclusion $y_0 \in S(t_0, \varepsilon)$ it follows that either $x(\tau, y_0) \in S(x(t, x_0), \varepsilon)$ for all $t \geq 0$ and $\tau \geq 0$, or for some $T > 0$ and $\tau_0 > 0$ we have

$$x(\tau, y_0) \in S(x(t, x_0), \varepsilon), \quad \forall \tau \in [0, \tau_0), \quad \forall t \in [0, T),$$
$$|x(\tau_0, y_0) - x(T, x_0)| = \varepsilon.$$

Since in these cases we have

$$\frac{\partial F}{\partial \tau} = f(x(\tau, y_0))^* f(x(t, y_0)) \geq \delta,$$

then by the implicit function theorem we obtain the existence of $\varphi(t)$ such that relations (9.2) are satisfied on $[0, T)$. In this case T is finite (if (9.4) is satisfied) or $T = +\infty$ (if (9.3) holds for all $t \geq 0$). By the implicit function theorem we have

$$\frac{d\varphi}{dt} = -\frac{\partial F}{\partial t}(t, \varphi(t)) \left[\frac{\partial F}{\partial \tau}(t, \varphi(t))\right]^{-1}.$$

This implies relation (9.5). □

We can similarly prove the following

Lemma 9.2. *If the trajectory $x(t, x_0)$ is Zhukovsky stable, then there exists $\delta > 0$ such that for any $y_0 \in S(x_0, \delta)$ there exists a differentiable function $\varphi(t) = \varphi(t, y_0)$ such that for all $t \geq 0$ relations (9.2) and (9.5) are valid.*

We write now an equation for the difference $z(t) = x(\varphi(t), y_0) - x(t, x_0)$ using (9.1), (9.2), and (9.5). By (9.1)

$$\frac{dz}{dt} = f(x(\varphi(t), y_0))\dot{\varphi}(t) - f(x(t, x_0)). \tag{9.6}$$

Use (9.5) to rewrite (9.6) as

$$\begin{aligned} \frac{dz}{dt} &= f(z + x(t, x_0)) \frac{|f(x(t, x_0)|^2 - z^* \dfrac{\partial f}{\partial x}(x(t, x_0)) f(x(t, x_0))}{f(z + x(t, x_0))^* f(x(t, x_0))} - f(x(t, x_0)), \\ & z^* f(x(t, x_0)) = 0. \end{aligned} \tag{9.7}$$

Represent (9.7) in the form

$$\begin{aligned} &\frac{dz}{dt} = A(x(t, x_0))z + g(t, z), \\ &z^* f(x(t, x_0)) = 0\,, \end{aligned} \tag{9.8}$$

where

$$A(x) = \frac{\partial f}{\partial x}(x) - \frac{f(x) f(x)^*}{|f(x)|^2} \left[\frac{\partial f}{\partial x}(x) + \left(\frac{\partial f}{\partial x}(x) \right)^* \right].$$

We shall show that for system (9.8) the relations

$$g(t, z)^* f(x(t, x_0)) = 0, \tag{9.9}$$

$$|g(t, z)| = O(|z|^2), \tag{9.10}$$

are valid. In fact, from the identity $f(x(t, x_0))^* z(t) \equiv 0$ we have

$$\dot{z}(t)^* f(x(t, x_0)) + z(t)^* \frac{\partial f}{\partial x}(x(t, x_0)) f(x(t, x_0)) \equiv 0.$$

Therefore

$$f(x(t, x_0))^* (\dot{z} - A(x(t, x_0))z) = 0.$$

This is equivalent to (9.9). Estimate (9.10) results at once from (9.7) and the definitions of the matrix $A(x)$. Thus, for system (9.8) we have the system of the first approximation

$$\frac{dv}{dt} = A(x(t, x_0))v, \quad f(x(t, x_0))^* v = 0. \tag{9.11}$$

It differs from the usual system of the first approximation

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}(x(t, x_0))w \tag{9.12}$$

in that we introduce here the projector

$$v = \left(I - \frac{f(x(t, x_0) f(x(t, x_0))^*}{|f(x(t, x_0)|^2} \right) w. \tag{9.13}$$

It is not hard to prove that the vector-function

$$y(t) = \frac{f(x(t, x_0))}{|f(x(t, x_0))|^2}$$

is the solution of the system

$$\dot{y} = A(x(t, x_0))y.$$

Therefore we can consider the fundamental matrix of this system

$$Y(t) = \left(\frac{f(x(t, x_0))}{|f(x(t, x_0))|^2}, y_2(t), \ldots, y_n(t) \right),$$

where the solution $y_j(t)$ satisfies the condition

$$f(x(t, x_0))^* y_j(t) \equiv 0, \quad \forall\, t \geq 0, \;\; \forall\, j = 2, \ldots, n.$$

Now we apply to the system of solutions, making up the matrix $Y(t)$, the orthogonalization procedure (7.5). Further, as in § 7.1, from the orthogonal vector-functions $v_j(t)$ $(j = 1, \ldots, n)$ we compose the unitary matrix

$$U(t) = \left(\frac{v_1(t)}{|v_1(t)|}, \ldots, \frac{v_n(t)}{|v_n(t)|} \right).$$

It is obvious that in the case considered we have

$$\frac{v_1(t)}{|v_1(t)|} = \frac{f(x(t, x_0))}{|f(x(t, x_0))|}. \tag{9.14}$$

Besides, as shown in § 7.1, the matrix $U(t)$ possesses all properties formulated in Theorem 7.1. It is clear that in this case

$$f(x(t, x_0))^* U(t) \equiv (|f(x(t, x_0))|, 0 \ldots 0). \tag{9.15}$$

Therefore by the unitary transformation $z = U(t)u$, system (9.8) can be reduced to the form

$$\dot{u} = B(t)u + U(t)^* g(t, U(t)u), \tag{9.16}$$

where

$$B(t) = U(t)^* A(x(t, x_0))U(t) - U(t)^* \dot{U}(t).$$

Relation (9.15) implies the equivalence of the identities

$$f(x(t, x_0))^* z(t) \equiv 0,$$

and

$$u_1(t) \equiv 0. \tag{9.17}$$

Here $z(t)$ is the solution of system (9.8) and $u_1(t)$ is the first component of the vector-function $u(t)$ being the solution of system (9.16).

Thus, system (9.8) can be reduced to system (9.16) of order $n-1$, where relation (9.17) is satisfied. The latter makes it possible to apply the theory of the first approximation, as developed in Chapters 3–7, to the study of Zhukovsky stability. For this purpose we give some simple propositions.

Proposition 9.1. *If the zero solution of system (9.8) is Lyapunov stable, then the trajectory $x(t, x_0)$ is Zhukovsky stable. If the zero solution of system (9.8) is asymptotically Lyapunov stable, then the trajectory $x(t, x_0)$ is asymptotically Zhukovsky stable.*

Proof. The assertion follows at once from Lemma 9.1 and transformations (9.6)–(9.8). Here as $\tau(t)$ (see Definition 2.6) we choose the reparametrization $\varphi(t) : \tau(t) = \varphi(t)$. □

Proposition 9.2. *If the zero solution of system (9.8) is Lyapunov unstable, then the trajectory $x(t, x_0)$ is Zhukovsky unstable.*

Proof. If the trajectory $x(t, x_0)$ is Zhukovsky stable, then by Lemma 9.2 there is a reparametrization $\varphi(t)$ for which relations (9.8) are valid and from the condition $|z(0)| \leq \delta$ the inequality $|z(t)| \leq \varepsilon, \ \forall\, t \geq 0$ follows. Hence the zero solution of system (9.8) is Lyapunov stable. This contradicts the condition of Proposition 9.2 and finishes the proof. □

Proposition 9.3. *Lyapunov stability, asymptotic Lyapunov stability, and Lyapunov instability for the zero solutions of systems (9.8) and (9.16)–(9.17) are equivalent.*

This results from the unitary transformation $U(t)$, which reduces system (9.8) to system (9.16)–(9.17). Propositions 9.1, 9.3, and Theorems 3.3, 6.3 imply the following

Proposition 9.4. *If for system (9.11) the inequality $\Lambda + \Gamma < 0$ is satisfied, then the trajectory $x(t, x_0)$ is asymptotically Zhukovsky stable. Here Λ is the higher characteristic exponent of system (9.11), and Γ is a coefficient of irregularity.*

Note that the vector-function $f(x(t, x_0))$ is the solution of system (9.12) and that

$$\lim_{t\to+\infty} \frac{1}{t} \ln |f(x(t, x_0))| = 0. \tag{9.18}$$

Hence

$$\liminf_{t\to+\infty} \frac{1}{t} \int_0^t \operatorname{Tr} A(x(s, x_0))\, ds = \lim_{t\to+\infty} \inf \left[\frac{1}{t} \int_0^t \left(\operatorname{Tr}(\frac{\partial f}{\partial x}(x(s, x_0))\right) - \right.$$

$$- \operatorname{Tr}\left(\frac{f(x(s, x_0))f(x(s, x_0))^*}{|f(x, s, x_0)|^2} \left(\frac{\partial f}{\partial x}(x(s, x_0)) \;+\right.\right.$$

$$\left.\left.\left.\left(\frac{\partial f}{\partial x}(x(s, x_0))\right)^*\right)\right)\right)\, ds\right] = \lim_{t\to+\infty} \inf \left[\frac{1}{t} \int_0^t \left(\operatorname{Tr}(\frac{\partial f}{\partial x}(x(s, x_0))\right) - \right.$$

$$- \operatorname{Tr}\left(\frac{1}{|f(x(s, x_0)|^2} (f(x(s, x_0))^{\bullet} f(x(s, x_0))^* + \right.$$

$$+ f(x(s, x_0))(f(x(s, x_0))^*)^{\bullet}\big)\Big)\, ds] =$$

$$= \liminf_{t\to+\infty} \left[\frac{1}{t} \int_0^t \left(\operatorname{Tr}\left(\frac{\partial f}{\partial x}(x(s, x_0))\right) \; - \frac{(|f(x(s, x_0))|^2)^{\bullet}}{|f(x(s, x_0))|^2}\right)\, ds\right] =$$

$$= \liminf_{t\to+\infty} \left[\frac{1}{t} \left(\int_0^t \left(\operatorname{Tr}\left(\frac{\partial f}{\partial x}(x(s, x_0))\right)\right)\, ds \; - \ln |f(x(t, x_0))|^2\right)\right] =$$

$$= \lim_{t\to+\infty} \inf \frac{1}{t} \int_0^t \operatorname{Tr}\left(\frac{\partial f}{\partial x}(x(s, x_0))\right)\, ds \tag{9.19}$$

and system (9.12) has the one null characteristic exponent λ_1. Denote by $\lambda_2 \geq \cdots \geq \lambda_n$ the remaining characteristic exponents of (9.12).

From (9.13) it follows that the characteristic exponents of system (9.11) are not greater than the corresponding characteristic exponents of system (9.12). From this and (9.19) we have

$$\gamma \geq \Gamma. \tag{9.20}$$

Here γ is the coefficient of irregularity of (9.12). Additionally, we have

$$\lambda_2 \geq \Lambda. \tag{9.21}$$

Proposition 9.4 and inequalities (9.20) and (9.21) give the following

Theorem 9.1. *If for system (9.12) the inequality*

$$\lambda_2 + \gamma < 0 \tag{9.22}$$

is satisfied, then the trajectory $x(t, x_0)$ is asymptotically Zhukovsky stable.

This result generalizes the well-known Andronov—Witt theorem.

Theorem 9.2 (Andronov, Witt [26]). *If the trajectory $x(t, x_0)$ is periodic, differs from equilibria, and for system (9.12) the inequality*

$$\lambda_2 < 0$$

is satisfied, then the trajectory $x(t, x_0)$ is asymptotically orbitally stable (asymptotically Poincaré stable).

Theorem 9.2 is a corollary of Theorem 9.1 since system (9.12) with the periodic matrix

$$\frac{\partial f}{\partial x}(x(t, x_0))$$

is regular [26].

Recall that for periodic trajectories, asymptotic stability in the senses of Zhukovsky and Poincaré are equivalent.

The theorem of Demidovich is also a corollary of Theorem 9.1.

Theorem 9.3 (Demidovich [27]). *If system (9.12) is regular (i.e. $\gamma = 0$) and $\lambda_2 < 0$, then the trajectory $x(t, x_0)$ is asymptotically orbitally stable.*

Now we show that, by the first of approximation (9.11) and a simple Lyapunov function technique, the well-known results of Poincaré [26, 84, 117] and Borg [18, 38, 84, 116] can be generalized.

We introduce the symmetric differentiable matrix $H(x(t,x_0))$ and the continuous function $\lambda(x(t,x_0))$ for which

$$\dot{V}(t) \le \lambda(x(t,x_0))V(t). \tag{9.23}$$

Here $V(t) = z(t)^* H(x(t,x_0))z(t)$, and $z(t)$ is the solution of system (9.8).

Assume also that for some sequence $t_j \to +\infty$ and number $\kappa > 0$ the relation $t_{j+1} - t_j \le \kappa$ is satisfied.

Lemma 9.3. *If for certain positive numbers ρ and δ the inequalities*

$$\int_{t_j}^{t_{j+1}} \lambda(x(t,x_0))\,dt \le -\rho, \quad \forall\, j, \tag{9.24}$$

$$z^* H(x(t,x_0))z \ge \delta |z|^2, \quad \forall t \ge 0, \quad \forall z \in \mathbb{R}^n, \tag{9.25}$$

are satisfied, then there exist positive numbers C and ε such that

$$|z(t)| \le Ce^{-\varepsilon t}|z(0)|, \quad \forall t \ge 0. \tag{9.26}$$

If for some positive numbers ρ and δ the inequalities

$$\int_{t_j}^{t_{j+1}} \lambda(x(t,x_0))\,dt \ge \rho, \quad \forall\, j, \tag{9.27}$$

$$z^* H(x(t,x_0))z \le -\delta |z|^2, \quad \forall t \ge 0, \quad \forall z \in \mathbb{R}^n, \tag{9.28}$$

are satisfied, then there exist positive numbers C and ε such that

$$|z(t)| \ge Ce^{\varepsilon t}|z(0)|, \quad \forall t \ge 0. \tag{9.29}$$

Proof. We remark first that boundedness of $\lambda(x(t,x_0))$ follows from that of $x(t,x_0)$ and the continuity of $\lambda(\cdot)$. Then there exists $\alpha \ge 0$ such that

$$\left| \int_{t_j}^{t} \lambda(x(t,x_0))\,dt \right| \le \alpha, \quad \forall j, \ \forall\, t \in [t_j, t_{j+1}]. \tag{9.30}$$

Relation (9.23) can be rewritten as

$$\left(V(t) \exp\left(- \int_0^t \lambda(x(\tau,x_0))\,d\tau \right) \right)^{\bullet} \le 0.$$

Hence we have the estimate

$$V(t) \le V(0)\exp\left(\int_0^t \lambda(x(\tau,x_0))\,d\tau\right). \tag{9.31}$$

Inequalities (9.24) and (9.30) imply that there exist $\varepsilon > 0$ and $\beta > 0$ such that

$$\int_0^t \lambda(x(\tau,x_0))\,d\tau \le -2\varepsilon t + \beta, \quad \forall t \ge 0. \tag{9.32}$$

Then by (9.31) and (9.25) we have estimate (9.26). If (9.27) and (9.30) are satisfied, then there exist $\varepsilon > 0$ and $\beta > 0$ such that

$$\int_0^t \lambda(x(\tau,x_0))d\tau \ge 2\varepsilon t - \beta, \quad \forall\, t \ge 0.$$

Therefore (9.28) and (9.31) imply (9.29). □

It is clear that relation (9.26) corresponds to asymptotic Zhukovsky stability and (9.29) to Zhukovsky instability.

It should be noted that if (9.23) is satisfied on the solutions of system (9.8) only in some neighborhood $\Omega(0)$ of the point $z = 0$, then the assertion of Lemma 9.3 about the validity of estimate (9.26) is true for $z(0)$ from some sufficiently small neighborhood $\Phi(0) \subset \Omega(0)$. Here, like the direct Lyapunov method, assuming that $z(0)$ is sufficiently small, we find that (9.23) holds on a sufficiently large time interval $[0, T]$ (this follows, for example, from the fact that the rate of change of the solution $\dot{z}(t)$ in the neighborhood $\Omega(0)$ is small). Then we find that for $t = T$ estimate (9.26) is satisfied, which implies that for large T the inequality $|z(T)| \le |z(0)|$ holds. Hence for small $z(0)$ and some $t > 0$, the solution $z(t)$ cannot leave the neighborhood $\Omega(0)$. Then for this $z(t)$ estimate (9.23) is valid for all $t \ge 0$. This implies that (9.26) is also valid for all $t \ge 0$.

Consider the symmetric matrix M and its eigenvalues $\lambda_1 \ge \cdots \ge \lambda_n$.

Lemma 9.4 [38]. *For all $z, y \in \mathbb{R}^n$ we have*

$$\begin{aligned}(\lambda_n + \lambda_{n-1})(|z|^2|y|^2 - (z^*y)^2) &\le z^*Mz|y|^2 + |z|^2y^*My - 2z^*yz^*My \\ &\le (\lambda_1 + \lambda_2)(|z|^2|y|^2 - (z^*y)^2)).\end{aligned} \tag{9.33}$$

Proof. Since the orthogonal transformation $z = S\widetilde{z}$, $y = S\widetilde{y}$, $S^* = S^{-1}$

does not change scalar products and can reduce M to the diagonal form

$$S^*MS = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \tag{9.34}$$

we can assume, without loss of generality, that in inequalities (9.33) the matrix M takes the form (9.34). Then these inequalities take the form

$$\begin{aligned}
(\lambda_n + \lambda_{n-1}) & \left(\sum_j z_j^2 \sum_i y_i^2 - (\sum_j z_j y_j)^2 \right) \\
&= \frac{1}{2}(\lambda_n + \lambda_{n-1}) \left(\sum_{j=1}^n \sum_{i=1}^n (z_j y_i - z_i y_j)^2 \right) \\
&\le \frac{1}{2} \sum_{i,j} (\lambda_i + \lambda_j)(z_j y_i - z_i y_j)^2 = \sum_{j=1}^n \sum_{i=1}^n \lambda_j (z_j y_i - z_i y_j)^2 \\
&= \sum_{j=1}^n \sum_{i=1}^n \lambda_j (z_j z_j y_i y_i + z_i z_i y_j y_j - 2 z_j z_i y_j y_i) \\
&= \frac{1}{2} \sum_{i,j} (\lambda_i + \lambda_j)(z_j y_i - z_i y_j)^2 \\
&\le \frac{1}{2}(\lambda_1 + \lambda_2) \left(\sum_{j=1}^n \sum_{i=1}^n (z_j y_i - z_i y_j)^2 \right) \\
&= (\lambda_1 + \lambda_2) \left(\sum_j z_j^2 \sum_i y_i^2 - \left(\sum_j z_j y_j \right)^2 \right).
\end{aligned}$$

Since the above inequalities are obvious, Lemma 9.4 is proved. □

Now let us take

$$V(t) = |f(x(t, x_0))|^2 |z(t)|^2.$$

Relations (9.8)–(9.10) imply that for any $\nu > 0$ there exists a neighborhood

$\Omega(0)$ of the point $z = 0$ such that

$$\dot{V}(t) \leq |f(x(t,x_0))|^2 z(t)^* \left(\frac{\partial f}{\partial x}(x(t,x_0)) + \left(\frac{\partial f}{\partial x}(x(t,x_0))\right)^*\right) z(t)$$
$$+ |z(t)|^2 f(x(t,x_0))^* \left(\frac{\partial f}{\partial x}(x(t,x_0)) + \left(\frac{\partial f}{\partial x}(x(t,x_0))\right)^*\right) f(x(t,x_0))$$
$$+ \nu |z(t)|^2.$$

By Lemma 9.4, assuming that $M = \frac{\partial f}{\partial x} + \left(\frac{\partial f}{\partial x}\right)^*$, $z = z(t)$, and $y = f(x(t,x_0))$, we obtain

$$\dot{V}(t) \leq \left(\lambda_1(x(t,x_0)) + \lambda_2(x(t,x_0)) + \frac{\nu}{|f(x(t,x_0))|^2}\right) V(t).$$

Since for some $\gamma > 0$ the estimate $|f(x(t,x_0))| \geq \gamma$, $\forall t \geq 0$ is valid and ν is arbitrary small, for the conditions of Lemma 9.3 to hold it is sufficient that

$$\int_{t_j}^{t_{j+1}} (\lambda_1(x(t,x_0)) + \lambda_2(x(t,x_0)))\, dt < -\alpha, \quad \forall j \tag{9.35}$$

for some positive α. Indeed, here

$$\rho = \alpha - \frac{\nu}{\gamma^2}\kappa.$$

Therefore for small ν conditions (9.23)–(9.25) in the certain neighborhood $\Omega(0)$ are satisfied. Thus by the characteristic values $\lambda_1(x) \geq \cdots \geq \lambda_n(x)$ of the matrix

$$\frac{\partial f}{\partial x} + \left(\frac{\partial f}{\partial x}\right)^*,$$

we can formulate the following

Theorem 9.4 (analog of Poincaré criterion). *If for the sequence $t_j \to +\infty$ inequalities $t_{j+1} - t_j \leq \kappa$ and (9.35) with some positive numbers α and κ are valid, then the trajectory $x(t,x_0)$ is asymptotically Zhukovsky stable.*

Similarly, considering the function

$$V(t) = -|f(x(t,x_0))|^2 |z(t)|^2$$

and applying Lemmas 9.3 and 9.4, we obtain the following

Theorem 9.5 (analog of Poincaré criterion). *If for the sequence $t_j \to +\infty$, inequalities $t_{j+1} - t_j \leq \kappa$ and*

$$\int_{t_j}^{t_{j+1}} (\lambda_n(x(t,x_0)) + \lambda_{n-1}(x(t,x_0)))\,dt \geq \alpha, \quad \forall j, \tag{9.36}$$

with some positive numbers α and κ are valid, then the trajectory $x(t,x_0)$ is Zhukovsky unstable.

For $n = 2$ and T-periodic solution $x(t,x_0)$ from Theorems 9.4 and 9.5 we obtain at once the well-known Poincaré criterion [26, 84, 117].

Theorem 9.6. *If for the trajectory $x(t,x_0)$ the inequality*

$$\int_0^T \operatorname{Tr} \frac{\partial f}{\partial x}(x(t,x_0))\,dt < 0$$

is satisfied, then it is asymptotically orbitally stable. If

$$\int_0^T \operatorname{Tr} \frac{\partial f}{\partial x}(x(t,x_0))\,dt > 0,$$

then it is orbitally unstable.

Consider now a function

$$V(t) = |z(t)|^2.$$

Relations (9.8)–(9.10) imply that for any $\nu > 0$ there exists a neighborhood $\Omega(0)$ of $z = 0$ such that

$$\dot{V}(t) \leq 2z(t)^* \frac{\partial f}{\partial x}(x(t,x_0))z(t) + \nu|z(t)|^2.$$

Here all conditions Lemma 9.3 with $\delta = 1$, $\kappa = 1$, and with small positive ρ are valid if

$$z^* \frac{\partial f}{\partial x}(x(t,x_0))z \leq -\alpha|z|^2, \quad \forall z \in \{z^* f(x(t,x_0)) = 0, z \neq 0\}, \tag{9.37}$$

for some positive α. Indeed, it suffices to put $\lambda(x(t,x_0)) = \alpha - \nu$ and $\rho = \alpha - \nu$. Thus we obtain the following well-known result of G. Borg.

Theorem 9.7 (Borg [18, 38, 84, 116]. *If condition (9.37) is satisfied, then the trajectory $x(t,x_0)$ is asymptotically orbitally stable.*

Further developments in the theory of the first approximation on Zhukovsky stability and instability can be found in [60, 67, 72, 78, 80, 84].

Chapter 10

Lyapunov Functions in the Estimates of Attractor Dimension

Harmonic oscillations are characterized by an amplitude, period, and frequency, and periodic oscillations by a period. Numerous investigations have shown that more complex oscillations have also numerical characteristics. These are the dimensions of attractors, corresponding to ensembles of such oscillations.

The theory of topological dimension [43, 46], developed in the first half of the 20th century, is of little use in giving the scale of dimensional characteristics of attractors. The point is that the topological dimension can take integer values only. Hence the scale of dimensional characteristics compiled in this manner turns out to be quite poor.

For investigating attractors, the Hausdorff dimension of a set is much better. This dimensional characteristic can take any nonnegative value, and on such customary objects in Euclidean space as a smooth curve, a surface, or a countable set of points, it coincides with the topological dimension. Let us proceed to the definition of Hausdorff dimension.

Consider a compact metric set X with metric ρ, a subset $E \subset X$, and numbers $d \geq 0$, $\varepsilon > 0$. We cover E by balls of radius $r_j < \varepsilon$ and denote

$$\mu_H(E, d, \varepsilon) = \inf \sum_j r_j^d,$$

where the infimum is taken over all such ε-coverings E. It is obvious that $\mu_H(E, d, \varepsilon)$ does not decrease with decreasing ε. Therefore there exists the limit (perhaps infinite), namely

$$\mu_H(E, d) = \lim_{\varepsilon \to 0} \mu_H(E, d, \varepsilon).$$

Definition 10.1. The function $\mu_H(\cdot, d)$ is called the *Hausdorff d-measure.*

For fixed d, the function $\mu_H(E,d)$ possesses all properties of outer measure on X. For a fixed set E, the function $\mu_H(E,\cdot)$ has the following property. It is possible to find $d_{kp} \in [0,\infty]$ such that

$$\begin{aligned} \mu_H(E,d) &= \infty, \quad \forall\, d < d_{kp}, \\ \mu_H(E,d) &= 0, \quad \forall\, d > d_{kp}. \end{aligned}$$

If $X \subset \mathbb{R}^n$, then $d_{kp} \le n$. Here $\mathbb{R}^n$ is an Euclidean n-dimensional space.

We put

$$\dim_H E = d_{kp} = \inf\{d \mid \mu_H(E,d) = 0\}.$$

Definition 10.2. We call $\dim_H E$ the *Hausdorff dimension* of the set E.

Example 10.1. Consider the Cantor set from Example 1.3:

$$E = \bigcap_{j=0}^{\infty} E_j,$$

where $E_0 = [0,1]$ and E_j consists of 2^j segments of length 3^{-j}, obtained from the segments belonging to E_{j-1} by eliminating from them the open middle segments of length 3^{-j} (Fig. 1.6). In the classical theory of topological dimension it is well known that $\dim_T E = 0$. From the definitions of Hausdorff dimension we deduce easily that $\mu_H(E,d) = 1$ for $d = \log 2/\log 3 = 0.63010\ldots$ and, therefore,

$$\dim_H E = \frac{\log 2}{\log 3}. \qquad \square$$

Topological dimension is invariant with respect to homeomorphisms. Hausdorff dimension is invariant with respect to diffeomorphisms, and noninteger Hausdorff dimension is not invariant with respect to homeomorphisms [43].

In studying the attractors of dynamical systems in phase space, the smooth change of coordinates is often used. Therefore, in such considerations it is sufficient to assume invariance with respect to diffeomorphisms.

It is well known that $\dim_T E \le \dim_H E$. The Cantor set E shows that this inequality can be strict.

We give now two equivalent definitions of fractal dimension. Denote by $\mathcal{N}_\varepsilon(E)$ the minimal number of balls of radius ε needed to cover the set

$E \subset X$. Consider the numbers $d \geq 0$, $\varepsilon > 0$ and put

$$\mu_F(E, d, \varepsilon) = \mathcal{N}_\varepsilon(E)\varepsilon^d,$$
$$\mu_F(E, d) = \limsup_{\varepsilon \to 0} \mu_F(E, d, \varepsilon).$$

Definition 10.3. The *fractal dimension* of the set E is the value

$$\dim_F E = \inf\{d \mid \mu_F(E, d) = 0\}.$$

Note that this definition is patterned after that for Hausdorff dimension. However in this case the covering is by the balls of the same radius ε only.

Definition 10.4. The fractal dimension of E is the value

$$\dim_F E = \limsup_{\varepsilon \to 0} \frac{\log \mathcal{N}_\varepsilon(E)}{\log(1/\varepsilon)}.$$

It is easy to see that

$$\dim_H E \leq \dim_F E.$$

Example 10.2. For $X = [0, 1]$ and $E = \{0, 1, 2^{-1}, 3^{-1}, \ldots\}$ we have

$$\dim_H E = 0, \quad \dim_F E = \frac{1}{2}.$$ □

The extension of the scheme for introducing the Hausdorff and fractal measures and dimensions and the definitions of different metric dimensional characteristics can be found in [110, 111]. It turns out [30, 59, 73, 84, 85] that the upper estimate of the Hausdorff and fractal dimension of invariant sets is the Lyapunov dimension, which will be defined below.

Consider the continuously differentiable map F of the open set $U \subset \mathbb{R}^n$ in $\mathbb{R}^n$. Denote by T_xF the Jacobian matrix of the map F at the point x. The continuous differentiability of F gives

$$F(x + h) - F(x) = (T_xF)h + o(h).$$

We shall assume further that the set $K \subset U$ is invariant with respect to the transformation F: $F(K) = K$.

Consider the singular values of the $n \times n$ matrix A

$$\alpha_1(A) \geq \cdots \geq \alpha_n(A).$$

Recall that a singular value of A is a square root of an eigenvalue of the matrix A^*A. Here the asterisk denotes either transposition (in the real case) or Hermitian conjugation. Further we shall often write

$$\omega_d(A) = \alpha_1(A) \cdots \alpha_j(A)\alpha_{j+1}(A)^s,$$

where $d = j + s$, $s \in [0, 1]$, j is an integer from the interval $[1, n]$.

Definition 10.5. The *local Lyapunov dimension* of the map F at the point $x \in K$ is the number

$$\dim_L(F, x) = j + s,$$

where j is the largest integer from the interval $[1, n]$ such that

$$\alpha_1(T_xF) \cdots \alpha_j(T_xF) \geq 1$$

and s is such that $s \in [0, 1]$ and

$$\alpha_1(T_xF) \cdots \alpha_j(T_xF)\alpha_{j+1}(T_xF)^s = 1.$$

By definition in the case $\alpha_1(T_xF) < 1$ we have $\dim_L(F, x) = 0$ and in the case

$$\alpha_1(T_xF) \cdots \alpha_n(T_xF) \geq 1$$

we have $\dim_L(F, x) = n$.

Definition 10.6. The *Lyapunov dimension* of the map F of the set K is the number

$$\dim_L(F, K) = \sup_K \dim_L(F, x).$$

Definition 10.7. The local Lyapunov dimension of the sequence of the maps F^i at the point $x \in K$ is the number

$$\dim_L x = \limsup_{i \to +\infty} \dim_L(F^i, x).$$

Definition 10.8. The Lyapunov dimension of the sequence of the maps F^i of the set K is the number

$$\dim_L K = \sup_K \dim_L x.$$

For the maps F_t, depending on the parameter $t \in \mathbb{R}^1$, we can introduce the following analog of Definitions 10.7 and 10.8.

Definition 10.9. The local Lyapunov dimension of the map F_t at the point $x \in K$ is the number

$$\dim_L x = \limsup_{t \to +\infty} \dim_L(F_t, x).$$

Definition 10.10. The Lyapunov dimension of the map F_t of the set K is the number

$$\dim_L K = \sup_K \dim_L x.$$

Again, the inequality [14, 42, 44] $\dim_F K \leq \dim_L K$ is an important property of Lyapunov dimension. Its proof can be found in [42, 44].

Thus, we have

$$\dim_T K \leq \dim_H K \leq \dim_F K \leq \dim_L K.$$

Note that the Lyapunov dimension can be used as the characteristic of the inner instability of the dynamical system, defined on the invariant set K and generated by the family of the maps F^i or F_t.

The Lyapunov dimension is not a dimensional characteristic in the classical sense. However, it does permit us to estimate from above a topological, Hausdorff, or fractal dimension. It is also the characteristic of instability of dynamical systems. Finally, it is well "adapted" for investigations by the methods of classical stability theory. We shall demonstrate this, introducing the Lyapunov functions in the estimate of Lyapunov dimension. The idea of introducing Lyapunov functions in the estimate of dimensional characteristics first appeared in [57], and was subsequently developed in [58, 59, 61, 63, 65, 66, 68, 71, 73, 75, 76, 78, 84, 85, 86]. Here we follow, in the main, these ideas.

Consider the $n \times n$ matrices $Q(x)$, depending on $x \in \mathbb{R}^n$. We assume that

$$\det Q(x) \neq 0, \quad \forall\, x \in U,$$

and that there exist c_1 and c_2 such that

$$\sup_K \omega_d(Q(x)) \leq c_1, \qquad \sup_K \omega_d(Q^{-1}(x)) \leq c_2.$$

Theorem 10.1. *Let $F(K) = K$ and suppose that for some matrix $Q(x)$*

$$\sup_K \omega_d\big(Q(F(x))T_x F Q^{-1}(x)\big) < 1. \tag{10.1}$$

Then

$$\dim_L(F^i, K) \le d \tag{10.2}$$

for sufficiently large natural numbers i.

Proof. For the matrix T_xF^i we have

$$T_xF^i = (T_{F^{i-1}(x)}F)(T_{F^{i-2}(x)}F)\cdots(T_xF).$$

This relation can be represented as

$$\begin{aligned} T_xF^i = Q(F^i(x))^{-1}&\big(Q(F^i(x))T_{F^{i-1}(x)}FQ(F^{i-1}(x))^{-1}\big) \\ &\times(Q(F^{i-1}(x))T_{F^{i-2}(x)}FQ(F^{i-2}(x))^{-1}) \\ &\times\cdots\times(Q(F(x))T_xFQ(x)^{-1})Q(x). \end{aligned}$$

From this and the well-known property [84, 85]

$$\omega_d(AB) \le \omega_d(A)\omega_d(B)$$

we obtain

$$\omega_d(T_xF^i) \le c_1c_2\left[\sup_K \omega_d(Q(F(x))T_xFQ(x)^{-1}\right]^i.$$

This estimate, the condition (10.1) of the theorem, and the definitions of Lyapunov dimension imply the estimate (10.2). □

Condition (10.1) is easily seen to be invariant with respect to the linear nonsingular change $y = Sx$, where S is a constant $n \times n$-matrix. It is clear that in the new basis condition (10.1) is also satisfied with the new matrix $Q_1(y)$:

$$Q_1(y) = Q(F(S^{-1}y))S.$$

Consider the important special case

$$Q(x) = p(x)S,$$

where S is a constant nondegenerate $n \times n$-matrix, $p(x)$ is the continuous function $\mathbb{R}^n \to \mathbb{R}^1$ for which

$$p_1 \le p(x) \le p_2, \quad \forall\, x \in K.$$

Here p_1 and p_2 are positive. In this case inequality (10.1) takes the form

$$\sup_K \omega_d\left(\frac{p(F(x))}{p(x)}\, ST_xFS^{-1}\right) < 1. \tag{10.3}$$

As will be shown below in condition (10.3) the multipliers of the type $p(F(x))/p(x)$ play the role of the Lyapunov type functions. This becomes especially clear in the case of the passage to the dynamical systems generated by differential equations.

Consider the system

$$\frac{dx}{dt} = f(t,x), \quad t \in \mathbb{R}^1, \quad x \in \mathbb{R}^n, \tag{10.4}$$

where $f(t,x)$ is the continuously differentiable T-periodic vector-function $\mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$, $f(t+T,x) = f(t,x)$. We assume that the solutions $x(t,x_0)$ of system (10.4) with the initial data $x(0,x_0) = x_0$ are defined on the interval $[0,T]$ and denote by G_T a shift operator along the solutions of system (10.4):

$$G_T q = x(T,q).$$

Suppose that the bounded set $K \subset \mathbb{R}^n$ is invariant with respect to the operator G_T, namely

$$G_T K = K.$$

Denote by $J(t,x)$ the Jacobian matrix of the vector-function $f(t,x)$:

$$J(t,x) = \frac{\partial f(t,x)}{\partial x}$$

and consider the nondegenerate $n \times n$ matrix S. Denote by $\lambda_1(t,x,S) \geq \cdots \geq \lambda_n(t,x,S)$ the eigenvalues of

$$\frac{1}{2}\left[SJ(t,x)S^{-1} + (SJ(t,x)S^{-1})^*\right].$$

Here the asterisk denotes transposition.

Theorem 10.2. *Suppose that for the integer $j \in [1,n]$ and $s \in [0,1]$ there exists a function $v(x)$, continuously differentiable on $\mathbb{R}^n$, and a nondegenerate $n \times n$ matrix S such that*

$$\sup_K \int_0^T [\lambda_1(t,x(t,q),S) + \ldots + \lambda_j(t,x(t,q),S) + s\lambda_{j+1}(t,x(t,q),S) + \dot{v}(x(t,q))]\,dt < 0. \tag{10.5}$$

Then for sufficiently large i the inequality

$$\dim_L(G_T^i, K) \leq j + s \tag{10.6}$$

holds.

Proof. Denote the Jacobian matrix by

$$H(t,q) = \frac{\partial x(t,q)}{\partial q}.$$

Substituting $x(t,q)$ in (10.4) and differentiating both sides of (10.4) with respect to q, we obtain

$$\frac{dH(t,q)}{dt} = J(t,x(t,q))H(t,q).$$

Represent this relation as

$$\frac{d}{dt}[SH(t,q)S^{-1}] = [SJ(t,x(t,q))S^{-1}][SH(t,q)S^{-1}].$$

For the singular values $\sigma_1(t) \geq \cdots \geq \sigma_n(t)$ of the matrix $SH(t,q)S^{-1}$ we have the inequality [30, 85]

$$\sigma_1 \cdots \sigma_k \leq \exp\left(\int_0^t (\lambda_1 + \cdots + \lambda_k)\, d\tau\right)$$

for any $k = 1, \ldots, n$. From this and the relation

$$\sigma_1 \cdots \sigma_j \sigma_{j+1}^s = (\sigma_1 \cdots \sigma_j)^{1-s}(\sigma_1 \cdots \sigma_{j+1})^s$$

we obtain the estimate

$$\sigma_1 \cdots \sigma_j \sigma_{j+1}^s \leq \exp\left(\int_0^t (\lambda_1 + \cdots + \lambda_j + s\lambda_{j+1})\, d\tau\right). \tag{10.7}$$

Put

$$p(x) = \big(\exp v(x)\big)^{1/(j+s)}$$

and multiply both sides of (10.7) by the relation

$$\left(\frac{p(x(t,q))}{p(q)}\right)^{j+s} = \exp\big[v(x(t,q) - v(q)\big] = \exp\left(\int_0^t \dot{v}(x(\tau,q))\, d\tau\right).$$

As a result we obtain

$$\left(\frac{p(x(t,q))}{p(q)}\right)^{j+s} \sigma_1 \ldots \sigma_j \sigma_{j+1}^s \\ \leq \exp\left(\int_0^t (\lambda_1 + \cdots + \lambda_j + s\lambda_{j+1} + \dot{v}(x(\tau,q))\, d\tau\right).$$

This implies the estimate

$$\begin{aligned}\alpha_1(t,q)\ldots\alpha_j(t,q)\alpha_{j+1}(t,q)^s \\ \leq \exp\Bigg(\int_0^t \big(\lambda_1(\tau,x(\tau,q),S)+\cdots+\lambda_j(\tau,x(\tau,q),S) \\ + s\lambda_{j+1}(\tau,x(\tau,q),S)+\dot{v}(x(\tau,q))\big)\,d\tau\Bigg),\end{aligned} \tag{10.8}$$

where $\alpha_k(t,q)$ are the singular values of the matrix

$$\frac{p(x(t,q))}{p(q)}\,SH(t,q)S^{-1}.$$

From estimate (10.8) and condition (10.5) of Theorem 10.2 it follows that there exists $\varepsilon > 0$ such that

$$\alpha_1(T,q)\cdots\alpha_j(T,q)\alpha_{j+1}(T,q)^s \leq \exp(-\varepsilon)$$

for all $q \in K$. Thus, in this case condition (10.3) with $F = G_T$

$$T_qF = T_qG_T = H(T,q)$$

is satisfied and, therefore, estimate (10.6) is valid. □

The following simple assertions will be useful in the sequel.

Lemma 10.1. *Suppose that the real matrix A can be reduced to the diagonal form*

$$SAS^{-1} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix},$$

where S is a real nonsingular matrix. Then there exist positive numbers c_1 and c_2 such that

$$c_1|\lambda_1\cdots\lambda_j\lambda_{j+1}^s|^i \leq \omega_d(A^i) \leq c_2|\lambda_1\cdots\lambda_j\lambda_{j+1}^s|^i.$$

Proof. It is sufficient to note that the singular values of the matrix

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

are the numbers $|\lambda_j|$, and for the singular values $\alpha_1 \geq \cdots \geq \alpha_n$ the inequalities

$$\alpha_n(C)\alpha_j(B) \leq \alpha_j(CB) \leq \alpha_1(C)\alpha_j(B)$$

are satisfied. □

Lemma 10.2. *Let $F(x) = x$ and the Jacobian matrix T_xF of the map F have the simple real eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Then the local Lyapunov dimension of the sequence of maps F^i at the point x is equal to $j+s$, where j and s are determined from*

$$|\lambda_1 \cdots \lambda_j \lambda_{j+1}^s| = 1.$$

Lemma 10.2 is a direct corollary of Lemma 10.1. A similar result holds for the map F_t.

Lemma 10.3. *Let $T_xF_t = e^{At}$ and the matrix A satisfy the condition of Lemma 10.1. Then the local Lyapunov dimension of the map F_t at the point x is equal to $j+s$, where j and s are determined from*

$$\lambda_1 + \cdots + \lambda_j + s\lambda_{j+1} = 0.$$

Lemma 10.3 is also a corollary of Lemma 10.1.

Now we apply Theorems 10.1 and 10.2 to the Henon and Lorenz systems in order to construct Lyapunov functions $p(x)$ (for the Henon system) and $v(x)$ (for the Lorenz system). Consider the Henon map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{aligned} x &\to a + by - x^2, \\ y &\to x, \end{aligned} \tag{10.9}$$

where $a > 0$, $b \in (0,1)$ are the parameters of mapping. Consider the bounded invariant set K of map (10.9), $FK = K$, involving stationary points of this map:

$$x_+ = \frac{1}{2}\left[b - 1 + \sqrt{(b-1)^2 + 4a}\,\right],$$
$$x_- = \frac{1}{2}\left[b - 1 - \sqrt{(b-1)^2 + 4a}\,\right].$$

Such a set K is shown in Fig. 2.5.

Theorem 10.3. *For the map F we have*

$$\dim_L K = 1 + \frac{1}{1 - \ln b / \ln \alpha_1(x_-)},$$

where

$$\alpha_1(x_-) = \sqrt{x_-^2 + b} - x_-.$$

Proof. Denote $\xi = \begin{pmatrix} x \\ y \end{pmatrix}$. The Jacobian matrix $T_\xi F$ of the map F takes the form

$$\begin{pmatrix} -2x & b \\ 1 & 0 \end{pmatrix}.$$

We introduce the matrix

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{b} \end{pmatrix}.$$

In this case

$$S T_\xi F S^{-1} = \begin{pmatrix} -2x & \sqrt{b} \\ \sqrt{b} & 0 \end{pmatrix}. \tag{10.10}$$

We shall show that the singular values of (10.10) are

$$\begin{aligned} \alpha_1(x) &= \sqrt{x^2 + b} + |x|, \\ \alpha_2(x) &= \sqrt{x^2 + b} - |x| = \frac{b}{\alpha_1(x)}. \end{aligned} \tag{10.11}$$

It is obvious that

$$\begin{aligned} \alpha_1(x)^2 &= 2x^2 + b + 2|x| \sqrt{x^2 + b}\ , \\ \alpha_2(x)^2 &= 2x^2 + b - 2|x| \sqrt{x^2 + b}\ . \end{aligned}$$

It is clear that $\alpha_k(x)^2$ are zeros of the polynomial

$$\lambda^2 - (4x^2 + 2b)\lambda + b^2,$$

which is the characteristic polynomial of the matrix

$$\begin{pmatrix} -2x & \sqrt{b} \\ \sqrt{b} & 0 \end{pmatrix} \begin{pmatrix} -2x & \sqrt{b} \\ \sqrt{b} & 0 \end{pmatrix}.$$

Thus, formulas (10.11) are proved.

From Theorem 10.1 it follows that if there exist $s \in [0,1)$ and a continuously differentiable function $p(\xi)$, positive on K and such that

$$\sup_{\xi \in K} \alpha_1(x)\alpha_2(x)^s \left(\frac{p(F(\xi))}{p(\xi)}\right)^{1+s} < 1, \tag{10.12}$$

then

$$\dim_L K \leq 1 + s.$$

Put

$$p(\xi)^{1+s} = e^{\gamma(1-s)(x+by)},$$

where γ is a positive parameter. It is not hard to prove that

$$\left(\frac{p(F(\xi))}{p(\xi)}\right)^{1+s} = e^{\gamma(1-s)(a+(b-1)x-x^2)}.$$

This implies that after taking the logarithm, condition (10.12) becomes

$$\begin{aligned}\sup_K &\big[\ln \alpha_1(x) + s \ln \alpha_2(x) + \gamma(1-s)(a+(b-1)x-x^2)\big] \\ &= \sup_K \big[(1-s)\ln \alpha_1(x) + s \ln b + \gamma(1-s)(a+(b-1)x-x^2)\big] < 0.\end{aligned}$$

This inequality is satisfied if

$$s \ln b + (1-s)\varphi(x) < 0, \quad \forall\, x \in (-\infty, +\infty),$$

where

$$\varphi(x) = \ln\left[\sqrt{x^2+b} + |x|\right] + \gamma(a+(b-1)x-x^2).$$

The inequalities $\gamma > 0$, $b-1 < 0$ result in the estimate

$$\varphi(-|x|) \geq \varphi(|x|).$$

Therefore it suffices to consider the extremum point of the functions $\varphi(x)$ for $x \in (-\infty, 0]$. It is clear that on this set we have

$$\varphi'(x) = \frac{-1}{\sqrt{x^2+b}} + \gamma\big[(b-1)-2x\big], \quad \varphi''(x) < 0.$$

Letting

$$\gamma = \frac{1}{(b-1-2x_-)\sqrt{x_-^2+b}},$$

we find that $\varphi'(x_-) = 0$ and therefore, for such a choice of γ,

$$\varphi(x) \le \ln\left(\sqrt{x_-^2 + b} + |x_-|\right) = \ln \alpha_1(x_-).$$

Thus, inequality (10.12) holds for all s satisfying

$$s > \frac{\ln \alpha_1(x_-)}{\ln \alpha_1(x_-) - \ln b}. \tag{10.13}$$

Hence the estimate

$$\dim_L K \le 1 + s$$

is valid for all s satisfying (10.13). Passing to the limit, we obtain

$$\dim_L K \le 1 + \frac{1}{1 - \ln b / \ln \alpha_1(x_-)}. \tag{10.14}$$

Note that the point $x = x_-$, $y = x_-$ is stationary for the map F. Then

$$\alpha_1(x_-)\alpha_2(x_-)^s = 1, \tag{10.15}$$

where

$$s = \frac{1}{1 - \ln b / \ln \alpha_1(x_-)}.$$

It is easily shown that $\alpha_1(x_-)$ and $\alpha_2(x_-)$ are the eigenvalues of the Jacobian matrix $T_\xi F$ of the map F at the fixed point $y = x = x_-$:

$$T_\xi F = \begin{pmatrix} -2x_- & b \\ 1 & 0 \end{pmatrix}.$$

From relation (10.15) by Lemma 10.2 we conclude that the local Lyapunov dimension of the sequence of maps F^i at this stationary point is equal to

$$1 + \frac{1}{1 - \ln b / \ln \alpha_1(x_-)}. \tag{10.16}$$

By inequality (10.14) we obtain the assertion of Theorem 10.3. □

Note that for $a = 1.4$, $b = 0.3$ from Theorem 10.3 we have

$$\dim_L K = 1.49532\ldots$$

Consider a Lorenz system

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y, \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy, \end{aligned} \tag{10.17}$$

where r, b, σ are positive. Suppose that the inequalities $r > 1$,

$$\sigma + 1 \geq b \geq 2 \tag{10.18}$$

are valid. Consider the shift operator along the trajectory of system (10.17) G_T, where T is an arbitrary positive number. Let K be an invariant set with respect to this operator G_T. Suppose that K involves the stationary point $x = y = z = 0$. Such a set is represented in Fig. 2.3 and 2.4. We provide a formula for the Lyapunov dimension $\dim_L K$ of the set K with respect to the sequence of maps $(G_T)^i$.

Theorem 10.4. *Suppose the inequalities* (10.18) *and*

$$r\sigma^2(4-b) + 2\sigma(b-1)(2\sigma - 3b) > b(b-1)^2 \tag{10.19}$$

are valid. Then

$$\dim_L K = 3 - \frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma-1)^2 + 4r\sigma}}. \tag{10.20}$$

Proof. The Jacobian matrix of the right-hand side of system (10.17) has the form

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}.$$

Introduce the matrix

$$S = \begin{pmatrix} -a^{-1} & 0 & 0 \\ -\sigma^{-1}(b-1) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $a = \dfrac{\sigma}{\sqrt{r\sigma + (b-1)(\sigma - b)}}$. In this case we obtain

$$SJS^{-1} = \begin{pmatrix} b - \sigma - 1 & \sigma/a & 0 \\ \dfrac{\sigma}{a} - az & -b & -x \\ ay + \dfrac{a(b-1)}{\sigma}x & x & -b \end{pmatrix}.$$

Therefore the characteristic polynomial of the matrix

$$\frac{1}{2}((SJS^{-1})^* + (SJS^{-1}))$$
$$= \begin{pmatrix} b-\sigma-1 & \dfrac{\sigma}{a}-\dfrac{az}{2} & \dfrac{1}{2}\left(ay+\dfrac{a(b-1)}{\sigma}x\right) \\ \dfrac{\sigma}{a}-\dfrac{az}{2} & -b & 0 \\ \dfrac{1}{2}\left(ay+\dfrac{a(b-1)}{\sigma}x\right) & 0 & -b \end{pmatrix}$$

takes the form

$$(\lambda+b)\left\{\left[\lambda^2+(\sigma+1)\lambda+b(\sigma+1-b)-\left(\frac{\sigma}{a}-\frac{az}{2}\right)^2\right]-\left[\frac{a(b-1)}{2\sigma}x+\frac{ay}{2}\right]^2\right\}.$$

This implies that eigenvalues of the matrix

$$\frac{1}{2}[(SJS^{-1})^* + (SJS^{-1})]$$

are the values

$$\lambda_2 = -b,$$

and

$$\lambda_{1,3} = -\frac{\sigma+1}{2} \pm \frac{1}{2}\left[(\sigma+1-2b)^2+\left(\frac{2\sigma}{a}-az\right)^2+\left(\frac{a(b-1)}{\sigma}x+ay\right)^2\right]^{1/2}.$$

From relations (10.18) it follows easily that $\lambda_1 \geq \lambda_2 \geq \lambda_3$.

Consider the Lyapunov type function

$$v(x,y,z) = \frac{1}{2}a\theta^2(1-s)\left(\gamma_1 x^2 + \gamma_2\left(y^2+z^2-\frac{(b-1)^2}{\sigma^2}x^2\right)+\gamma_3 z\right),$$

where $s \in (0,1)$,

$$\theta^2 = \left(2\sqrt{(\sigma+1-2b)^2 + \left(\frac{2\sigma}{a}\right)^2}\right)^{-1},$$

$$\gamma_3 = -\frac{4\sigma}{ab}, \quad \gamma_2 = \frac{a}{2},$$

$$\gamma_1 = -\frac{1}{2\sigma}\left[2\gamma_2\,\frac{r\sigma-(b-1)^2}{\sigma} + \gamma_3 + 2\,\frac{a(b-1)}{\sigma}\right].$$

Consider the relation

$$2\big[\lambda_1+\lambda_2+s\lambda_3+\dot v\big] = -(\sigma+1+2b) - s(\sigma+1) + (1-s)\varphi(x,y,z),$$

where

$$\begin{aligned}\varphi(x,y,z) = &\left((\sigma+1-2b)^2 + \left(\frac{2\sigma}{a} - az\right)^2\right.\\ &\left.+\left(\frac{a(b-1)}{\sigma}x + ay\right)^2\right)^{1/2}\\ &+\theta^2\Bigg\{\left(-2a\gamma_1\sigma + 2\gamma_2\frac{a(b-1)^2}{\sigma}\right)x^2 - 2a\gamma_2 y^2\\ &-2a\gamma_2 b z^2 + a\left(2\sigma\gamma_1 + 2\gamma_2\frac{r\sigma-(b-1)^2}{\sigma} + \gamma_3\right)xy - \gamma_3 abz\Bigg\}.\end{aligned}$$

By using the obvious inequality

$$\sqrt{u} \le \frac{1}{4\theta^2} + \theta^2 u,$$

we obtain the estimate

$$\begin{aligned}\varphi(x,y,z) \le \frac{1}{4\theta^2} + \theta^2\Bigg\{&(\sigma+1-2b)^2 + \left(\frac{2\sigma}{a}\right)^2\\ &+\left[-2a\gamma_1\sigma + 2\gamma_2\frac{a(b-1)^2}{\sigma} + \frac{a^2(b-1)^2}{\sigma^2}\right]x^2\\ &+[a^2-2a\gamma_2]y^2 + [a^2-2\gamma_2 ab]z^2\\ &+\left[a\left(2\sigma\gamma_1 + 2\gamma_2\frac{r\sigma-(b-1)^2}{\sigma} + \gamma_3\right)\right.\\ &\left.+2a^2\frac{b-1}{\sigma}\right]xy - [\gamma_3 ab + 4\sigma]z\Bigg\}.\end{aligned}$$

Note that the parameters γ_1, γ_2, and γ_3 are chosen in such a way that

$$\varphi(x,y,z) \le \frac{1}{4\theta^2} + \theta^2 \Big\{ (\sigma+1-2b)^2 + \Big(\frac{2\sigma}{a}\Big)^2 + \Big[-2a\gamma_1\sigma + 2\gamma_2 \frac{a(b-1)^2}{\sigma} + \frac{a^2(b-1)^2}{\sigma^2} \Big] x^2 \Big\}.$$

It is not hard to prove that for the above parameters $\gamma_1, \gamma_2, \gamma_3$ under condition (10.19) we have

$$-2a\gamma_1\sigma + 2\gamma_2 \frac{a(b-1)^2}{\sigma} + \frac{a^2(b-1)^2}{\sigma^2} \le 0.$$

Thus, for all x, y, z we have

$$\varphi(x,y,z) \le \sqrt{4r\sigma + (\sigma-1)^2}\ .$$

This implies that for any number

$$s < s_0 = \frac{\sqrt{4r\sigma + (\sigma-1)^2} - 2b - \sigma - 1}{\sqrt{4r\sigma + (\sigma-1)^2} + \sigma + 1}$$

there exists $\varepsilon > 0$ such that for all x, y, z the estimate

$$\lambda_1(x,y,z) + \lambda_2(x,y,z) + s\lambda_3(x,y,z) + \dot{v}(x,y,z) < -\varepsilon$$

is satisfied. Letting $s \to s_0$ on the right, by Theorem 10.2 we obtain

$$\dim_L K \le 3 - \frac{2(\sigma+b+1)}{\sigma+1+\sqrt{(\sigma-1)^2+4r\sigma}}. \tag{10.21}$$

By Lemma 10.3 we see that the local Lyapunov dimension of the stationary point $x = y = z = 0$ of system (10.17) is equal to

$$3 - \frac{2(\sigma+b+1)}{\sigma+1+\sqrt{(\sigma-1)^2+4r\sigma}}. \tag{10.22}$$

Relations (10.21) and (10.22) yield the formula (10.20). □

By using a similar approach to the construction of the Lyapunov functions, we can obtain formulas for the Lyapunov dimension of the attractors of the dissipative Chirikov map [76].

Chapter 11

Homoclinic Bifurcation

When the parameters of a dynamical system are varied, the structure of the minimal global attractor can vary as well. Such changes are the subject of bifurcation theory. Here we describe one particular phenomenon: the homoclinic bifurcation.

The first important results, concerning homoclinic bifurcations in dissipative dynamical systems, were obtained in 1933 by the outstanding Italian mathematician Franchesko Tricomi [133]. Here we give Tricomi's results and similar theorems for the Lorenz systems.

Consider the second-order differential equation

$$\ddot{\theta} + \alpha\dot{\theta} + \sin\theta = \gamma, \tag{11.1}$$

where α and γ are positive. This describes the motion of a pendulum with a constant moment of force, the operation of a synchronous electrical machine, and the phase-locked loop [11, 62, 64, 83, 84, 85, 87, 88, 89, 92, 130, 134, 138]. For $\gamma < 1$ the equivalent system

$$\begin{aligned} \dot{\theta} &= z, \\ \dot{z} &= -\alpha z - \sin\theta + \gamma, \end{aligned} \tag{11.2}$$

has the saddle equilibria $z = 0$, $\theta = \theta_0 + 2k\pi$ (Fig. 11.1). Here θ_0 is a number for which $\sin\theta_0 = \gamma$ and $\cos\theta_0 < 0$.

Consider the trajectory $z(t), \theta(t)$ of (11.2) for which

$$\lim_{t\to+\infty} z(t) = 0, \quad \lim_{t\to+\infty} \theta(t) = \theta_0, z(t) > 0, \qquad \forall\, t \geq T.$$

Here T is a certain number. In Fig 11.1, such a trajectory is denoted by S. It is often called a separatrix of the saddle.

Fix $\gamma > 0$ and vary the parameter α. For $\alpha = 0$ the system (11.2) is integrable. It is easily shown that in this case, for the trajectory $S =$

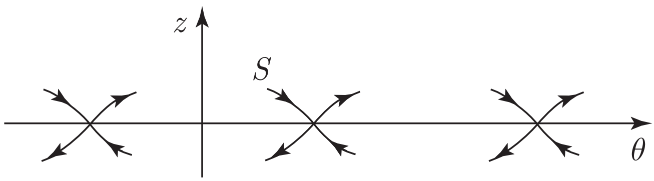

Fig. 11.1 Separatrix of the saddle.

$\{z(t), \theta(t)\}$ there exists τ such that

$$\begin{aligned} &z(\tau) = 0, \quad \theta(\tau) \in (\theta_0 - 2\pi, \theta_0) \\ &z(t) > 0, \quad \forall\, t > \tau. \end{aligned} \tag{11.3}$$

See Fig. 11.2.

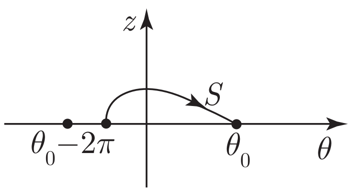

Fig. 11.2 Relations (11.3).

Consider now the line segment $z = -K(\theta - \theta_0)$, $\theta \in [\theta_0 - 2\pi, \theta_0]$. It is not hard to prove that on this segment for system (11.2) the relations

$$(z + K(\theta - \theta_0))^{\bullet} = -\alpha z + Kz - \sin\theta + \gamma = (\theta - \theta_0)\left(-K(K - \alpha) + \frac{\gamma - \sin\theta}{\theta - \theta_0}\right)$$

are valid. We make use of the obvious inequality

$$\left|\frac{\gamma - \sin\theta}{\theta - \theta_0}\right| \leq 1, \quad \forall\, \theta \neq \theta_0.$$

If the conditions

$$\alpha > 2, \qquad \frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - 1} < K < \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - 1}$$

are satisfied, we obtain the estimate

$$(z + K(\theta - \theta_0))^{\bullet} < 0$$

for $z = -K(\theta - \theta_0)$, $\theta \in (\theta_0 - 2\pi, \theta_0)$. See Fig. 11.3.

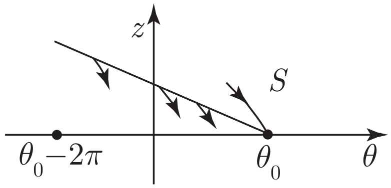

Fig. 11.3 Estimate of separatrix.

The figure shows that there does not exist τ such that conditions (11.3) are satisfied (Fig. 11.4).

It is well known that the piece of the trajectory $S : \{z(t), \theta(t) \mid t \geq \tau\}$ is continuously dependent on the parameter α. Here τ satisfies (11.3).

Then from the disposition of the trajectory S for $\alpha > 2$ (Fig. 11.4) it follows that there exists $\alpha_0 \in (0, 2)$ such that the trajectory S of system (2) with $\alpha = \alpha_0$ satisfies the relation

$$\lim_{t \to -\infty} z(t) = 0, \quad \lim_{t \to -\infty} \theta(t) = \theta_0 - 2\pi. \tag{11.4}$$

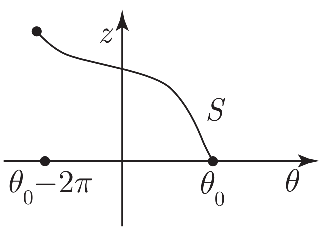

Fig. 11.4 Behavior of separatrix.

Thus, $\alpha = \alpha_0$ is a bifurcational parameter. To this parameter corresponds the heteroclinic trajectory $S = \{z(t), \theta(t) \mid t \in \mathbb{R}^1\}$. Recall that the trajectory $x(t)$ of the system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{11.5}$$

is said to be *heteroclinic* if

$$\lim_{t \to +\infty} x(t) = c_1, \quad \lim_{t \to -\infty} x(t) = c_2, \quad c_1 \neq c_2.$$

In the case $c_1 = c_2$, the trajectory $x(t)$ is called *homoclinic*.

Sometimes for systems involving angular coordinates, the cylindrical phase space is introduced. We do this for system (11.2).

It is obvious that the properties of system (11.2) are invariant with respect to the shift $x + d$. Here

$$x = \begin{pmatrix} \theta \\ z \end{pmatrix}, \quad d = \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}.$$

In other words, if $x(t)$ is a solution of system (11.2), then so is $x(t) + d$.

Consider a discrete group

$$\Gamma = \{x = kd \mid k \in Z\}.$$

We consider the factor group R^2/Γ, the elements of which are the classes of the residues $[x] \in R^2/\Gamma$. They are defined as

$$[x] = \{x + u \mid u \in \Gamma\}.$$

We introduce the so-called plane metric

$$\rho([x],[y]) = \inf_{\substack{u\in[x] \\ v\in[y]}} |u - v|.$$

Here, as above, $|\cdot|$ is a Euclidean norm in $\mathbb{R}^2$.

It is obvious that $[x(t)]$ is a solution and the metric space R^2/Γ is a phase space of system (11.2). This space is partitioned into the nonintersecting trajectories $[x(t)]$, $t \in \mathbb{R}^1$.

It is easy to establish the following diffeomorphism between R^2/Γ and a surface of the cylinder $R^1 \times C$. Here C is a circle of unit radius.

Consider the set $\Omega = \{x \mid \theta \in (0, 2\pi],\ z \in \mathbb{R}^1\}$, in which exactly one representer of each class $[x] \in \mathbb{R}^2/\Gamma$ is situated. Cover the surface of cylinder by the set Ω, winding Ω round this surface (Fig. 11.5)

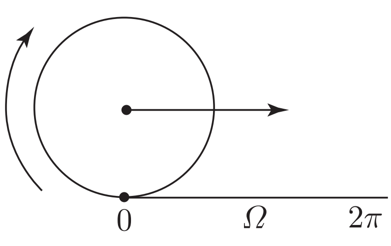

Fig. 11.5 Cylindrical space.

It is obvious that the map constructed is a diffeomorphism. Therefore, the surface of the cylinder is also partitioned into nonintersecting trajectories. Such a phase space is called *cylindrical*.

Note that heteroclinic trajectory (11.4) in the phase space $\mathbb{R}^2$ becomes homoclinic in the cylindrical phase space and in the phase space R^2/Γ since we have

$$\lim_{t\to+\infty}[x(t)] = \lim_{t\to-\infty}[x(t)] = \begin{bmatrix} \theta_0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} \theta_0 \\ 0 \end{bmatrix} = \left\{ \begin{pmatrix} \theta_0 + 2k\pi \\ 0 \end{pmatrix} \mid k \in Z \right\}.$$

Now the assertion obtained is formulated in the following way. Consider the smooth path $\alpha(s)\,(s \in [0,1])$ such that $\alpha(0) = 0,\ \alpha(s) > 0, \forall\, s \in (0,1),\ \alpha(1) > 2.$

Theorem 11.1 (Tricomi). *For any $\gamma > 0$ there exists $s_0 \in (0,1)$ such that system (11.2) with the parameters $\gamma, \alpha(s_0)$ has a homoclinic trajectory in the phase space R^2/Γ.* See Fig. 11.6.

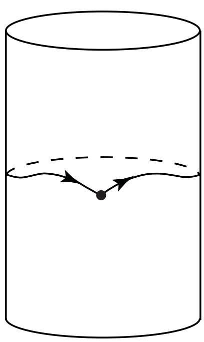

Fig. 11.6 Homoclinic trajectory on cylinder.

We proceed to obtain a similar assertion for the Lorenz system

$$\begin{aligned} \dot{x} &= \sigma(x-y), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy, \end{aligned} \tag{11.6}$$

where σ, b, r are positive. The function

$$V(x,y,z) = y^2 + z^2 + \frac{1}{\sigma}x^2$$

satisfies

$$\dot{V}(x(t), y(t), z(t)) = -2bz(t)^2 - 2y(t)^2 - 2x(t)^2 + 2(r+1)x(t)y(t).$$

From this we easily find that for $r \leq 1$, all the solutions of system (11.6) tend to zero as $t \to +\infty$. Therefore we consider further the case $r > 1$.

Using the transformation

$$\begin{array}{lll} \theta = \frac{\varepsilon x}{\sqrt{2\sigma}}, & \eta = \varepsilon^2\sqrt{2}(y-x), & \xi = \varepsilon^2(z - \frac{x^2}{b}), \\ t = t_1\frac{\sqrt{\sigma}}{\varepsilon}, & \varepsilon = \frac{1}{\sqrt{r-1}}, & \end{array} \tag{11.7}$$

we reduce system (11.6) to the form

$$\begin{aligned} \dot{\theta} &= \eta, \\ \dot{\eta} &= -\mu\eta - \xi\theta - \varphi(\theta), \\ \dot{\xi} &= -\alpha\xi - \beta\theta\eta. \end{aligned} \tag{11.8}$$

Here

$$\varphi(\theta) = -\theta + \gamma\theta^3, \quad \mu = \frac{\varepsilon(\sigma+1)}{\sqrt{\sigma}}, \quad \alpha = \frac{\varepsilon b}{\sqrt{\sigma}}, \quad \beta = 2\left(\frac{2\sigma}{b} - 1\right), \quad \gamma = \frac{2\sigma}{b}.$$

It follows easily that if the conditions

$$\begin{aligned} \lim_{t\to+\infty} \theta(t) = \lim_{t\to-\infty} \theta(t) = \lim_{t\to+\infty} \eta(t) \\ = \lim_{t\to-\infty} \eta(t) = \lim_{t\to+\infty} \xi(t) = \lim_{t\to-\infty} \xi(t) = 0 \end{aligned}$$

are satisfied, then

$$\begin{aligned} \lim_{t\to+\infty} x(t) = \lim_{t\to-\infty} x(t) = \lim_{t\to+\infty} y(t) \\ = \lim_{t\to-\infty} y(t) = \lim_{t\to+\infty} z(t) = \lim_{t\to-\infty} z(t) = 0. \end{aligned}$$

Thus, a homoclinic trajectory of system (11.8) corresponds to a homoclinic trajectory of system (11.6).

Denote by $\theta^+(t), \eta(t)^+, \xi(t)^+$ a separatrix of the saddle $\theta = \eta = \xi = 0$, outgoing in the half-plane $\{\theta > 0\}$. See Fig. 11.7.

In other words, we consider a solution of system (11.6) such that

$$\lim_{t\to-\infty} \theta(t)^+ = \lim_{t\to-\infty} \eta(t)^+ = \lim_{t\to-\infty} \xi(t)^+ = 0$$

and $\theta(t)^+ > 0$ for $t \in (-\infty, T)$. Here T is a certain number or $+\infty$.

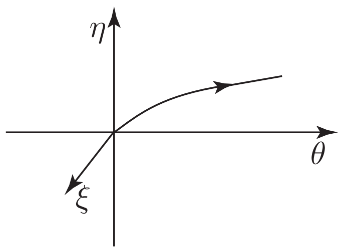

Fig. 11.7 Separatrix of the Lorenz system.

Consider the smooth path $b(s), \sigma(s), r(s)$ $(s \in [0,1])$ in a space of the parameters $\{b, \sigma, r\}$. It is clear that in this case the parameters $\alpha, \beta, \gamma, \mu$ are also smooth functions of $s \in [0,1]$.

Theorem 11.2. *Let $\beta(s) > 0$, $\forall\, s \in [0,1]$ and for $s \in [0, s_0)$ suppose there exist $T(s) > \tau(s)$ such that the relations*

$$\theta(T)^+ = \eta(\tau)^+ = 0, \tag{11.9}$$

$$\theta(t)^+ > 0, \quad \forall\, t < T, \tag{11.10}$$

$$\eta(t)^+ \neq 0, \quad \forall\, t < T, \;\; t \neq \tau, \tag{11.11}$$

are satisfied. Suppose also that for $s = s_0$ there does not exist the pair $T(s_0) > \tau(s_0)$ such that relations (11.9)–(11.11) are satisfied. Then system (11.8) with the parameters $b(s_0), \sigma(s_0), r(s_0)$ has the homoclinic trajectory $\theta(t)^+, \eta(t)^+, \xi(t)^+$:

$$\lim_{t\to+\infty} \theta(t)^+ = \lim_{t\to+\infty} \eta(t)^+ = \lim_{t\to+\infty} \xi(t)^+ = 0.$$

To prove this theorem we need the following

Lemma 11.1. *If for system (11.8) the conditions*

$$\eta(\tau)^+ = 0, \quad \eta(t)^+ > 0, \quad \forall\, t \in (-\infty, \tau),$$

are valid, then $\dot\eta(\tau)^+ < 0$.

Proof. Suppose to the contrary that $\dot\eta(\tau)^+ = 0$. In this case from the last two equations of system (11.8) we obtain

$$\ddot\eta(\tau)^+ = \alpha\xi(\tau)^+\theta(\tau)^+. \tag{11.12}$$

From the relations $\eta(t)^+ > 0$, $\theta(t)^+ > 0$, $\forall\, t \in (-\infty, \tau)$ and the last equation of system (11.8) we obtain $\xi(t)^+ < 0\ \forall\, t \in (-\infty, \tau]$. Then (11.12) yields the inequality $\ddot{\eta}(\tau)^+ < 0$, which contradicts the assumption $\dot{\eta}(\tau)^+ = 0$ and the hypotheses of the lemma. □

Lemma 11.2. *Let $\beta(s) > 0$, $\forall\, s \in [0, 1]$. Suppose that for system (11.8), relations (11.9), (11.10) and the inequalities*

$$\begin{aligned} &\eta(t)^+ > 0, \quad \forall\, t \in (-\infty, \tau), \\ &\eta(t)^+ \leq 0, \quad \forall\, t \in (\tau, T), \end{aligned} \tag{11.13}$$

are valid. Then inequality (11.11) is also valid.

Proof. Assuming the contrary, we see that there exists $\rho \in (\tau, T)$ such that the relations

$$\begin{aligned} &\eta(\rho)^+ = \dot{\eta}(\rho)^+ = 0, \\ &\ddot{\eta}(\rho)^+ = \alpha\theta(\rho)^+\xi(\rho)^+ < 0, \\ &\eta(t)^+ < 0, \quad \forall\, t \in (\rho, T), \end{aligned}$$

are satisfied. Then from conditions (11.9), (11.10) and from the fact that the trajectories $\theta(t) = \eta(t) = 0$, $\xi(t) = \xi(0)\exp(-\alpha t)$ belong to a stable manifold of the saddle $\theta = \eta = \xi = 0$ we obtain the crossing of the separatrix $\theta(t)^+, \eta(t)^+, \xi(t)^+$ and this stable manifold. Therefore the separatrix belongs completely to the stable manifold of the saddle. In addition, the condition $\theta(t)^+ > 0$, $\forall\, t \geq \rho$ holds. The latter is in the contrast to condition (11.9). This contradiction proves Lemma 11.2. □

It is possible to give the following geometric interpretation of the proof of Lemma 11.2 in the phase space with coordinates θ, η, ξ. "Under" the set $\{\theta > 0, \eta = 0, \xi \leq 1 - \gamma\theta^2\}$ is situated the piece of stable two-dimensional manifold of the saddle $\theta = \eta = \xi = 0$. This does not allow the trajectories with the initial data from this set to attain the plane $\{\theta = 0\}$ if they remain in the quadrant $\{\theta \geq 0, \eta \leq 0\}$.

Consider the polynomial

$$p^3 + ap^2 + bp + c, \tag{11.14}$$

where a, b, c are positive.

Lemma 11.3. *Either all zeros of (11.14) have negative real parts, or two zeros of (11.14) have nonzero imaginary parts.*

Proof. It is well known [88] that all zeros of (11.14) have negative real parts if and only if $ab > 0$. For $ab = c$, polynomial (11.14) has two pure imaginary zeros.

Suppose now that for the certain a, b, c such that $ab < c$, polynomial (11.14) has real zeros only. From the positiveness of coefficients it follows that these zeros are negative. The latter yields $ab > c$, which contradicts the assumption. □

Proof of Theorem 11.2. We shall show that to the values of the parameters $b(s_0), \sigma(s_0), r(s_0)$ there corresponds a homoclinic trajectory.

First note that for these parameters for the certain τ the relations

$$\begin{aligned} &\eta(t)^+ > 0, \quad \forall\, t < \tau, \quad \eta(t)^+ \le 0, \quad \forall\, t \ge \tau, \\ &\theta(t)^+ > 0, \quad \forall\, t \in (-\infty, +\infty), \end{aligned} \tag{11.15}$$

hold. Actually, if there exist $T_2 > T_1 > \tau$ such that

$$\begin{gathered} \theta(t)^+ > 0, \quad \forall\, t \in (-\infty, T_2); \quad \theta(T_2)^+ = 0, \;\; \eta(T_1)^+ > 0, \\ \eta(t)^+ > 0, \quad \forall\, t < \tau; \quad \eta(\tau)^+ = 0, \;\; \dot\eta(\tau)^+ < 0, \end{gathered}$$

then for the values $s < s_0$ and for the values s sufficiently close to s_0 the inequality $\eta(T_1)^+ > 0$ holds true. This is in the contrast to the definition of s_0. If there exist $T_1 > \tau$ such that

$$\begin{gathered} \eta(T_1)^+ > 0, \quad \eta(t)^+ > 0, \quad \forall\, t < \tau, \\ \eta(\tau)^+ = 0, \quad \dot\eta(\tau)^+ < 0, \quad \theta(t)^+ > 0, \quad \forall\, t \in (-\infty, +\infty), \end{gathered}$$

then for $s < s_0$, which is sufficiently closed to s_0, the inequality $\eta(T_1)^+ > 0$ holds true, which is in contrast to the definition of s_0. If there exist $T > \tau$ such that

$$\begin{gathered} \theta(t)^+ > 0, \quad \forall\, t < T, \quad \theta(T)^+ = 0, \quad \eta(t)^+ > 0, \quad \forall\, t < \tau, \\ \eta(t)^+ \le 0, \quad \forall\, t \in [\tau, T], \end{gathered}$$

then by Lemma 11.2 inequality (11.11) holds. Therefore for $s = s_0$ relations (11.9)–(11.11) are valid, which is in the contrast to the hypotheses of the theorem. This contradiction proves inequality (11.15).

From (11.15) it follows that only one of the equilibria can be the ω-limit set of the trajectory $\theta(t)^+, \eta(t)^+, \xi(t)^+$ for $s = s_0$. We shall show that the equilibrium $\theta = 1/\sqrt{\gamma}$, $\eta = \xi = 0$ cannot be the ω-limit point of the considered trajectory.

Having performed the linearization in the neighborhood of this equilibrium, we obtain the characteristic polynomial

$$p^3 + (\alpha + \mu)p^2 + (\alpha\mu + 2/\gamma)p + 2\alpha.$$

Suppose, for $s = s_0$ the separatrix $\theta(t)^+, \eta(t)^+, \xi(t)^+$ has in its ω-limit set the point $\theta = 1/\sqrt{\gamma}, \eta = \xi = 0$. By Lemma 11.3 and from a continuous dependence of the semitrajectories $\{\theta(t)^+, \eta(t)^+, \xi(t)^+ | \, t \in (-\infty, t_0)\}$ on the parameter s we obtain that for the values s sufficiently close to s_0, the separatrices $\theta(t)^+, \eta(t)^+, \xi(t)^+$ either tend to the equilibrium $\theta = 1/\sqrt{\gamma}, \eta = \xi = 0$ as $t \to +\infty$ or oscillate on the certain time interval with the sign reversal of the coordinate η. Both possibilities are in contrast to properties (11.9)–(11.11).

Thus, for system (11.8) with the parameters $b(s_0), \sigma(s_0), r(s_0)$ the trajectory $\theta(t)^+, \eta(t)^+, \xi(t)^+$ tends to zero equilibrium as $t \to +\infty$. □

Remark 11.1. It is well known that the semitrajectory

$$\{\theta(t)^+, \eta(t)^+, \xi(t)^+ \mid t \in (-\infty, t_0)\}$$

depends continuously on the parameter s. Here t_0 is a certain fixed number. Then Lemma 11.1 implies that if for system (11.8) with the parameters $b(s_1), \sigma(s_1), r(s_1)$ relations (11.9)–(11.11) are satisfied, then these relations are also satisfied for $b(s), \sigma(s), r(s)$. Here $s \in (s_1 - \delta, s_1 + \delta)$ where δ is sufficiently small. □

Theorem 11.2 and Remark 11.1 result in the following

Theorem 11.3. *Let be $\beta(s) > 0$, $\forall\, s \in [0, 1]$. Suppose, for system (11.8) with the parameters $b(0), \sigma(0).r(0)$ there exist $T > \tau$ such that relations (11.9)–(11.11) are valid. Suppose also that for system (11.8) with the parameters $b(1), \sigma(1), r(1)$ the inequality*

$$\theta(t)^+ > 0, \quad \forall\, t \in (-\infty, +\infty),$$

holds. Then there exists $s_0 \in [0, 1]$ such that system (11.8) with the parameters $b(s_0), \sigma(s_0), r(s_0)$ has the homoclinic trajectory $\theta(t)^+, \eta(t)^+, \xi(t)^+$.

We shall show that if

$$3\sigma - 2b > 1, \tag{11.16}$$

then for sufficiently large r the relations (11.9)–(11.11) are valid. Consider the system

$$Q\frac{dQ}{d\theta} = -\mu Q - P\theta - \varphi(\theta),$$
$$Q\frac{dP}{d\theta} = -\alpha P - \beta Q\theta, \tag{11.17}$$

which is equivalent to (11.8) in the sets $\{\theta \geq 0, \eta > 0\}$ and $\{\theta \geq 0, \eta < 0\}$. Here P and Q are the solutions of system (11.17). It is clear that P and Q are functions of $\theta : P(\theta), Q(\theta)$.

We perform the asymptotic integration of the solutions of system (11.17) with the small parameter ε, which corresponds to the separatrix $\theta(t)^+, \eta(t)^+, \xi(t)^+$. For this purpose we transform (11.17) as follows:

$$\frac{1}{2}\frac{d}{d\theta}(Q(\theta))^2 = -\mu Q(\theta) - P(\theta)\theta - \varphi(\theta),$$
$$\frac{dP(\theta)}{d\theta} = -\alpha\frac{P(\theta)}{Q(\theta)} - \beta\theta.$$

Here α and μ are small parameters. In the first approximation the solutions considered can be represented in the form

$$Q_1(\theta)^2 = \theta^2 - \tfrac{\theta^4}{2} - 2\mu\int_0^\theta \theta\sqrt{1-\tfrac{\theta^2}{2}}d\theta - 2\alpha\beta\int_0^\theta \theta\left(1-\sqrt{1-\tfrac{\theta^2}{2}}\right)d\theta,$$
$$Q_1(\theta) \geq 0, \quad P_1(\theta) = -\left(\tfrac{\beta}{2}\right)\theta^2 + \alpha\beta\left(1-\sqrt{1-\tfrac{\theta^2}{2}}\right),$$
$$Q_2(\theta)^2 = \theta^2 - \tfrac{\theta^4}{2} - 2\mu\int_\theta^{\sqrt{2}} \theta\sqrt{1-\tfrac{\theta^2}{2}}\,d\theta - \tfrac{4}{3}\mu$$
$$+2\alpha\beta\int_\theta^{\sqrt{2}} \theta\left(1+\sqrt{1-\tfrac{\theta^2}{2}}\right)d\theta - \tfrac{2}{3}\alpha\beta,$$
$$Q_2(\theta) \leq 0, \quad P_2(\theta) = -\left(\tfrac{\beta}{2}\right)\theta^2 + \alpha\beta\left(1+\sqrt{1-\tfrac{\theta^2}{2}}\right).$$

This implies that if inequality (11.16) is satisfied, then for the certain $T > \tau$ relations (11.9)–(11.11) are valid. In addition we have

$$\xi(T)^+ = P_2(0) = 2\alpha\beta,$$
$$\eta(T)^+ = Q_2(0) = -\sqrt{8(\alpha\beta-\mu)/3} = -\sqrt{8\varepsilon(3\sigma-2b-1)/3\sqrt{\sigma}}.$$

Thus, if inequality (11.16) is satisfied, then for sufficiently large r relations (11.9)–(11.11) are valid . □

We now obtain conditions such that relations (11.9)–(11.11) do not hold and

$$\theta(t)^+ > 0, \;\; \forall\, t \in (-\infty, +\infty). \tag{11.18}$$

Consider first the case $\beta < 0$. Here for the function

$$V(\theta,\eta,\xi) = \eta^2 - \frac{1}{\beta}\xi^2 + \int_0^\theta \varphi(\theta)d\theta$$

we have

$$\dot V(\theta(t),\eta(t),\xi(t)) = -2\left(\mu\eta(t)^2 - \frac{\alpha}{\beta}\xi(t)^2\right). \tag{11.19}$$

Thus, for $\beta < 0$ the function V is the Lyapunov function for system (11.8). From the conditions (11.19) and $\beta < 0$ we obtain

$$V(\theta(t)^+,\eta(t)^+,\xi(t)^+) < V(\theta(-\infty)^+,\eta(-\infty)^+,\xi(-\infty)^+) = V(0,0,0) = 0,$$
$$\forall\, t \in (-\infty,+\infty).$$

This implies (11.18). In this case the separatrix $\theta(t)^+,\eta(t)^+,\xi(t)^+$ does not tend to zero as $t \to +\infty$. For $\beta = 0$ we have $\xi(t)^+ \equiv 0$ and from the first two equations of system (11.8) we obtain at once (11.18). In this case the separatrix $\theta(t)^+,\eta(t)^+,\xi(t)^+$ does not tend to zero as $t \to +\infty$. □

Consider the case

$$\beta = \frac{2}{b}(2\sigma - b) > 0.$$

In this case by using the change of variables

$$\eta = \sigma(x-y), \quad Q = z - x^2/(2\sigma),$$

we can reduce system (11.6) to the form

$$\begin{aligned} &\dot x = \eta, \\ &\dot\eta = -(\sigma+1)\eta + \sigma\{(r-1) - Q - \tfrac{x^2}{2\sigma}\}x, \\ &\dot Q = -bQ + (1 - \tfrac{b}{2\sigma})x^2. \end{aligned} \tag{11.20}$$

Consider the separatrix $x(t)^+,\eta(t)^+,Q(t)^+$ of zero saddle equilibrium such that

$$\begin{aligned} &\lim_{t\to-\infty} x(t)^+ = \lim_{t\to-\infty}\eta(t)^+ = \lim_{t\to-\infty} Q(t)^+ = 0, \\ &x(t)^+ > 0, \quad \forall\, t \in (-\infty, T). \end{aligned} \tag{11.21}$$

Find the estimates of this separatrix.

Lemma 11.3. *The estimate*

$$Q(t)^+ \geq 0, \quad \forall\, t \in (-\infty,+\infty), \tag{11.22}$$

is valid.

Proof. From the inequality $2\sigma > b$ and from the last equation of (11.20) we have

$$\dot{Q}(t) \geq -bQ(t).$$

This implies that

$$Q(t) \geq \exp(-bt)Q(0).$$

Therefore (11.22) holds. □

Lemma 11.4. *From condition (11.21) follows the inequality*

$$\eta(t)^+ \leq Lx(t)^+, \quad \forall\, t \in (-\infty, T), \tag{11.23}$$

where

$$L = -\frac{\sigma+1}{2} + \sqrt{\frac{(\sigma+1)^2}{4} + \sigma(r-1)}.$$

Proof. Relation (11.22) and the first two equations of system (11.20) give

$$\eta(t)^+ \leq \widetilde{\eta}(t)^+, \quad \forall\, t \in (-\infty, T). \tag{11.24}$$

Here $\widetilde{\eta}(t)^+, \widetilde{x}(t)^+$ is the separatrix of zero saddle of the system

$$\begin{aligned} \dot{x} &= \eta, \\ \dot{\eta} &= -(\sigma+1)\eta + \sigma(r-1)x. \end{aligned}$$

Obviously, $\widetilde{\eta}(t)^+ = L\widetilde{x}(t)^+$. The lemma follows from (11.24). □

Lemma 11.5. *From condition (11.21) follows the estimate*

$$Q(t)^+ \geq a(x(t)^+)^2, \quad \forall\, t \in (-\infty, T), \tag{11.25}$$

where

$$a = \frac{(2\sigma - b)}{(2\sigma(2L+b))}.$$

Proof. Estimate (11.23) gives the differential inequality

$$\begin{aligned} &\big(Q(t)^+ - a(x(t)^+)^2\big)^{\bullet} + b(Q(t)^+ - a(x(t)^+)^2) \\ &\qquad \geq \left[(1 - \frac{b}{2\sigma}) - 2aL - ab\right](x(t)^+)^2 = 0. \end{aligned}$$

This implies (11.25). □

Consider now the Lyapunov-type function introduced in [24]:

$$V(x,\eta,Q) = \eta^2 + \sigma x^2\left(\frac{x^2}{4\sigma} + Q - (r-1)\right) + (\sigma+1)x\eta. \qquad (11.26)$$

It can easily be checked that for the solutions $x(t), \eta(t), Q(t)$ of system (11.20) we have

$$\dot V(x(t),\eta(t),Q(t)) = -(\sigma+1)V(x(t),\eta(t),Q(t)) + \frac{3}{4}\left(\sigma - \frac{2b+1}{3}\right)x(t)^4 - b\sigma Q(t)x(t)^2. \qquad (11.27)$$

Lemma 11.6. *Let the inequality*

$$3\sigma - (2b+1) < \frac{2b(2\sigma-b)}{2L+b} \qquad (11.28)$$

be valid. Then condition (11.21) results in the estimate

$$\dot V(x(t)^+,\eta(t)^+,Q(t)^+) + (\sigma+1)V(x(t)^+,\eta(t)^+,Q(t)^+) < 0, \quad \forall\, t \in (-\infty, T). \qquad (11.29)$$

Proof. From (11.28) and (11.25) we have

$$\frac{3}{4}\left(\sigma + \frac{2b+1}{3}\right)(x(t)^+)^4 - b\sigma Q(t)(x(t)^+)^2 < 0, \quad \forall\, t \in (-\infty, T).$$

Then (11.27) yields estimate (11.29). Note that relation (11.29) results in the inequality

$$V(x(T)^+,\eta(T)^+,Q(T)^+) < 0.$$

It is easy to see that

$$V(0,\eta,Q) \geq 0, \quad \forall \eta \in R^1, \ \forall Q \in R^1.$$

Therefore, if (11.28) is satisfied, then (11.29) is satisfied for all $T \in \mathbb{R}^1$. □

Thus, we can formulate

Theorem 11.4 [70]. *If inequality (11.28) holds, then so does (11.18) and the separatrix $x(t)^+, \eta(t)^+, Q(t)^+$ does not tend to zero as $t \to +\infty$.*

This implies

Theorem 11.5. *If*

$$2b + 1 \geq 3\sigma,$$

then for any $r > 1$ *the homoclinic trajectory of system (11.6) does not exist.*

Theorem 11.6. *If*

$$2b + 1 < 3\sigma,$$

then for the values $r > 1$ *and sufficiently close to* 1 *the conditions (11.9)–(11.11) are not valid.*

Theorems 11.3, 11.5, 11.6 imply the following

Theorem 11.7. *Given* b *and* σ *fixed, for the existence of* $r \in (1, +\infty)$, *corresponding to the homoclinic trajectory of the saddle* $x = y = z = 0$, *it is necessary and sufficient that*

$$2b + 1 < 3\sigma. \tag{11.30}$$

The sufficiency of condition (11.30) was first proved in [55]. It was proved by another method (the shooting method [39–41]) in [24]. The papers [41, 24] involve the notes, added in the proof, about a priority of the assertion from [55].

In the papers [55, 56] the conjecture was asserted that (11.30) is a necessary condition for the existence of a homoclinic trajectory. This conjecture is proved in [24] on the basis of constructing the Lyapunov-type function (11.26).

We remark that the consideration of the smooth paths, in the space of parameters of nonlinear dynamic systems, on which there exist the points of homoclinic bifurcation, is a fruitful direction in the development of the analytic theory of global bifurcations.

We formulate now one more assertion of the same type, obtained for the Lorenz system in the paper [74].

Theorem. *Let be* $\sigma = 10, r = 28$. *Then there exists* $b \in (0, +\infty)$ *such that (11.6) has a homoclinic trajectory.*

Chapter 12

Frequency Criterion for Weak Exponential Instability on Attractors of Discrete Systems

In this chapter we suggest an approach using lower frequency estimates of the Lyapunov exponents [63, 65, 68, 69, 86]. This is developed in the spirit of classical stability theory and modern nonlinear control: the basic results are formulated in terms of transfer functions, and the frequency responses of linear portions of the systems are considered. The conditions imposed on the characteristics of the nonlinear blocks $\varphi(\sigma)$ are close to the traditional "inequalities of the sector type" [83–88, 115, 138], which are assumed for the theory of absolute stability of nonlinear control systems. However for the attractors to be unstable it is necessary turn such sectors conditions "inside out". In place of the usual conditions $\varphi(\sigma)/\sigma \in [k_1, k_2]$ or $\varphi'(\sigma) \in [k_1, k_2]$ for all $\sigma \in \mathbb{R}^1$, it is required that $\varphi' \notin [k_1, k_2]$ for almost all σ. In this case on some sets of nonzero measure we have $\varphi'(\sigma) < k_1$ or $\varphi'(\sigma) > k_2$. For continuous systems such conditions and the corresponding frequency inequalities for the transfer functions of the linear part separate out empty sets in the space of parameters. However for discrete systems such consideration permits us to obtain the frequency criteria for instability on attractors. Here we give some examples of concrete first- and second-order discrete dynamical systems for which the frequency criterion considered gives the condition of instability on attractors.

For discrete systems we can seldom visualize a representation in phase space, which is characteristic for continuous systems having smooth trajectories filling this set. However we can partially avoid this deficiency if rather than observing the change of distance between two solutions at discrete moments we consider at the initial moment the line segment connecting the initial data and its iterates, i.e., the sequence of continuous curves. The decreasing or increasing of the lengths of these curves provides information on the stability or instability, respectively, and that on the other properties of solutions of discrete systems. The consideration of the lengths of such continuous curves can also be used as the basis for the definitions of stability and instability, which is especially important for the noninjective maps, defining the corresponding discrete systems (namely, at the present time they are the widely-known generators of chaos). The point is that sometimes two trajectories can "stick together" for a finite number of iterations. However the length of iteratable segment, connecting the initial data, can increase even with provision for such sticking together.

In some cases the stretching mechanism turns out to be sufficiently intense and we can find and estimate it using the rich arsenal of the direct Lyapunov method, which can be combined with the effective algebraic apparatus: the Kalman—Szego lemma and the Shepeljavy lemma. The lemmas gives the estimate of the stretching of the lengths of iteratable curves in terms of the frequency inequalities, which operate on the transfer functions of linear blocks.

The effect of "sticking together", which is less intense than "stretching" in the small neighborhoods of the trajectories considered, is strengthened with increasing size of this neighborhood. At a certain time the neighborhood turns out to be large and the "sticking together" becomes so intense that it completely compensates for the stretching.

Such a situation permits the coexistence of the local exponential instability on attractor and the attractive properties (or, at least, the compactness) of the attractor itself. This also permits us to apply the rich arsenal of methods, devices, and results of the nonlinear analysis of discrete control systems to the analysis of the strange attractors of discrete systems.

12.1 The Yakubovich—Kalman and Kalman—Szego Lemmas

Proceeding with the parallel consideration of continuous and discrete dynamic systems, we can state the lemmas of Yakubovich—Kalman [10, 84, 85, 138] and Kalman—Szego [10, 89, 113] and clarify the relationship between these two assertions.

Let P and q be matrices of dimensions $n \times n$ and $n \times m$, respectively, $F(x, \xi)$ be the Hermitian form of the vector variables $x \in \mathbb{C}^n$, $\xi \in \mathbb{C}^m$. Here $\mathbb{C}^n$ is an n-dimension complex-valued linear space with the Euclidean norm

$$|x| = \sqrt{|x_1|^2 + \ldots + |x_n|^2},$$

where x_j are the components of the vector x. Recall that any Hermitian form $F(x,\xi)$ can be represented as

$$F(x,\xi) = \begin{pmatrix} x \\ \xi \end{pmatrix}^* M \begin{pmatrix} x \\ \xi \end{pmatrix}, \tag{12.1}$$

where the asterisk denotes the conjugate transpose and M is a self-adjoint (Hermitian) $(n+m)\times(n+m)$-matrix (i.e., $M^* = M$).

From (12.1) we conclude that for any $x \in \mathbb{C}^n$ and $\xi \in \mathbb{C}^m$ the Hermitian form is purely real. For real M and for $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^m$ the form $F(x,\xi)$ becomes quadratic. Below we shall show that in studying quadratic forms of the type (12.1) it is often useful to "go" in the complex-valued space $\mathbb{C}^{n+m}$ ($x \in \mathbb{C}^n$, $\xi \in \mathbb{C}^m$) while preserving the Hermitian form (i.e., preserving M as a symmetric real matrix.

To proceed further we need the following definition [10, 84, 85, 138].

Definition 12.1. The pair (P,q) is said to be *controllable* if the rank of the matrix

$$(q, Pq, \ldots, P^{n-1}q)$$

is equal to n.

Note that in the most interesting case when q is a vector the condition of controllability can be written as

$$\det(q, Pq, \ldots, P^{n-1}q) \neq 0.$$

Lemma 12.1 (Yakubovich—Kalman). *Let the pair (P,q) be controllable. For the existence of the Hermitian matrix $H = H^*$ such that for any $x \in \mathbb{C}^n$ and $\xi \in \mathbb{C}^m$ the inequality*

$$2\,\mathrm{Re}[x^* H(Px + q\xi)] + F(x,\xi) \leq 0 \tag{12.2}$$

holds, it is necessary and sufficient that for any $\omega \in \mathbb{R}^1$ and any $\xi \in \mathbb{C}^m$ the inequality

$$F((i\omega I - P)^{-1}q\xi, \xi) \leq 0 \tag{12.3}$$

be satisfied. If P, q, and the coefficients of the form F are real, then the elements of H can be chosen real.

All known proofs of this fact are complicated. They can be found in many books [85, 115, 138] and will therefore be omitted. As a corollary, we obtain the similar Kalman—Szego lemma.

Lemma 12.2. *For the matrices P, q, H, and the number ρ such that $|\rho| = 1$ and $\det(P + \rho I) \neq 0$ we have*

$$(Pz + q\xi)^* H(Pz + q\xi) - z^* Hz = 2\,\mathrm{Re}\, x^* H(P_0 x + q_0 \xi)$$

for all $x \in \mathbb{C}^n$ and $\xi \in \mathbb{C}^m$, where

$$z = \rho^{-1}(I - P_0)(\frac{1}{\sqrt{2}}x - \frac{1}{2}q\xi), \tag{12.4}$$
$$P_0 = (P + \rho I)^{-1}(P - \rho I),$$
$$q_0 = \frac{1}{\sqrt{2}}(I - P_0)q.$$

Proof. It is easy to see that

$$P = \rho(I - P_0)^{-1}(I + P_0).$$

Therefore

$$\begin{aligned}(Pz + q\xi) &= \rho(I - P_0)^{-1}(I + P_0)\rho^{-1}(I - P_0)\left(\frac{1}{\sqrt{2}}x - \frac{1}{2}q\xi\right) + q\xi \\ &= \frac{1}{\sqrt{2}}(I + P_0)x + \frac{1}{2}(I - P_0)q\xi. \end{aligned} \tag{12.5}$$

This implies that

$$\begin{aligned}
&(Pz + q\xi)^* H(Pz + q\xi) - z^* Hz \\
&\quad = \frac{1}{2}x^*[(I + P_0)^* H(I + P_0) - (I - P_0)^* H(I - P_0)]x \\
&\quad = \frac{1}{4}\xi^*[q^*(I - P_0)^* H(I - P_0)q - q^*(I - P_0)^* H(I - P_0)q]\xi \\
&\qquad + \frac{1}{2\sqrt{2}}x^*[(I + P_0)^* H(I - P_0)q + (I - P_0)^* H(I - P_0)q]\xi \\
&\qquad + \frac{1}{2\sqrt{2}}\xi^*[q^*(I - P_0)H(I + P_0) + q^*(I - P_0)^* H(I - P_0)]x \\
&\quad = x^*(P_0^* H + HP_0)x + \sqrt{2}\,\mathrm{Re}(x^* H(I - P_0)q\xi) \\
&\quad = 2\,\mathrm{Re}(x^*(P_0 x + \frac{1}{\sqrt{2}}(I - P_0)q\xi)).
\end{aligned}$$

Lemma is proved. □

Note that from (12.5) and the relation

$$P_0 x + q_0 \xi = i\omega x \tag{12.6}$$

we have

$$Pz + q\xi = \frac{1}{\sqrt{2}}[(I + P_0)x + q_0\xi] = \frac{1}{\sqrt{2}}(1 + i\omega)x. \tag{12.7}$$

Relation (12.4) can be rewritten as

$$\sqrt{2}\rho z = x - P_0x - q_0x. \tag{12.8}$$

By (12.6)–(12.8) we obtain

$$Pz + q\xi = \rho\frac{1 + i\omega}{1 - i\omega}z. \tag{12.9}$$

By Lemma 12.2 and the equivalence of relations (12.6) and (12.9) we can state now the Kalman—Szego lemma as a corollary of the Yakubovich—Kalman lemma.

Lemma 12.3 (Kalman—Szego). *Let the pair (P, q) be controllable. For the existence of the Hermitian matrix $H = H^*$ such that for any $x \in \mathbb{C}^n$ and $\xi \in \mathbb{C}^m$ the inequality holds*

$$(Px + q\xi)^*H(Px + q\xi) - x^*Hx + F(x, \xi) \leq 0 \tag{12.10}$$

it is necessary and sufficient that for any values $p \in \mathbb{C}^1$ such that $|p| = 1$ and for any $\xi \in \mathbb{C}^m$ the inequality

$$F((pI - P)^{-1}q\xi, \xi) \leq 0 \tag{12.11}$$

is satisfied.

Note that if P, q, and the coefficients of the form F are real and for the Hermitean matrix $H = H_1 + iH_2$ (H_1 and H_2 are real) inequality (12.10) is valid, then for the matrix $H = H_1$ relation (12.10) is also satisfied for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^m$. Then if condition (12.11) is valid for real P, q, and F, then there exists the real symmetric matrix $H = H_1$ such that inequality (12.10) is satisfied for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^m$.

The following result is often useful.

Lemma 12.4 [126]. *Let under the conditions of the Kalman—Szego lemma be $F(x, 0) > 0$, $\forall x \in \mathbb{C}^n$, $x \neq 0$ and the matrix P have m eigenvalues outside the unit circle and $n - m$ eigenvalues inside it. Then the matrix H has m negative and $n - m$ positive eigenvalues.*

Proof. It is sufficient to make use of Lemma 12.2 and the inequality

$$\operatorname{Re} x^*HP_0x < 0, \quad \forall x \in \mathbb{C}^n, \quad x \neq 0, \tag{12.12}$$

where

$$P_0 = (P + \rho I)^{-1}(P - \rho I).$$

It is obvious that the eigenvalues λ_j of the matrix P and the eigenvalues ν_j of the matrix P_0 are related as

$$\nu_j = \frac{\lambda_j - \rho}{\lambda_j + \rho}.$$

Therefore we have

$$\operatorname{Re} \nu_j = \frac{|\lambda_j|^2 - |\rho|^2}{|\lambda_j + \rho|^2}.$$

Whence it follows that under the condition of lemma m eigenvalues of the matrix P_0 have positive real parts and $n - m$ the negative ones. It is well known [84, 85, 88, 138] that if (12.12) and the above restrictions on the spectrum of the matrix P_0 are satisfied the assertion of lemma is true. □

12.2 Definitions of Curve Length and Weak Exponential Instability

Consider a system

$$x(t+1) = f(x(t), t), \qquad x \in \mathbb{R}^n, \quad t \in \mathbb{Z}, \tag{12.13}$$

where $f(x,t)$ is a vector-function: $\mathbb{R}^n \times \mathbb{Z} \to \mathbb{R}^n$, $\mathbb{R}^n$ is an Euclidean space, $\mathbb{Z}$ is a set of integers. Denote by $x(t, x_0)$ the solution of system (12.13) with the initial data $x(0, x_0) = x_0$. Denote by $S(r, x_0)$ the sphere of radius r centered at the point $x_0 \in \mathbb{R}^n$.

In the sequel we shall need the notion of the length $l(\gamma)$ of the curve $\gamma(s)$, $s \in [0,1]$, where $\gamma(s)$ is a continuous vector-function: $[0,1] \to \mathbb{R}^n$.

For discrete systems (12.13) on the curve γ with the parametrization $\gamma(s)$ the iterations

$$\gamma_t(s) = f(\ldots(f(\gamma(s), 0)\ldots t)$$

will be performed. Even in the one-dimensional case the complete information on $\gamma_t(s)$ is, as a rule, lacking. In the cases in question the maps $f(\cdot)$ are noninjective and therefore we have the stick together of some points γ due to the mapping f:

$$\gamma_t(s_1) \neq \gamma_t(s_2),$$

$$f(\gamma_t(s_1), t) = f(\gamma_t(s_2), t).$$

The definition of a length of the path $\gamma_t(s)$ without stick together [45, 127, 144] does not give the necessary information on the separation of trajectories from each other. Therefore it is necessary to define the length of γ taking into account the points, which stick together.

Consider first the case that $\gamma(s)$ is the injective map of the segment $[0,1]$ in $\mathbb{R}^n$. In other words, $\gamma(s_1) \neq \gamma(s_2)$ if $s_1 \neq s_2$. In this case the length $l(\gamma)$ of the curve γ is defined in the following way. Consider the various finite or countable partitions of the segment $Q = [0,1]$ on the set of the type $Q_j = (\alpha_j, \beta_j]$, where

$$Q_i \cap Q_j = \emptyset, \quad \forall\, i \neq j, \quad Q \setminus \{0\} = \bigcup_j Q_j.$$

Here $\emptyset$ is an empty set. In this case we shall say that the length $l(\gamma)$ of the curve $\gamma(s)$ is the following number (or $+\infty$):

$$l(\gamma) = \sup \sum_j \left|\gamma(\beta_j) - \gamma(\alpha_j)\right|, \tag{12.14}$$

where the supremum is taken by the various partitions $\{Q_j\}$ of the segment $[0,1]$. If γ is injective and $l(\gamma) < \infty$, then this is a standard definition of a length of the curve γ [32, 104, 114].

In standard definitions of length the finite partitions are used only. However it is easily shown that if the countable partitions will be added to these partitions, we obtain the same value $l(\gamma)$.

Suppose now that the segment $Q = [0,1]$ can be partitioned into a finite or countable number of sets $L_j \quad (j = 1, \dots, m)$ of the type (α_j, β_j), where

$$L_i \cap L_j = \emptyset, \quad \forall\, i \neq j, \quad Q = \bigcup_j \overline{L}_j, \quad \overline{L}_j = [\alpha_j, \beta_j]$$

and the map $\gamma(\cdot)$ is injective on each set L_j.

In other words, here we consider the curves, which are partitioned into the injective continuous segments $\gamma_j(s) : L_j \to \mathbb{R}^n$.

We now assume that the partition $\{L_j\}$ has one more property: for the subset $\{L_j\}^1$ of this partition the condition

$$\gamma(L_j) \cap \gamma(L_i) = \emptyset$$

is satisfied for any $L_j \in \{L_j\}^1$ and for any $i \neq j$.

For the subset $\{L_j\}^2$ of the partition $\{L_j\}$ the segments $\gamma(L_j)$ shall twice "stick to each other", i.e. for any $L_j \in \{L_j\}^2$ there exists a unique $L_i \in \{L_j\}^2$ $i \neq j$ such that $\gamma(L_j) = \gamma(L_i)$ and $\gamma(L_j) \cap \gamma(L_k) = \emptyset, \ \forall\, k \neq i, \ k \neq j$.

For the subset $\{L_j\}^3$ the corresponding segments $\gamma(L_j)$ shall "triply stick to each other", i.e. for any $L_j \in \{L_j\}^3$ there exist exactly two different $L_i \in \{L_j\}^3, \ L_k \in \{L_j\}^3, \ i \neq j, k \neq j, i \neq k$, such that

$$\gamma(L_j) = \gamma(L_i) = \gamma(L_k)$$

and $\gamma(L_j) \cap \gamma(L_g) = \emptyset, \ \forall\, g \neq j, \ g \neq i, \ g \neq k$.

Proceeding this process, we can define the subset $\{L_j\}^4, \ldots, \{L_j\}^m$. Here m is a certain natural number such that

$$\{L_j\} = \bigcup_{1}^{m} \{L_j\}^k.$$

We assume that such m exists. Naturally, the certain subsets $\{L_j\}^k$ can be empty.

Definition 12.2. We shall say that the *length* $l(\gamma)$ of the curve $\gamma(s)$, having the above properties, is the following number (or $+\infty$):

$$l(\gamma) = \sum_{k=1}^{m} \frac{1}{k} \sum_{\{L_j\}^k} l(\gamma(\overline{L}_j)).$$

Here the notation $\sum\limits_{\{L_j\}^k} l(\gamma(\overline{L}_j))$ indicates summation over all L_j that belong to the subsets $\{L_j\}^k$.

It is clear that when $\gamma(\cdot)$ is a diffeomorphism, the length from Definition 12.2 coincides with the usual length of the curve. The multiplier k^{-1} makes it possible to consider the cases when a certain part of curve "is passed two, three, and more times with increasing s". In studying chaos the necessity to exclude "the case of multiplicity" is typical.

The simple estimate

$$l(\gamma) \geq \frac{1}{m} l_0(\gamma), \quad l_0(\gamma) = \sum_{\{L_j\}} l(\gamma(\overline{L}_j)),$$

is the basis for the proof of instability criterion. Here $\ell_0(\gamma)$ is a length of the path $\gamma(s)$ [45, 144]. The value $l_0(\gamma)$ can also be regarded as the length of the curve "taking into account the multiplicity of stick together" since $l_0(\gamma)$

can be defined by (12.14) for any continuous curve $\gamma(s)$. In the case when the consideration of the value $l(\gamma)$ for noninjective $\gamma(s)$ is either difficult or impossible, we can introduce the value $l_m(\gamma) = l_0(\gamma)/m$, where m is a certain upper estimate of the maximal numbers of preimages of points of the curve γ. We shall say that $l_m(\gamma)$ is the m-estimate of a length of the curve γ.

We introduce now the following notation for a straight line segment in $\mathbb{R}^n$:

$$\Gamma(s, x_0, y_0) = s(y_0 - x_0) + x_0, \qquad s \in [0, 1].$$

Definition 12.3. The solution $x(t, x_0)$ is said to be *weakly exponentially unstable* if there exist the positive numbers C, $\rho > 1$, ε such that for any $\delta \in (0, \varepsilon)$ it can be found the point $y_0 \in S(\delta, x_0)$, satisfying the following conditions:

(1) the positive numbers $t \in \mathbb{Z}$ and $s \in [0, 1]$ exist such that

$$\big|x(t, x_0) - x(t, \Gamma(s, x_0, y_0))\big| > \varepsilon; \tag{12.15}$$

(2) if for all $t \in [0, N]$, $t \in \mathbb{Z}$, and $s \in [0, 1]$ the inequality

$$\big|x(t, x_0) - x(t, \Gamma(s, x_0, y_0))\big| \leq \varepsilon \tag{12.16}$$

is satisfied, then for these values t we have either the estimate

$$l\big(x(t, \Gamma(s, x_0, y_0))\big) \geq C\rho^t\delta, \tag{12.17}$$

or the estimate

$$l_m\big(x(t, \Gamma(s, x_0, y_0))\big) \geq C\rho^t\delta. \tag{12.18}$$

Note that for noninjective maps $f(x, t)$ Definition 12.3 is a certain weakening of usual definitions of exponential instability with the Lyapunov exponent $\ln \rho$, which is widely used for diffeomorfismes, continuous dynamical systems, and differential equations. Note also that here inequalities (12.17), (12.18) not necessarily implies property 1. Generally speaking, for nonsmooth noninjective maps the case is possible when $\delta \ll \varepsilon$ and for any $t > 0$ the curves $x(t, \Gamma(s, x_0, y_0))$ remain in the ε-neighborhood of the solution $x(t, x_0)$ but at the same time in this neighborhood the exponential stretching along the directions, depending nonsmoothly on t, occurs.

12.3 Frequency Instability Criterion

Consider system (12.13) with

$$f(x,t) = Ax + b\varphi(c^*x), \tag{12.19}$$

where A is a constant $n \times n$ matrix, b and c are constant n-vectors, $\varphi(\sigma)$ is a continuous piecewise-differentiable function, having the derivative discontinuities of the first kind in the points $\sigma = \sigma_j$ $(j = 1, \ldots, m)$. Here m is either a certain number or ∞. We assume that all of these points are isolated and $\sigma_{j+1} > \sigma_j$. In other words, we can find $\tau > 0$ such that

$$|\sigma_j - \sigma_i| \geq \tau, \quad \forall\, i \neq j. \tag{12.20}$$

Suppose, the inequalities $\det(A + \varphi'(\sigma)bc^*) \neq 0$, $\forall\, \sigma \neq \sigma_j$,

$$(\varphi'(\sigma) - \mu_2)(\mu_1 - \varphi'(\sigma)) \leq 0, \quad \forall \sigma \neq \sigma_j, \tag{12.21}$$

are valid. Here $\mu_1 < 0$, $\mu_2 > 0$ are certain numbers.

Introduce the transfer function of system (12.13), (12.19), namely

$$W(p) = c^*(A - pI)^{-1}b, \qquad p \in \mathbb{C}^1.$$

Theorem 12.1 [69]. *Let the pair (A, b) be controllable and for the number $\lambda > 2$ the following conditions are valid:*

(1) the matrix $\lambda^{-1}(A + \mu_1 bc^*)$ has the one eigenvalue, situated outside circle of unit radius, and $n - 1$ eigenvalues inside this circle;
(2) for all values $p \in \mathbb{C}^1$ such that $|p| = 1$, the inequality

$$\mathrm{Re}\left[(1 + \mu_1 \overline{W(\lambda p)})(1 + \mu_2 W(\lambda p))\right] < 0 \tag{12.22}$$

is satisfied.

Then any solution of system (12.13), (12.19) is weakly exponentially unstable.

Proof of Theorem 12.1. By the Kalman—Szego lemma from condition 2) of theorem it follows that there exists the symmetric real matrix H such that the inequality holds

$$\frac{1}{\lambda^2}[Az + b\xi]^* H [Az + b\xi] - z^* H z - (\xi - \mu_2 c^* z)(\mu_1 c^* z - \xi) < 0,$$
$$\forall\, z \in \mathbb{R}^n, \quad \forall\, \xi \in \mathbb{R}^1, \quad z \neq 0. \tag{12.23}$$

Indeed, we have $P = \lambda^{-1}A$, $q = \lambda^{-1}b$, $x = z$, and

$$F(z,\xi) = \frac{1}{2}[(\mu_2 c^* z - \xi)^*(\mu_1 c^* z - \xi) \\ + (\mu_2 c^* z - \xi)(\mu_1 c^* z - \xi)^*] + \varepsilon|z|^2, \quad z \in \mathbb{C}^n, \quad \xi \in \mathbb{C}^1,$$

where ε is sufficiently small. It is obvious that

$$F((pI - P)^{-1}q\xi, \xi) = \mathrm{Re}\left[(1 + \mu_1 \overline{W(\lambda p)})(1+ \right. \\ \left. + \mu_2 W(\lambda p))\right] |\xi|^2 + \varepsilon|(\lambda pI - A)^{-1}b\xi|^2.$$

Condition 2 of Theorem 12.1 implies that for sufficiently small $\varepsilon > 0$ condition (12.11) of Lemma 12.3 is satisfied. Therefore there exists the real matrix $H = H^*$ such that (12.23) is satisfied.

Letting $\xi = \mu_1 c^* z$ in (12.23), we obtain

$$\left(\frac{1}{\lambda}(A + \mu_1 bc^*)z\right)^* H \left(\frac{1}{\lambda}(A + \mu_1 bc^*)z\right) - z^* H z < 0, \\ \forall\, z \in \mathbb{R}^n, \quad z \neq 0.$$

Then by condition 1 of the theorem and Lemma 12.4 we conclude that the matrix H has the one negative and $n-1$ positive eigenvalues.

Without loss of generality, it can be assumed that the vector z and the matrix H take the form

$$H = \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}, \qquad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

where I is the unit $(n-1)\times(n-1)$ matrix, $z_1 \in \mathbb{R}^1$, $z_2 \in \mathbb{R}^{n-1}$. In this case the quadratic form $V(z) = z^* H z$ is as follows $V(z) = z^* H z = |z_2|^2 - z_1^2$.

Take ε satisfying the inequality

$$\varepsilon < \tau/(2|c|). \tag{12.24}$$

The nondegeneracy of the matrix $A + \varphi'(\sigma)bc^*$ in the bands $\{\sigma_j \le \sigma < \sigma_{j+1}\}$ results in that the set, consisting of either one or two points, can be the preimage of any point of the map $Ax + b\varphi(c^*x)$, situated in any ε-neighborhood of any point $\mathbb{R}^n$. This implies at once the estimate of the numbers m in the definition of the length of the images of segment, situated in such ε-neighborhoods, after t iterations: $m \le 2^t$.

Hold the vector x_0 fixed, and choose y_0 for which $z_1(0) = \delta$, $z_2(0) = 0$, where $z(0) = x_0 - y_0$, $z_1(0)$ is the first component of the vector $z(0)$, $z_2(0)$

is an $(n-1)$-vector of the rest of the vectors $z(0)$. Here δ is a certain fixed number such that $\delta < \varepsilon$. Thus, we have

$$\Gamma(s, x_0, y_0) = \begin{pmatrix} s\, z_1(0) \\ 0 \end{pmatrix} + x_0, \qquad s \in [0,1].$$

Now we prove (12.18), assuming that for $t \in [0, N]$, $t \in \mathbb{Z}$ inequality (12.16) is satisfied. Consider the function $\psi_t(s) = c^* x(t, \Gamma(s, x_0, y_0)) - \sigma_{i(t)}$. Here $\sigma_{i(t)}$ is a point of derivative discontinuity of the functions $\varphi(\sigma)$ such that for some $s \in [0,1]$ the inequality $\psi_t(s) \geq 0$ is satisfied and for some other values s the inequality of opposite sign is satisfied. By (12.20), (12.24) the number $\sigma_{i(t)}$ such that $\psi_t(s)$ changes the sign is either unique or does not exist. We shall assume further that such $\sigma_{i(t)}$ exists. In the case that $\sigma_{i(t)}$ does not exist the consideration is essentially simplified: in this case it is not required to partition the segment $[0,1]$.

Consider the following partition of the segment $Q = [0,1]$ into nonintersecting segments of the type $(\alpha_{0j}, \beta_{0j})$. Here α_{0j} is a point such that $\psi_0(\alpha_{0j}) = 0$ and $\psi_0(s) > 0$ (or $\psi_0(s) < 0$) in a certain sufficiently small right neighborhood α_{0j} (i.e., $s > \alpha_{0j}$). For the point β_{0j} the relations $\psi_0(\beta_{0j}) = 0$ and $\psi_0(s) < 0$ (or $\psi_0(s) > 0$) are valid in the certain sufficiently small right neighborhood β_{0j} (i.e., $s > \beta_{0j}$). In other words, α_{0j} and β_{0j} are the exit points of the graph $\psi_0(s)$ in the upper and lower half-planes when increased s. Besides, if necessary, the intervals $(0, \beta_{0j})$ and $(\alpha_{0j}, 1)$ can be added.

It follows easily that the constructed system of the intervals $M_{0j} = (\alpha_{0j}, \beta_{0j})$ has the properties

$$M_{0i} \cap M_{0j} = \emptyset, \quad \bigcup_j \overline{M}_{0j} = Q.$$

For $\psi_1(s)$ each set M_{0j} can, in turn, be partitioned similarly into the intervals $M_{0j,1j} = (\alpha_{0j,1j}, \beta_{0j,1j})$. Proceeding this process for $k = 2, \ldots, N$ and renumbering $M_{0j,1j,\ldots}$, we obtain, as a result, the partition of the segment $Q = [0,1]$ into the intervals $M_j = (\alpha_j, \beta_j)$:

$$M_j \cap M_i = \emptyset, \quad \bigcup_j \overline{M}_j = Q.$$

It is important for us that on any $\overline{M}_j$ there exists the derivative $\varphi'(\sigma)$, where $\sigma = c^* x(t, \Gamma(s, x_0, y_0))$, $s \in \overline{M}_j$, $t = 0, \ldots, N$.

It is clear that under these conditions inequality (12.21) is also satisfied. Therefore the above relations and inequalities (12.23) imply that for the difference $x(t,\Gamma(\beta_j,x_0,y_0)) - x(t,\Gamma(\alpha_j,x_0,y_0))$, $t = 0,\dots,N-1$ the estimate

$$\begin{aligned} &V\big(x(t+1,\Gamma(\beta_j,x_0,y_0)) - x(t+1,\Gamma(\alpha_j,x_0,y_0))\big) \\ &\qquad\qquad \le \lambda^2 V\big(x(t,\Gamma(\beta_j,x_0,y_0)) - x(t,\Gamma(\alpha_j,x_0,y_0))\big) \end{aligned} \tag{12.25}$$

is valid. Indeed, for two solutions $x_1(t)$ and $x_2(t)$ of system (12.13), (12.19) by (12.23) we have

$$\begin{aligned} &\frac{1}{\lambda^2}[A(x_1(t)-x_2(t)) + b(\varphi(c^*x_1(t)) - \varphi(c^*x_2(t))]^* H \\ &\quad \times [A(x_1(t)-x_2(t)) + b(\varphi(c^*x_1(t)) - \varphi(c^*x_2(t))] \\ &\quad - (x_1(t)-x_2(t))H(x_1(t)-x_2(t)) \\ &\quad - [(\varphi(c^*x_1(t)) - \varphi(c^*x_2(t)) - \mu_2(c^*x_1(t)) - c^*x_2(t))] \\ &\quad \times [\mu_1(c^*x_1(t)) - c^*x_2(t)) - (\varphi(c^*x_1(t)) - \varphi(c^*x_2(t))] \le 0. \end{aligned} \tag{12.26}$$

Inequality (12.21) yields the relation

$$(\varphi(\widetilde{\sigma}) - \varphi(\widetilde{\widetilde{\sigma}}) - \mu_2(\widetilde{\sigma} - \widetilde{\widetilde{\sigma}}))(\mu_1(\widetilde{\sigma} - \widetilde{\widetilde{\sigma}}) - (\varphi(\widetilde{\sigma}) - \varphi(\widetilde{\widetilde{\sigma}})) \le 0$$

for any pair of points $\widetilde{\sigma}$ and $\widetilde{\widetilde{\sigma}}$ from the interval $[\sigma_j,\sigma_{j+1}]$ (either $(-\infty,\sigma_1]$ or $[\sigma_m,+\infty)$). Recall that σ_j are the points of derivative discontinuity of the function $\varphi(\sigma)$. If $c^*x_1(t)$ and $c^*x_2(t)$ are from the intervals considered, then

$$\begin{aligned} &[(\varphi(c^*x_1(t)) - \varphi(c^*x_2(t)) - \mu_2(c^*x_1(t)) - c^*x_2(t))] \\ &\qquad \times [\mu_1(c^*x_1(t)) - c^*x_2(t)) - (\varphi(c^*x_1(t)) - \varphi(c^*x_2(t))] \le 0. \end{aligned}$$

In this case (12.26) yields (12.25).

Relation (12.25) and the matrix H of special type imply at once that

$$\begin{gathered} |x(t,\Gamma(\beta_j,x_0,y_0)) - x(t,\Gamma(\alpha_j,x_0,y_0))| \ge \lambda^t(\beta_j-\alpha_j)\delta, \\ \forall\ j, \qquad \forall t = 0,\dots,N. \end{gathered} \tag{12.27}$$

Indeed, by (12.25) we have

$$\begin{aligned} &(x(t,\Gamma(\beta_j,x_0,y_0)) - x(t,\Gamma(\alpha_j,x_0,y_0)))^* H(x(t,\Gamma(\beta_j,x_0,y_0)) \\ &\qquad - x(t,\Gamma(\alpha_j,x_0,y_0))) \le \lambda^{2t}(\Gamma(\beta_j,x_0,y_0) \\ &\qquad\qquad - \Gamma(\alpha_j,x_0,y_0))^* H(\Gamma(\beta_j,x_0,y_0) - \Gamma(\alpha_j,x_0,y_0)). \end{aligned}$$

Since $V(z) = |z_2|^2 - z_1^2$ and $z_2(0) = 0$, we obtain (12.27).

Summing over j and noting that $m \leq 2^t$, we obtain

$$l_m\big(x(t,\Gamma(s,x_0,y_0))\big) \geq \frac{1}{2^t}\,\lambda^t\delta, \quad \forall t = 0,\ldots,N. \tag{12.28}$$

Since $\lambda > 2$, estimate (12.18) is proved.

Inequality (12.15) is proved similarly by taking into account a special form of the matrix H. All of the curves $\gamma_t(s) = x(t,\Gamma(s,x_0,y_0))$ can be projected on the straight line $\{z_2 = 0\}$. Since for the length of projection an estimate similar to (12.28) is satisfied, we conclude that for the certain t and s inequality (12.15) is valid. The theorem is proved. □

Let system (12.13) have the form

$$z(t+1) = Pz(t) + q\varphi(\sigma(t)), \quad \sigma(t+1) - \sigma(t) = r^*z(t) + \rho\varphi(\sigma(t)), \tag{12.29}$$

where P is a constant $(n-1)\times(n-1)$-matrix, r and q are constant n-vectors, and $\varphi(\sigma)$ is a continuous function. Then the transfer function $W(p)$ for system (12.9) can be represented as

$$W(p) = \frac{1}{p-1}\,\big[r^*(P - pI)^{-1}q - \rho\big].$$

It is well known that if the rational function $W(p)$ is nondegenerate (i.e., its denominator is a polynomial of degree n and has no common zeros with the numerator) [84, 85, 88, 138], then both systems considered above are equivalent for $\det A = 0$.

The following is well known and simply proved.

Lemma 12.5. *If all eigenvalues of the matrix P are situated inside the unit circle and $\varphi(\sigma)$ is a 2π-periodic function, then system (12.29) is dissipative by Levinson in the cylindrical phase space $\{z \in \mathbb{R}^n,\ \sigma \mod 2\pi\}$.*

Recall that *dissipativity by Levinson* means that there exists R such that for any solution $z(t,z_0,\sigma_0)$, $\sigma(t,z_0,\sigma_0)$ of system (12.29) the inequality

$$\limsup_{t\to+\infty} \big|z(t,z_0,\sigma_0)\big| \leq R \tag{12.30}$$

holds. It is clear that in the domain of dissipativity (12.30) there exists a certain global attractor (i.e., a compact invariant globally attractive set). This is, for example, the family of all ω-limit points of system (12.29).

Definition 12.4. The global attractor K is said to be *strange* if for any $x_0 \in K$ the solution $x(t,x_0)$ is weakly exponentially unstable.

The theorem and the lemma imply the following

Corollary. *Let all the hypothesis of the theorem and lemma be satisfied. Then there exists a strange attractor of system (12.13).*

The other conditions of dissipativity by Levinson or the boundedness of the solutions of discrete systems can be found in [82]. Note also that the different generalizations and the weakening of the conditions of theorem can be obtained in the case of more precise estimates of the sets L_j.

Example 12.1. Consider system (12.13), (12.19) with $n = 1$, $A = 1$, $b = c = 1$, and $\varphi(\sigma)$, which is a continuous piecewise-differentiable 2π-periodic function satisfying (12.21). Here $W(p) = (1 - p)^{-1}$. Condition 1 of the theorem is

$$|1 + \mu_1| > \lambda > 2 \tag{12.31}$$

and condition 2 has the form

$$(\lambda \cos \omega)^2 + (1 - \lambda \sin \omega)^2 + \mu_1\mu_2 + +(\mu_1 + \mu_2)(1 - \lambda \sin \omega) < 0, \ \forall \omega \in \mathbb{R}^1. \tag{12.32}$$

This inequality is equivalent to the condition

$$\lambda^2 + 1 + \mu_1\mu_2 + \mu_1 + \mu_2 + \lambda|2 + \mu_1 + \mu_2| < 0. \tag{12.33}$$

If we assume that $\mu_2 = -\mu_1 = \mu$, then (12.31) and (12.33) take the simple form $\mu > \lambda + 1 > 3$.

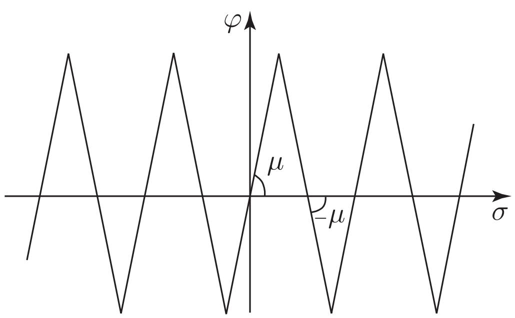

Fig. 12.1 Graph of $\varphi(\sigma)$.

Thus, if $\mu > 3$, then there exists $\lambda > 2$ such that all conditions of theorem are valid. So, if $\mu > 3$, then in the phase space, which is a circle, a strange attractor occurs. A function $\varphi(\sigma)$ for which all conditions of this example are valid is shown in Fig. 12.1. □

Example 12.2. Consider system (12.13), (12.19) with $n = 1$, $A = 0$, $b = c = 1$, where $\varphi(\sigma)$ is a continuous piecewise-differentiable function, satisfying conditions (12.21) with $\mu_2 = -\mu_1 = \mu$ such that $\varphi(\sigma) \in \in [\alpha, \beta]$,

$\forall\sigma \in [\alpha, \beta]$. Here we can consider the compact phase space $[\alpha, \beta]$ and apply the theorem. It can easily be checked that condition 1 of the theorem is satisfied if $\mu > \lambda > 2$. For the validity of condition 2 it is also sufficient to satisfy this inequality. Thus, for any functions under condition (12.21) with $\mu_2 = -\mu_1 = \mu > 2$, which map the segment $[\alpha, \beta]$ into itself, all conditions of theorem are satisfied and, therefore, the weak exponential instability occurs.

We remark that for the system considered, the global stability of the fixed point occurs for $\mu < 1$.

We emphasize that the conditions of the theorem can be weakened in the presence of more detailed information regarding the set L_j (see Definition 12.2). □

Example 12.3. Consider system (12.13), (12.19) with $n = 2$ and the transfer function

$$W(p) = \frac{\beta p}{(p-1)(p-\alpha)}, \qquad p \in \mathbb{C}^1,$$

where α and β are numbers with $\alpha \neq 0$, $|\alpha| < 1$, $\beta > 0$, and $\varphi(\sigma)$ is a 2π-periodic, continuous, piecewise-differentiable function satisfying condition (12.21) with $-\mu_1 = \mu_2 = \mu$. Systems of this type describe certain classes of rotators, periodic with impulses [123], and phase-locked loops [82, 83, 87, 89]. We see that all conditions of Lemma 12.5 are satisfied. Consider now the conditions under which the hypotheses of the theorem are satisfied. Note that the eigenvalues of the matrix $\lambda^{-1}(A + \mu_1 bc^*)$ are zeros of the

$$(\lambda p - 1)(\lambda p - \alpha) - \mu\beta\lambda p.$$

It is easily shown that condition 1 of theorem takes the form

$$\mu\beta + \alpha + 1 + \sqrt{(\mu\beta + \alpha + 1)^2 - 4\alpha} > 2\lambda,$$

and condition 2 is satisfied if

$$\mu\beta > \frac{(1+\lambda)(\alpha+\lambda)}{\lambda}.$$

Hence the system has a strange attractor in the cylindrical phase space if

$$(\mu\beta + \alpha + 1) + \sqrt{(\mu\beta + \alpha + 1)^2 - 4\alpha} > 4, \quad \mu\beta > \frac{3}{2}\,(\alpha + 2).$$

We remark that the stated frequency criterion can be extended to a discrete system with many nonlinearities. □

Chapter 13

Estimates of Oscillation Period in Nonlinear Discrete Systems

In the last 50 years one more interesting direction of stability theory, the method of a priori integral estimates [64, 83–88, 113], was developed. The basis of this method is the application of the Fourier transform as a unitary operator in certain functional spaces. We shall show that by this method effective estimates for the oscillation periods of discrete dynamical systems can be found. The application of this criterion to various one-dimensional mappings yields estimates for a lack of oscillation with period 3. It is well known [91, 124, 125] that, for one-dimensional discrete dynamical systems, "period three implies chaos". So nontrivial estimates of the period of oscillatory motions permit us to put forward quantitative propositions concerning the scenario of the passage to chaotic dynamics.

We formulate here the criterion for the lack of a given oscillation period for the discrete dynamical systems of the following type:

$$x(t+1) = Ax(t) + b\varphi(t, c^*x(t)), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{Z}. \tag{13.1}$$

Here A is a constant $n \times n$-matrix, b and c are constant n-dimensional vectors, and $\varphi(t,\sigma)$ is a scalar function satisfying on some set $\Omega \subset \mathbb{R}^1$ the condition

$$0 < \varphi(t,\sigma)\sigma < k\sigma^2, \quad \forall t \in \mathbb{Z}, \quad \sigma \in \Omega, \tag{13.2}$$

where k is some positive number. We introduce the following transfer function of system (13.1):

$$W(p) = c^*(A - pI)^{-1}b,$$

where p is a complex variable.

Theorem 13.1. *Suppose that for the natural numbers $\mathcal{N}$ the inequality*

$$\frac{1}{k} + \operatorname{Re} W\left(\exp\left(\frac{2\pi}{\mathcal{N}} ij\right)\right) \geq 0 \qquad (13.3)$$

holds for all $j = 0, \ldots, \mathcal{N} - 1$. Then there does not exist an $\mathcal{N}$-periodical sequence $x(t)$, satisfying equation (13.1), and the inclusion

$$c^* x(t) \in \Omega, \quad \forall t \in \mathbb{Z}. \qquad (13.4)$$

Proof. Consider the sequence $u(t)$, $t = 0, \ldots, \mathcal{N} - 1$ and the discrete Fourier transformation

$$v(j) = \sum_{t=0}^{\mathcal{N}-1} u(t) \exp\left(\frac{2\pi}{\mathcal{N}} ijt\right). \qquad (13.5)$$

Recall that

$$\sum_{t=0}^{\mathcal{N}-1} \exp\left(\frac{2\pi}{\mathcal{N}} ikt\right) = \begin{cases} \mathcal{N}, & k = 0, \\ 0, & k = 1, \ldots, \mathcal{N} - 1. \end{cases}$$

Therefore for the Fourier transforms of the sequences $u_1(t)$ and $u_2(t)$ the relation

$$\sum_{j=0}^{\mathcal{N}-1} v_1(j)^* v_2(j) = \mathcal{N} \sum_{t=0}^{\mathcal{N}-1} u_1(t)^* u_2(t) \qquad (13.6)$$

is satisfied. In the case $u(t + \mathcal{N}) = u(t)$, $\forall t \in \mathcal{Z}$ we have

$$\sum_{t=0}^{\mathcal{N}-1} u(t+1) \exp\left(\frac{2\pi}{\mathcal{N}} ijt\right) = \exp\left(-\frac{2\pi}{\mathcal{N}} ij\right) v(j). \qquad (13.7)$$

Denote by $y(j)$ and $w(j)$ the Fourier transforms of the sequences $x(t)$ and $\varphi(t, c^* x(t))$, respectively, where $t = 0, 1, \ldots, \mathcal{N} - 1$. Assuming that $x(t + \mathcal{N}) = x(t)$, by (13.7) we obtain

$$c^* y(j) = -W\left(\exp\left(\frac{-2\pi}{\mathcal{N}} ij\right)\right) w(j). \qquad (13.8)$$

From (13.6) and (13.8) we have the following chain of relations:

$$\sum_{t=0}^{\mathcal{N}-1} \varphi(t, c^*x(t)) \left(c^*x(t) - \frac{\varphi(t, c^*x(t))}{k} \right)$$
$$= \frac{1}{\mathcal{N}} \sum_{j=0}^{\mathcal{N}-1} \overline{w(j)} \left(-W \left(\exp \left(-\frac{2\pi}{\mathcal{N}} ij \right) \right) - \frac{1}{k} \right) w(j)$$
$$= -\frac{1}{\mathcal{N}} \sum_{j=0}^{\mathcal{N}-1} \left(\operatorname{Re} W \left(\exp \left(\frac{2\pi}{\mathcal{N}} ij \right) \right) + \frac{1}{k} \right) |w(j)|^2.$$

Condition (13.2) implies that the first sum is positive and (13.3) that the last one is not positive. This contradiction concludes the proof. □

Example 13.1. Consider the equation

$$\theta(t+1) = \theta(t) - \alpha \sin \theta(t), \tag{13.9}$$

describing the dynamics of a discrete phase-locked loop [82, 83, 87, 88, 89]. Here α is a positive parameter. Represent (13.9) in the form of (13.1) with $c = 1$, $b = -1$, $A = 1 - \varepsilon$,

$$\varphi(t, \sigma) = \alpha \sin \sigma - \varepsilon \sigma.$$

In this case we have

$$W(p) = \frac{1}{p - 1 + \varepsilon}.$$

It is well known [89] that for $\alpha < \alpha_1$, where α_1 is a root of the equation

$$\sqrt{\alpha^2 - 1} = \pi + \arccos \frac{1}{\alpha},$$

the interval $[-\pi, \pi]$ is mapped into itself: if $\theta(t) \in [-\pi, \pi]$, then $\theta(t+1) \in [-\pi, \pi]$. Besides, in this case for $\theta(0) \in (-\pi, \pi)$ we have

$$\limsup_{t \to +\infty} |\theta(t)| \leq \theta_0 = \left| \sqrt{\alpha^2 - 1} - \left(\arccos \frac{1}{\alpha} \right) \right| \tag{13.10}$$

for $\alpha \geq 2$, and

$$\lim_{t \to +\infty} |\theta(t)| = 0$$

for $\alpha < 2$.

Further we shall consider the case $\alpha > 2$. Define

$$\Omega = [-\theta_0, 0) \cup (0, \theta_0].$$

From estimate (13.10) we obtain that for the nonzero $\mathcal{N}$, which is a periodic solution of equation (13.9) with the initial data $\theta_0 \in (-\pi, \pi)$, the inclusion $\theta(t) \in \Omega$, $\forall t \in \mathbb{Z}$, $t \geq 0$ is valid. For $k = \alpha - \varepsilon$ the inequalities (13.2) are satisfied if

$$\varepsilon < \frac{\alpha \sin \theta_0}{\theta_0}. \tag{13.11}$$

Consider the case $\mathcal{N} = 3$. Inequality (13.3) takes the form

$$\frac{1}{\alpha - \varepsilon} + \frac{\cos \omega - 1 + \varepsilon}{(\cos \omega - 1 + \varepsilon)^2 + (\sin \omega)^2} \geq 0 \tag{13.12}$$

for $\omega = 0$, $\omega = 2\pi/3$, $\omega = 4\pi/3$. It is easy to see that this holds if

$$\frac{1}{\alpha - \varepsilon} + \frac{\varepsilon - \frac{3}{2}}{(\varepsilon - \frac{3}{2})^2 + \frac{3}{4}} \geq 0.$$

The last inequality can be rewritten as

$$\varepsilon > \frac{\frac{3}{2}\alpha - 3}{\alpha - \frac{3}{2}}. \tag{13.13}$$

For certain ε, conditions (13.11) and (13.13) are satisfied if

$$\frac{\alpha \sin \theta_0}{\theta_0} > \frac{\frac{3}{2}\alpha - 3}{\alpha - \frac{3}{2}}. \tag{13.14}$$

Inequality (13.14) is satisfied for $\alpha \in (2, 3.7)$.

Thus, for $\alpha \in (2, 3.7)$ equation (13.9) cannot have a cycle of period 3. Note that for equation (13.9) on this interval we observe the passage to chaos via period doubling bifurcations [89]. □

Example 13.2. Consider a logistic equation [123]

$$\theta(t+1) = \mu\theta(t)(1 - \theta(t)), \quad \theta(0) \in [0, 1], \tag{13.15}$$

where μ a positive parameter. Let $\mu \in (1, 4)$. (For $\mu < 1$ all trajectories $\theta(t)$ of (13.15) tend to the stable equilibrium $\theta(t) \equiv 0$ as $t \to +\infty$.) In this case (13.15) has two equilibria $\theta(t) \equiv 0$, $\theta(t) \equiv \theta_0 = 1 - \mu^{-1}$. Obviously, for $\theta(0) \in (0, 1)$ the inclusion

$$\theta(1) \in (0, \mu/4]$$

is valid. From this and the inequality

$$\mu\theta(1-\theta) > \theta, \quad \forall\, \theta \in (0, \theta_0),$$

we have

$$\liminf_{t\to+\infty} \theta(t) > \frac{\mu^2}{16}(4-\mu). \tag{13.16}$$

Suppose that

$$\frac{\mu^2}{16}(4-\mu) > \frac{1}{\mu}. \tag{13.17}$$

In this case (13.16) and (13.17) yield the estimate

$$\frac{\mu\theta(t)(1-\theta(t)) - \theta_0}{\theta(t) - \theta_0} < -a_1 \tag{13.18}$$

for any nontrivial $\mathcal{N}$-periodic sequence $\theta(t)$. Here

$$a_1 = \frac{\frac{\mu^3}{16}(4-\mu)(1-\frac{\mu^2}{16}(4-\mu)) - (1-\frac{1}{\mu})}{1-\frac{1}{\mu}-\frac{\mu^2}{16}(4-\mu)}.$$

Since for this sequence the inequality $\theta(t) \leq \mu/4$ is valid, we obtain the estimate

$$\frac{\mu\theta(t)(1-\theta(t)) - \theta_0}{\theta(t) - \theta_0} > -a_2, \tag{13.19}$$

where

$$a_2 = \frac{1-\frac{1}{\mu}-\frac{\mu^2}{16}(4-\mu)}{\frac{1}{\mu}+\frac{\mu}{4}-1}.$$

By the theorem with $A = -a_1$, $b = -1$, $c = 1$, $\sigma = \theta - \theta_0$, $\varphi(t, \sigma) = -(\mu\theta(1-\theta) - \theta_0) + a_1(\theta - \theta_0)$ and

$$W(p) = \frac{1}{p + a_1},$$

we find that for any nontrivial $\mathcal{N}$-periodic solution, condition (13.2) with $k = a_2 - a_1$ is satisfied. Condition (13.3) takes the form

$$\frac{1}{k} + \frac{a_1 + \cos\frac{2\pi}{\mathcal{N}}j}{(a_1+\cos\frac{2\pi}{\mathcal{N}}j)^2 + (\sin\frac{2\pi}{\mathcal{N}}j)^2} \geq 0, \quad \forall j = 0, \ldots, \mathcal{N}-1.$$

Thus, for $\mathcal{N} = 3$ we obtain the following condition for a lack of 3-period cycles:

$$\frac{1}{a_2 - a_1} + \frac{a_1 - \frac{1}{2}}{(a_1 - \frac{1}{2})^2 + \frac{3}{4}} \geq 0.$$

The last inequality can be written as

$$2(1 + a_1 a_2) \geq a_1 + a_2.$$

This inequality holds for

$$\mu \leq 3.635. \tag{13.20}$$

Thus, if (13.20) holds, then (13.15) has no cycles of period 3.

It is well known [123] that for the bifurcation parameter $\mu = 3$ the equilibrium $\theta(t) \equiv \theta_0$ loses stability and the 2-period cycle occurs. For $\mu = 3.44948\ldots$ the 2-period cycle loses stability and the 4-period cycle occurs. Then on the interval $(3.52, 3.57)$ similar period doubling bifurcations occur. As a result of the numerical analysis of equation (13.15) for $\mu = 1 + \sqrt{8} = 3.8284\ldots$ the 3-period cycle is found [123]. $\square$

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Published by

St. Petersburg University Press
11/21 6th Line, St. Petersburg, 199004, Russia

Leonov G. A.
Strange Attractors and Classical Stability Theory

ISBN 978-5-288-04500-4

Printed in Russia by St. Petersburg University Press