Local Voting Protocol in Decentralized Load Balancing Problem with Switched Topology, Noise, and Delays

Natalia Amelina, Oleg Granichin, and Aleksandra Kornivetc

Saint Petersburg State University

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MAIN CONCEPTS OF THE GRAPH THEORY

- Associate with each arc $(j,i) \in E$ the weight $a^{i,j} > 0$ and denote **the adjacency matrix** $A = [a^{i,j}]$ of the graph $\mathscr{G}_A = (N, E)$.
- **The in-degree** of i is the sum of ith row of matrix A: $d^i = \sum_{j=1}^{n} a^{i,j}$;
- $d_{\max}(A)$ is the maximum in-degree of graph $\mathscr{G}_A$;
- $D(A) = \mathrm{diag}\{d^i(A)\}$;
- $\mathscr{L}(A) = D(A) - A$ is the graph **Laplacian matrix**.
- **The directed path** from i_1 to i_s is a sequence of nodes $i_1, \ldots, i_s$, $s \geq 2$ such that $(i_k, i_{k+1}) \in E, k \in \{1, 2, \ldots, s-1\}$.
- The digraph is said to be **strongly connected** if from any node to any other node, there exists a directed path.
- **A directed tree** is a digraph where each node i, except the root, has exactly one parent node j so that $(j,i) \in E$. The digraph G is said to contain **a spanning tree** if there exists a directed tree as a subgraph of G.

PROBLEM FORMULATION - I

- Let the network system be composed by n agents (processors, machines, *etc.*), and a set of the same type of tasks, that have to be executed in the system.
- Agents perform incoming tasks in parallel. Tasks can be redistributed among agents based on a feedback. Tasks came to the system in different discrete time instants $t = 0, 1, \dots, T$.
- Agents communicate to each other according to the digraph (N, E) where E is a set of arcs.
- The dynamic network topology is modeled by a sequence of digraphs $\{(N, E_t)\}_{t \geq 0}$ where $E_t \subseteq E$ changes with time.

PROBLEM FORMULATION - II

At each time instant t the state of each agent $i \in N$ could be described by two characteristics:

- q_t^i is the queue length of atomic elementary tasks of agent i at time instant t;
- p_t^i is the productivity of agent i at time instant t;

Assume, that the dynamic model of the system is described by following equations:

$$q_{t+1}^i = q_t^i - p_t^i + z_t^i + u_t^i; \;\; i \in N, \; t = 0, 1, \ldots, T, \tag{1}$$

where $u_t^i \in \mathbb{R}$ are control actions (redistributed tasks to agent i at time instant t), which could (and should) be chosen, and z_t^i is an amount of new tasks received by agent i at time instant t.

Assume, that the following condition is satisfied

$$\mathbf{A1}: \quad p_t^i \geq p_{\min} > 0,\ \forall i \in N, t = 0, 1, \ldots.$$

If we take $x_t^i = q_t^i / p_t^i$ as a state of agent i of a dynamic network at time $t = 0, 1 \ldots, T$, then the control goal of achieving consensus in network will correspond to the optimal redistribution of tasks among agents.
Under this notation, the dynamics of each agent can be rewritten as

$$x_{t+1}^i = x_t^i + \bar{u}_t^i + f_t^i, \tag{2}$$

where $\bar{u}_t^i = u_t^i / p_t^i$, $i \in N$ are "normalized" control actions, $f_t^i = -1 + z_t^i / p_t^i$ are "normalized" perturbations.

OBSERVATIONS

We assume, that to form the control strategy u_t^i each agent $i \in N$ has noisy data about its own state:

$$y_t^{i,i} = x_t^i + w_t^{i,i}, \tag{3}$$

and, if the set $N_t^i \neq \emptyset$, noisy and possibly delayed observations about its neighbors' states:

$$y_t^{i,j} = x^j_{t-d_t^{i,j}} + w_t^{i,j},\ j \in N_t^i, \tag{4}$$

where $w_t^{i,i}, w_t^{i,j}$ are noises, $0 \leq d_t^{i,j} \leq \bar{d}$ are integer-valued delays, and $\bar{d}$ is a maximum of possible delays.

TASK REDISTRIBUTION PROTOCOL - I

Consider local voting protocol:

$$\tilde{u}_t^i = \gamma \sum_{j \in \bar{N}_t^i} b_t^{i,j}(y_t^{i,j} - y_t^{i,i}), \tag{5}$$

where $\gamma > 0$ is a step-size of the control protocol, $\bar{N}_t^i \subset N_t^i$, $b_t^{i,j} > 0 \ \forall j \in \bar{N}_t^i$. We set $b_t^{i,j} = 0$ for other pairs (i,j). The matrix of the control protocol is denoted by $B_t = [b_t^{i,j}]$.

The dynamics of the closed loop system with protocol (5) is as follows:

$$x_{t+1}^i = x_t^i + \gamma \sum_{j \in \bar{N}_t^i} b_t^{i,j}(y_t^{i,j} - y_t^{i,i}) + f_t^i, \ i \in N. \tag{6}$$

TASK REDISTRIBUTION PROTOCOL - II

If $\bar{d} = 0$, then the state vector is $X = (x_t^1, x_t^2, \ldots, x_t^n)$ and dynamics equation can be written in the vector-matrix form:

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \mathscr{L}(B_t)\mathbf{x}_t + w_t + f_t \tag{7}$$

If $\bar{d} > 0$ we "artificially" add $n\bar{d}$ new agents to the current network topology and receive an extended state vector: $\bar{X} = (\bar{x}_t, \bar{x}_{t-1}, \ldots, \bar{x}_{t-\bar{d}})$
Introduce the extended $\bar{n} \times \bar{n}$ matrix $\bar{B}_t$ of the control protocol (5) which consist of zeros at all places except $|\bar{N}_t^i|$ entries $\bar{b}_t^{i,j+nd_t^{i,j}}$ in each $i \in N$ of n first lines, which are equal to $b_t^{i,j}$.

TASK REDISTRIBUTION PROTOCOL - III

Due to the view of the Laplacian matrix $\mathscr{L}(\bar{B}_t)$ we can rewrite Equation (6) in the vector-matrix form:

$$\bar{\mathbf{x}}_{t+1} = U\bar{\mathbf{x}}_t - \gamma\mathscr{L}(\bar{B}_t)\bar{\mathbf{x}}_t + \gamma\begin{pmatrix} \mathbf{w}_t \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbf{f}_t \\ 0 \end{pmatrix}, \tag{8}$$

where U is a $\bar{d}n \times \bar{d}n$ matrix:

$$U = \begin{pmatrix} I_n & 0 & \dots & 0 & 0 \\ I_n & 0 & \dots & 0 & 0 \\ 0 & I_n & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & I_n & 0 \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix, and n-vectors $\mathbf{w}_t$ consist of the corresponding elements $\sum_{j\in\bar{N}_t^1} b_t^{1,j}(w_t^{1,j} - w_t^{1,1}), \dots, \sum_{j\in\bar{N}_t^n} b_t^{n,j}(w_t^{n,j} - w_t^{n,n})$.

ASSUMPTIONS OF STOCHASTIC PROPERTIES - I

Let $(\Omega, \mathscr{F}, P)$ be the underlying probability space. The symbol E stands for the mathematical expectation, and the following assumptions are satisfied:

A2. **a)** For all $i \in N$, $j \in N_t^i \cup \{i\}$ observations noises $w_t^{i,j}$ are zero-mean, independent identically distributed (i.i.d.) random variables with bounded variances: $\mathrm{E}(w_t^{i,j})^2 \leq \sigma_w^2$.

b) For all $i \in N, j \in N_{\max}^i$ the appearance of "variables" edges (j,i) in the graph $\mathscr{G}_{A_t}$ is independent random event (i. e. matrices A_t are i.i.d. random matrices).For all $i \in N$, $j \in \bar{N}_t^i$ weights $b_t^{i,j}$ in the control protocol are independent random variables with expectations: $\mathrm{E}b_t^{i,j} = b^{i,j}$, and bounded variances: $\mathrm{E}(b_t^{i,j} - b^{i,j})^2 \leq \sigma_b^2$.

c) For all $i \in N, j \in N^i$ there exists a finite value $\bar{d} \in \mathbb{N}$: $d_t^{i,j} \leq \bar{d}$ with probability 1, and the integer-valued delay $d_t^{i,j}$ is i.i.d. random variables taking values $k = 0, \ldots, \bar{d}$ with probability $p_k^{i,j}$.

d) For all $i \in N$, $t = 0, 1, \ldots$ variables f_t^i from (2) are i.i.d. random variables with expectations: $\mathrm{E}f_t^i = \bar{f}$, and variances: $\mathrm{E}(f_t^i - \bar{f})^2 = \sigma_f^2$.

Additionally, all these random variables are mutually independent.

- Matrix $A_{\max}$ of size $\bar{n} \times \bar{n}$ is denoted by

$$a_{\max}^{i,j} = p_{j \div \bar{d}}^{i,j \bmod \bar{d}} b^{i,j \bmod \bar{d}},\ i \in N,\ j = 1, 2, \ldots, \bar{n}, \tag{9}$$

  Here, the operation $\bmod$ is a remainder of division, and $\div$ is a division without remainder.
- Note, that if $\bar{d} = 0$, then the definition of network topology (of matrix $A_{\max}$ of size $n \times n$) is as follows $a_{\max}^{i,j} = b^{i,j}$, $i, j \in N$.

Assume, that the following assumption is satisfied for the averaged matrix of the network topology:

A3: Graph $\mathscr{G}_{A_{\max}}$ is strongly connected, and for any edge $(j, i) \in E_{\max}$ there exists at least one non-zero element of the matrix $A_{\max}$ among elements $a_{\max}^{i,j}, a_{\max}^{i,j+n}, \ldots, a_{\max}^{i,j+\bar{d}n}$.

ASSUMPTION ABOUT STEP-SIZE

A4: For the step-size of the control protocol $\gamma > 0$ the following conditions are satisfied:

$$\gamma \leq \frac{1}{d_{\max}(A_{\max})}, \tag{10}$$

and

$$\lambda_{\max}(Q)\gamma \leq Re(\lambda_2(A_{\max})), \tag{11}$$

where $Re(\lambda_2(A_{\max}))$ is the real part of the second eigenvalue of matrix $A_{\max}$ ordered by absolute magnitude, and $\lambda_{\max}(Q)$ is the maximum eigenvalue of matrix

$$Q = \mathrm{E}(\mathscr{L}(A_{\max}) - \mathscr{L}(\bar{B}_t))^{\mathrm{T}}(\mathscr{L}(A_{\max}) - \mathscr{L}(\bar{B}_t)).$$

DEFINITION

Definition

n agents are said to achieve *the asymptotic mean square ε-consensus*, if $\mathrm{E}||x_0^i||^2 < \infty$, $i \in N$, and there exists a sequence $\{x_t^\star\}$ such that $\overline{\lim}_{t\to\infty} \mathrm{E}||x_t^i - x_t^\star||^2 \leq \varepsilon$ for all $i \in N$.

Here and below, $||\cdot||$ is an Euclidean norm of a vector.
Let $x_0^\star$ be the average value of the initial data

$$x_0^\star = \frac{1}{n}\sum_{i=1}^{n} x_0^i$$

and $\{x_t^\star\}$ is the trajectory of the averaged system

$$x_{t+1}^\star = x_t^\star + \bar{f}, \tag{12}$$

where $\bar{f}$ is the mean value from the assumption **A2d**.

MAIN RESULT

Theorem

If Assumptions **A1–A4** *are satisfied* **then** *for trajectories of systems (6) and (12) the following inequality holds:*

$$\mathrm{E}||\mathbf{x}_{t+1} - x^{\star}_{t+1}\mathbf{1}_n||^2 \leq \frac{\Delta}{\rho} + (1-\rho)^t \left(||\mathbf{x}_0 - x^{\star}_0\mathbf{1}_n||^2 - \frac{\Delta}{\rho} \right),$$

where $\mathbf{1}_n$ *is n-vector of ones,*

$$\rho = \gamma Re(\lambda_2(A_{\max})) - \gamma^2 \lambda_{\max}(Q),$$

$$\Delta = 2\sigma_w^2\gamma^2(n^2\sigma_b^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}(b^{i,j})^2) + n\sigma_f^2,$$

i. e. if additionally $\mathrm{E}||x_0^i||^2 < \infty$, $i \in N$, **then** *the asymptotic mean square* ε*-consensus in (6) is achieved with*

$$\varepsilon \leq \frac{\Delta}{\rho}.$$

Consider vectors $\bar{\mathbf{x}}^\star_t \in \mathbb{R}^{\bar{n}}$, $t = 0, 1, \ldots$, which consist of $x^\star_t \mathbf{1}_n, x^\star_{t-1}\mathbf{1}_n, \ldots, x^\star_{t-\bar{d}}\mathbf{1}_n$ and satisfy the equation:

$$\bar{\mathbf{x}}^\star_{t+1} = U\bar{\mathbf{x}}^\star_t + \begin{pmatrix} \bar{f}\mathbf{1}_n \\ 0 \end{pmatrix}. \tag{13}$$

Due to the definition of matrixes $\bar{\mathscr{L}}_t$ for differences of trajectories of systems (8) and (13) we have

$$D_{t+1} = \bar{\mathbf{x}}_{t+1} - \bar{\mathbf{x}}^\star_{t+1} = U\bar{\mathbf{x}}_t - \gamma\mathscr{L}(\bar{B}_t)\bar{\mathbf{x}}_t + \gamma\begin{pmatrix} \mathbf{w}_t \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbf{f}_t \\ 0 \end{pmatrix} -$$

$$-U\bar{\mathbf{x}}^\star_t - \begin{pmatrix} \bar{f}\mathbf{1}_n \\ 0 \end{pmatrix} = D_t - \bar{\mathscr{L}}_t D_t + \gamma\begin{pmatrix} \mathbf{w}_t \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbf{f}_t - \bar{f}\mathbf{1}_n \\ 0 \end{pmatrix}.$$

Further, by adding and subtracting $\tilde{\mathscr{L}} D_t$ we get

$$D_{t+1} = (I_{\bar{n}} - \tilde{\mathscr{L}})D_t + (\tilde{\mathscr{L}} - \bar{\mathscr{L}}_t)D_t + \gamma\begin{pmatrix} \mathbf{w}_t \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbf{f}_t - \bar{f}\mathbf{1}_n \\ 0 \end{pmatrix}.$$

SKETCH OF THE PROOF - II

Let $\mathscr{F}_t$ denotes σ-algebra of all probabilistic events, generated by the random elements $A_0,\dots,A_t,x_0^i,w_0^{i,j},w_1^{i,j},\dots,w_{t-1}^{i,j},f_0^i,f_1^i,\dots,f_{t-1}^i,b_0^{i,j},b_1^{i,j},\dots,\ b_t^{i,j}$, $d_0^{i,j},d_1^{i,j},\dots,d_t^{i,j}$, $i,j\in N$, Consider the conditional mathematical expectation of the squared norm D_{t+1}:

$$\mathrm{E}_{\mathscr{F}_t}||D_{t+1}||^2 = ||(I_{\bar{n}}-\tilde{\mathscr{L}})D_t||^2 + 2D_t^{\mathrm{T}}(I_{\bar{n}}-\tilde{\mathscr{L}})^{\mathrm{T}}(\tilde{\mathscr{L}}-\bar{\mathscr{L}}_t)D_t +$$
$$+2D_t^{\mathrm{T}}(I_{\bar{n}}-\bar{\mathscr{L}}_t)^{\mathrm{T}}\begin{pmatrix}\gamma\mathrm{E}_{\mathscr{F}_t}\mathbf{w}_t+\mathrm{E}_{\mathscr{F}_t}(\mathbf{f}_t-\bar{f}\mathbf{1}_n)\\0\end{pmatrix}+ \tag{14}$$
$$+2\gamma\mathrm{E}_{\mathscr{F}_t}\mathbf{w}_t^{\mathrm{T}}(\mathbf{f}_t-\bar{f}\mathbf{1}_n)+D_t^{\mathrm{T}}(\tilde{\mathscr{L}}-\bar{\mathscr{L}}_t)^{\mathrm{T}}(\tilde{\mathscr{L}}-\bar{\mathscr{L}}_t)+$$
$$+\gamma^2\mathrm{E}_{\mathscr{F}_t}||\mathbf{w}_t||^2+\mathrm{E}_{\mathscr{F}_t}||\mathbf{f}_t-\bar{f}\mathbf{1}_n||^2.$$

SKETCH OF THE PROOF - III

Taking into account features of conditional mathematical expectation and denoting by $\bar{\mathbf{b}}_t$ the vector, consisting of the components $\sum_{j \in N_t^1} (b_t^{1,j})^2, \ldots, \sum_{j \in N_t^n} (b_t^{n,j})^2$, from (14) we derive

$$\mathrm{E}_{\mathscr{F}_t} ||D_{t+1}||^2 = ||(I_{\bar{n}} - \tilde{\mathscr{L}})D_t||^2 + 2D_t^{\mathrm{T}}(I_{\bar{n}} - \tilde{\mathscr{L}})^{\mathrm{T}}(\tilde{\mathscr{L}} - \bar{\mathscr{L}}_t)D_t +$$

$$+ D_t^{\mathrm{T}}(\tilde{\mathscr{L}} - \bar{\mathscr{L}}_t)^{\mathrm{T}}(\tilde{\mathscr{L}} - \bar{\mathscr{L}}_t) + 2\sigma_w^2 \gamma^2 \bar{\mathbf{b}}_t + n\sigma_f^2. \qquad (15)$$

Let $\tilde{\mathscr{F}}_t$ denotes the σ-algebra of probabilistic events, generated by all random elements as in $\mathscr{F}_t$, except those that were implemented in/after time t. Consider conditional expectations of both sides of (15). Due to stochastic properties of the uncertainties **A2b,c** and the independence of $\bar{B}_t$ and $\bar{\mathbf{b}}_t$ from σ-algebra $\tilde{\mathscr{F}}_t$, after some transformations we obtain:

$$\mathrm{E}_{\tilde{\mathscr{F}}_t} ||D_{t+1}||^2 = (1-\rho)||D_t||^2 + \Delta.$$

SKETCH OF THE PROOF - IV

We take the unconditional expectation and get:

$$\mathrm{E}||D_{t+1}||^2 \leq (1-\rho)\mathrm{E}||D_t||^2 + \Delta,$$

By Lemma 2 [Amelina, Fradkov 2012] it follows that inequality (13), which is the first part of Theorem 1, holds.
The second conclusion about the asymptotic mean square ε-consensus follows from inequality (13) if $t \to \infty$. Since Assumption **A4** is satisfied, we obtain that $|1-\rho| < 1$, and, therefore, the second term of (13) exponentially tends to zero.

SIMULATION RESULT - I

- We consider the open queuing network of 1024 servers (agents).The number of incoming tasks is 10^6.
- Time between new tasks in the input stream and lengths of its implementation are exponentially distributed.
- Choice of an agent, which receives the next task is performed randomly by the uniform distribution of 1024 agents.

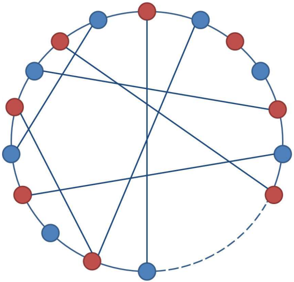

Fig. 1. An example of network topology

SIMULATION RESULT - II

Consider the case, when all tasks arrive at different time instants in the interval from 1 to 2000.

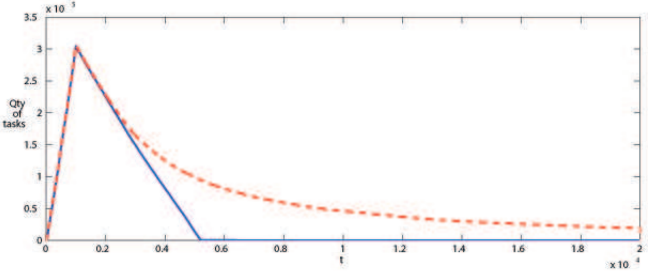

Fig. 2. The number of tasks in the queue

SIMULATION RESULT - III

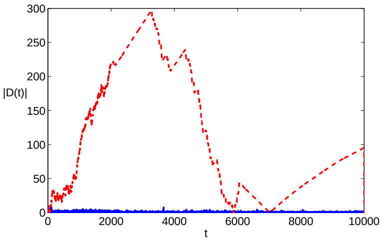

Fig. 3. The average deviation from the average load on the network

THANK YOU
FOR ATTENTION!