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TOPICAL ISSUE

Cyclic Stochastic Approximation with Disturbance on Input in the Parameter Tracking Problem Based on a Multiagent Algorithm

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Abstract—We consider the possibilities of using cyclic stochastic approximation for solving optimization problems of a nonstationary functional of the data produced by distributed observers (sensors) under constraints on the possibilities of simultaneous communication between the observers themselves. We achieve tracking the optimal value of parameters up to a certain level of quality with a multi-agent algorithm. The efficiency of the proposed approach is illustrated by an example of modeling the process of tracking the trajectories of a group of moving objects using a set of spatially distributed sensors.

Keywords: stochastic approximation, object tracking, nonstationary functional, parameter tracking, cyclic approach, multi-agent algorithm

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1. INTRODUCTION

Recently, optimization tasks in distributed systems have attracted considerable attention due to their wide spread and relevance in practical applications. Areas of their application include wireless sensor networks (mobile and/or stationary), distributed electrical networks, logistical networks, Internet of things, and much more (see, e.g., [1]). A common problem inherent in the above applications is the decentralized distribution of resources in a multi-agent system where agents (devices, robots, programs etc.) jointly solve optimization problems under a lack or incomplete knowledge of the overall structure of the task and constraints on communication channels. Under such conditions, agents can interact with each other to clarify the information necessary for globally efficient resource allocation. Decentralization is reflected in the ability of agents to "see" only a part of the entire network, with each agent exchanging messages only with its neighbors.

As a rule, solving a stochastic optimization problem under uncertainty implies finding a set of system parameters where the minimum or maximum value of a certain average risk functional is achieved, which in practical applications can often have different forms during system operation. Below we call such a functional nonstationary. In systems analysis, the points where the functional reaches its optimal value are often associated with certain reference values of system parameters. With a nonstationary functional of medium risk, the task is to monitor changes in system parameters. To optimize functionals of average risk type, the maximum likelihood estimator and stochastic approximation algorithms are actively used with step size decreasing to zero [2–4]. Derevitsky and Fradkov in [5, 6], in their analysis of the dynamics of adaptation algorithms based on constructing approximate averaged models, justified the use of stochastic approximation algorithms with nondecreasing step size. Later, randomized stochastic approximation algorithms with trial simultaneous input perturbation and step size that does not decrease to zero were used to solve optimization problems for nonstationary functionals in [7–10]. The works [11–13] studied, using the averaged model method, algorithms for consensus multi-agent decentralized network load control based on the stochastic approximation method with nondecreasing step size for agents in a network with nonlinear state dynamics with a randomly changing structure of constraints and observations with random delays and interference.

The goal of distributed optimization is usually to find the minimum or maximum of some objective function $\bar{f}(\mathbf{x}) = \sum_{j=1}^{n} f^j(\mathbf{x})$ via interaction between agents. Here $\mathbf{x} \in \mathbb{R}^d$ is the solution vector, and $f^j(\mathbf{x}) : \mathbb{R}^d \to \mathbb{R}$ is the objective function (loss function) of agent j, which is generally known only to the agent himself. First appearances of distributed optimization algorithms can be traced to the papers from the 1970s and 1980s [14–16]. To date, there exist a number of approaches for the case when function $f^j(\mathbf{x})$ is convex. In particular, the *Alternating Direction Method of Multipliers* [17], as well as the subgradient method [18, 19] were proposed. For non-convex tasks, the works [20, 21] develop a large class of distributed algorithms based on the use of various "functional-surrogate units." In cases when the gradient of the objective function is unknown or can be measured only approximately (with noise), stochastic approximation methods have been used in [22–24]. As the complexity of systems increases, "traditional" approaches based on centralized elements turns out to be inadequate. Limited computing power of the central node does not allow to scale the system as a whole. In addition, in a large system one has to deal with serious communication problems such as high delays in data transmission, packet loss, and so on. For such tasks, the spectral clustering method for large-scale datasets has been proposed in [25], in [26] the method of data aggregation from a large number of devices is considered, and in [27] the method for diagnosing faults in distributed systems.

Spall in [28, 29] investigated the methods of cyclic stochastic approximation where the vector of parameters being evaluated is divided into two or more parts, which we call subvectors in what follows, and the process of updating the estimate consists in sequentially evaluating each subvector while keeping the values of the other parts in the current state, and the values of the estimate vector are then joined in synchronous or asynchronous mode. Recently, these methods have been under active development for multi-agent systems [30–33].

One characteristic feature of cyclic optimization is that at each iteration one has to compute the "direction" for the next step only for some part of the parameter vector. Thanks to this feature, it becomes possible to reduce the computational complexity of the optimization procedure in high-dimensional problems, which in turn leads to an increase in numerical efficiency (increasing the computational speed and reducing the amount of memory required to store the data during calculations). These aspects are extremely important in solving the problem of estimating the trajectories of a large group of moving objects with multiple spatially distributed sensors. As the number of observers (sensors) and targets (moving objects) increases, the computational complexity of the optimization procedure grows significantly. Despite significant expansion of the capabilities of robotic devices, their computing and communication capabilities are often limited, which motivates the development of new approaches that from the theoretical point of view will be quite strict, but at the same time "simple" for practical application.

The paper is organized as follows. In Section 2 we give a general formulation of the optimization problem for a nonstationary average risk functional, illustrated by the problem of estimating the trajectories of moving objects by a group of observers. Next, we describe an approach for optimizing a nonstationary functional based on cyclic stochastic approximation and formulate a theoretical result about an upper bound on the mean-square estimation error. Section 3 describes in more detail the problem of distributed estimation of the trajectories of moving objects and shows a distributed cyclic algorithm for which we prove a theoretical result on the asymptotic properties of the sequence of resulting estimates. Section 4 presents the results of simulation modeling showing how the algorithm performs in practice. Section 5 sums up the work and briefly discusses plans for the future.

2. CYCLIC STOCHASTIC APPROXIMATION WITH PERTURBATION AT THE INPUT

2.1. Optimization of a Nonstationary Average Risk Functional

Many practical applications require to optimize some average risk functional. Although sometimes extreme values can be found analytically, technical systems often have to deal with a situation where the optimized function itself is not fully defined, and its gradient is unknown. In this case, it is possible to use only the values of the function at the selected points (where one can measure or calculate them).

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a set of elementary events Ω, σ-algebra of events $\mathcal{F}$, and probability measure $\mathbb{P}$; let E denote expectation, and let $\mathbb{W}$ be some set (for example, $\mathbb{W} = \mathbb{N}$ or $\mathbb{W} \subset \mathbb{R}^p$). We consider the family of differentiable functions $\{\bar{f}_w(\boldsymbol{\theta})\}_{w\in\mathbb{W}}$, $\bar{f}_w(\boldsymbol{\theta}) : \mathbb{R}^d \to \mathbb{R}$. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be a sequence of observation points (measurements) chosen by the experimenter (observation plan), where at every time moment $t = 1, 2, \ldots$ the values $y_1, y_2, \ldots$ of the functions $\bar{f}_w(\cdot)$ are available with additive external disturbances v_t

$$y_t = \bar{f}_{w_t}(\mathbf{x}_t) + v_t, \tag{1}$$

where $\{w_t\}$ is an uncontrolled sequence, $w_t \in \mathbb{W}$.

We denote by $\mathcal{F}_{t-1}$ the σ-algebra of probability events generated by those quantities from $w_0, \ldots, w_{t-1}, x_0, \ldots, x_{t-1}, v_0, \ldots, v_{t-1}$ that are random; let $\mathrm{E}_{\mathcal{F}_{t-1}}$ denote conditional expectation with respect to the σ-algebra $\mathcal{F}_{t-1}$. Suppose that if w_t is a random variable, then function $\bar{f}_{w_t}(\boldsymbol{\theta})$ as a function of w_t is measurable for each $\boldsymbol{\theta}$ with respect to the σ-algebra $\mathcal{F}_{t-1}$.

Nonstationary problem setting: find the "drifting" minimum point $\boldsymbol{\theta}_t$ *of the function*

$$\bar{F}_t(\boldsymbol{\theta}) = E_{\mathcal{F}_{t-1}} \bar{f}_{w_t}(\boldsymbol{\theta}) \to \min_{\boldsymbol{\theta}}. \tag{2}$$

More precisely: suppose that function $\bar{F}_t(\boldsymbol{\theta})$ has a minimum; then, using observations $y_1, \ldots, y_t$ and inputs $\mathbf{x}_1, \ldots, \mathbf{x}_t$, the problem is to construct an estimate $\widehat{\boldsymbol{\theta}}_t$ of the unknown vector $\boldsymbol{\theta}_t$ that minimizes the nonstationary (time-dependent) *average risk functional* (2).

2.2. Estimating Trajectories of Moving Objects by a Group of Observers

To illustrate the general problem setting, consider a distributed network of n observers (sensors) that have in their zone of visibility m objects whose state vectors are to be estimated.

Let $N = \{1, 2, \ldots, n\}$ be the set of observers (sensors), $M = \{1, 2, \ldots, m\}$, the set of objects, $\mathbf{s}_t^j \in \mathbb{R}^m$, the vector of the current state of sensor j, $j \in N$, at time t, $\mathbf{r}_t^i \in \mathbb{R}^p$, the state of object i, $i \in M$, at time t. The states $\mathbf{r}_t^i$ of objects are accessible to observers through measurements obtained from the observation model

$$\mathbf{z}_t^{i,j} = \varphi(\mathbf{s}_t^j, \mathbf{r}_t^i) + \boldsymbol{\varepsilon}_t^{i,j}, \tag{3}$$

where $\mathbf{z}_t^{i,j} \in \mathbb{R}^q$ are noisy observations about object i available to sensor j at time t, $\varphi(\cdot,\cdot) : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^q$ is the observation function that reflects the measurements of object i by sensor j according to the current sensor and object states, $\{\boldsymbol{\varepsilon}_t^{i,j}\}$ is independent noise in the measurements with zero mean $\mathrm{E}\boldsymbol{\varepsilon}_t^{i,j} = 0$ and covariance $\mathrm{E}\boldsymbol{\varepsilon}_t^{i,j}(\boldsymbol{\varepsilon}_t^{i,j})^{\mathrm{T}} = \Sigma_t^{i,j}$. (Here and below, $\cdot^{\mathrm{T}}$ denotes the transposition of a vector or matrix.)

We assume that there exists an inverse function with respect to the second argument $\varphi^{-1}(\mathbf{s}_t^j,\cdot):\mathbb{R}^q\to\mathbb{R}^p$ such that for any $i\in M$, $j\in N$ and independent centered $\boldsymbol{\varepsilon}_t^{i,j}$ with covariances $\Sigma_t^{i,j}$

$$\varphi^{-1}\left(\mathbf{s}_t^j,\varphi(\mathbf{s}_t^j,\mathbf{r}_t^i)+\boldsymbol{\varepsilon}_t^{i,j}\right)=\mathbf{r}_t^i+\boldsymbol{\xi}_t^{i,j}, \tag{4}$$

where $\boldsymbol{\xi}_t^{i,j}$ are independent with zero mean, $E\boldsymbol{\xi}_t^{i,j}=0$, covariance $\mathrm{E}\boldsymbol{\xi}_t^{i,j}(\boldsymbol{\xi}_t^{i,j})^{\mathrm{T}}=\Xi_t^{i,j}$ and a bounded fourth moment $\mathrm{E}\|\boldsymbol{\xi}_t^{i,j}\|^4\leqslant M_4$. In addition, we assume that with a certain probability p_σ the mean values of the traces $\mathrm{trace}(\Xi_t^{i,j})$ (sums of diagonal matrix elements $\Xi_t^{i,j}$) do not exceed a certain threshold value $(\bar{\sigma}_{\min})^2>0$, and their average values, if the thresholds $(\bar{\sigma}_{\min})^2$ are exceeded, are equal to $(\bar{\sigma}_t^{i,j})^2$.

The simplest typical example is $\varphi(\mathbf{s},\mathbf{r})=\mathbf{r}-\mathbf{s}$ and $\varphi^{-1}(\mathbf{s},\mathbf{z})=\mathbf{s}+\mathbf{z}$. For a different example, when the objects $\mathbf{r}_t^i=\begin{bmatrix}r_t^{i,1}\\ r_t^{i,2}\end{bmatrix}$ and sensors $\mathbf{s}_t^j=\begin{bmatrix}s_t^{j,1}\\ s_t^{j,2}\end{bmatrix}$ are located on a plane, and measurements are of angles and distances to objects, we can consider functions $\varphi(\cdot,\cdot)$ of the form

$$\varphi(\mathbf{s}_t^j,\mathbf{r}_t^i)=\begin{bmatrix}\psi(\mathbf{s}_t^j,\mathbf{r}_t^i)\\ \rho(\mathbf{s}_t^j,\mathbf{r}_t^i)\end{bmatrix}\in\mathbb{R}^2, \tag{5}$$

where

$$\psi(\mathbf{s}_t^j,\mathbf{r}_t^i)=\arctan\left[\frac{r_t^{i,1}-s_t^{j,1}}{r_t^{i,2}-s_t^{j,2}}\right] \tag{6}$$

is the angle between the direction from the sensor to the north and direction to the observed object, also called the azimuth angle, or directional angle,

$$\rho(\mathbf{s}_t^j,\mathbf{r}_t^i)=\sqrt{\left(r_t^{i,1}-s_t^{j,1}\right)^2+\left(r_t^{i,2}-s_t^{j,2}\right)^2} \tag{7}$$

is the distance from the location of the sensor to the object. The inverse function of the second argument $\varphi^{-1}(\mathbf{s}_t^j,\cdot)$ looks like

$$\varphi^{-1}\left(\mathbf{s}_t^j,\mathbf{z}_t^{i,j}\right)=\mathbf{s}_t^j+\begin{bmatrix}z_t^{i,j,2}\sin z_t^{i,j,1}\\ z_t^{i,j,2}\cos z_t^{i,j,1}\end{bmatrix}, \tag{8}$$

where $z_t^{i,j,1}$ and $z_t^{i,j,2}$ are the first and second coordinates of vector $\mathbf{z}_t^{i,j}$. If the error covariance matrices $\boldsymbol{\varepsilon}_t^{i,j}$ are equal to $\Sigma_t^{i,j}=\begin{bmatrix}\sigma_\psi^2 & 0\\ 0 & \left(z_t^{i,j,2}\sigma_\rho\right)^2\end{bmatrix}$, then for errors $\boldsymbol{\xi}_t^{i,j}$ we have

$$\Xi_t^{i,j}=R\left(z_t^{i,j,1}\right)\begin{bmatrix}\left(z_t^{i,j,2}\sigma_\psi\right)^2 & 0\\ 0 & \left(z_t^{i,j,2}\sigma_\rho\right)^2\end{bmatrix}R(z_t^{i,j,1})^{\mathrm{T}}, \tag{9}$$

where $R(\psi)=\begin{bmatrix}\sin\psi & -\cos\psi\\ \cos\psi & \sin\psi\end{bmatrix}$ is the rotation matrix for the angle ψ. Note that the trace of the resulting matrix equals $\mathrm{trace}(\Xi_t^{i,j})=(z_t^{i,j,2}\sigma_\psi)^2+(z_t^{i,j,2}\sigma_\rho)^2$.

We denote by $\boldsymbol{\theta}_t=\mathrm{col}\left(\mathbf{r}_t^1,\ldots,\mathbf{r}_t^m\right)$ the general state vector of all objects. Let $\widehat{\mathbf{r}}_t^i$ be an estimate for the state of object i at time t, $\widehat{\boldsymbol{\theta}}_t=\mathrm{col}(\widehat{\mathbf{r}}_t^1,\ldots,\widehat{\mathbf{r}}_t^m)$, the cumulative total vector of estimates. In a sufficiently general case, the problem of estimating unknown object states can be formulated as the problem of minimizing the functional

$$\bar{F}_t(\widehat{\boldsymbol{\theta}}_t) = \frac{1}{2}\sum_{i\in M}\left\|\mathbf{r}_t^i - \widehat{\mathbf{r}}_t^i\right\|^2 \to \min_{\widehat{\boldsymbol{\theta}}_t} \tag{10}$$

at observations

$$y_t = \frac{K}{2n}\sum_{j\in N}\sum_{i\in M}\left\|\varphi^{-1}\left(\mathbf{s}_t^j, \mathbf{z}_t^{i,j}\right) - \widehat{\mathbf{r}}_t^i\right\|^2 \Big/ \left(\sigma_t^{i,j}\right)^2, \tag{11}$$

where $\|\cdot\|$ denotes the Euclidean norm of a vector, $K = \mathrm{p}_\sigma(\bar{\sigma}_{\min})^2 + (1-\mathrm{p}_\sigma)\sum_{j\in N}(\bar{\sigma}_t^{i,j})^2$, $(\sigma_t^{i,j})^2 = \max\{\mathrm{trace}(\Xi_t^{i,j})\}$ and the corresponding terms are assumed to be zero if $(\sigma_t^{i,j})^2 = \infty$. Normalization with respect to $(\sigma_t^{i,j})^2$ in such problems is quite natural and allows one to rank observations according to their level of reliability.

The observation model (11) naturally "fits" into the general scheme (1) if one selects

$$\mathbb{W} = \otimes_{i=1}^m \otimes_{j=1}^n \mathbb{R}^q \otimes_{j=1}^n \mathbb{R}^p$$

and denotes $w_t = \mathrm{col}(\ldots, \boldsymbol{\varepsilon}_t^{i,j}, \ldots \mathbf{s}_t^j, \ldots)$, $\mathbf{x}_t = \widehat{\boldsymbol{\theta}}_t$,

$$\bar{f}_{w_t}(\mathbf{x}_t) = \frac{K}{2n}\sum_{j\in N}\sum_{i\in M}\left\|\varphi^{-1}\left(\mathbf{s}_t^j, \mathbf{z}_t^{i,j}\right) - \widehat{\mathbf{r}}_t^i\right\|^2 \Big/ \left(\sigma_t^{i,j}\right)^2.$$

The general scheme of (1) also allows for additional errors in observations v_t.

With the above notation, functional (10) is actually an average risk functional of type (2), $\bar{F}_t(\mathbf{x}_t) = E_{\mathcal{F}_{t-1}}\bar{f}_{w_t}(\mathbf{x}_t)$, because by virtue of independence and centering of $\boldsymbol{\xi}_t^{i,j}$ we have

$$\begin{aligned}
&\mathrm{E}_{\mathcal{F}_{t-1}}\frac{K}{2n}\sum_{j\in N}\sum_{i\in M}\left\|\varphi^{-1}\left(\mathbf{s}_t^j, \mathbf{z}_t^{i,j}\right) - \widehat{\mathbf{r}}_t^i\right\|^2 \Big/ \left(\sigma_t^{i,j}\right)^2 \\
&= \mathrm{E}_{\mathcal{F}_{t-1}}\frac{K}{2n}\sum_{i\in M}\sum_{j\in N}\left\|\mathbf{r}_t^i + \boldsymbol{\xi}_t^{i,j} - \widehat{\mathbf{r}}_t^i\right\|^2 \Big/ \left(\sigma_t^{i,j}\right)^2 \\
&= \frac{1}{2}\sum_{i\in M}\left\|\mathbf{r}_t^i - \widehat{\mathbf{r}}_t^i\right\|^2 \sum_{j\in N}\mathrm{p}_\sigma + (1-\mathrm{p}_\sigma)E_{\left\{\mathrm{trace}(\Xi_t^{i,j}) > (\bar{\sigma}_{\min})^2\right\}}\frac{\mathrm{trace}\left(\Xi_t^{i,j}\right)}{\left(\sigma_t^{i,j}\right)^2} \\
&= \frac{1}{2}\sum_{i\in M}\left\|\mathbf{r}_t^i - \widehat{\mathbf{r}}_t^i\right\|^2.
\end{aligned}$$

2.3. Cyclic Stochastic Approximation

The works [28, 33] introduced a cyclic algorithm of stochastic approximation type to find an estimate of vector $\boldsymbol{\theta}_t$, and the authors used step size decreasing to zero. However, in order to solve problem (2), which consists in monitoring the changes in parameter $\boldsymbol{\theta}_t$, one has to use a constant step size due to the nonstationarity of the minimized functional. This justifies the study of properties of estimates of the method of cyclic stochastic approximation with nondecreasing step size. The cyclic approach allows us to move from a centralized formulation of the problem to a distributed one. One characteristic feature of the cyclic approach is that the unknown vector $\boldsymbol{\theta}_t$ is divided into several subvectors, and at time t only the selected subvector is updated. Nevertheless, it is implied that over a certain time interval each subvector will be updated separately.

Let us consider the application of the cyclic approach in more detail. We break the time axis into a sequence of cycles of length $2k$: $2(T-1)k+1, 2(T-1)k+2, \ldots, 2Tk$, and on each cycle we split the set of indices $\mathbb{D} = \{1, \ldots, d\}$ into k disjoint subsets $\mathbb{I}_u$, $u = 1, \ldots, k$, that show the selection of active parameters at time moments $t = 2(T-1)k + 2u - 1$ and $t = 2(T-1)k + 2u$, $u = 1, \ldots, k$, and that satisfy conditions

$$\bigcup_{u=1}^{k} \mathbb{I}_u = \mathbb{D}, \quad \mathbb{I}_{u'} \bigcap \mathbb{I}_{u''} = \emptyset \quad \text{for} \quad u' \neq u''. \tag{12}$$

For each $t = 1, 2, \ldots$ we define diagonal matrices A_t that form a sparse vector $A_t \mathbf{x}_t$ with respect to the vector $\mathbf{x}_t$ with zeros at positions whose indices do not belong to $\mathbb{I}_{(t \bmod (2k)) \div 2}$, where mod denotes the remainder of a division, $\div$ denotes exact division. By a cyclic sequence of matrices $\{A_t\}$ we define a polynomial

$$\mathcal{A}(\lambda) = \sum_{u=1}^{k} A_{2kT+2u} \lambda^u,$$

which will later be convenient to use together with the index shift operation $\lambda \boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-2k+2}$.

Taking into account the above notation, the obtained observations $y_1, y_2, \ldots$ can be represented as

$$y_t = f_{w_t}(A_t \mathbf{x}_t) + v_t. \tag{13}$$

To track changes in $\boldsymbol{\theta}_t$, we use a randomized stochastic approximation algorithm with a perturbation at the input [9], modified with the cyclic approach. Let $\widehat{\boldsymbol{\theta}}_0 \in \mathbb{R}^d$ be a non-random initial vector, $\boldsymbol{\Delta}_T$, $T = 0, 1, \ldots$, be the observable sequence of Bernoulli random vectors from $\mathbb{R}^d$ that take values ± 1 with equal probabilities $\frac{1}{2}$, called *randomized sample disturbances*. To construct observation points $\{\mathbf{x}_t\}$ and estimates $\{\widehat{\boldsymbol{\theta}}_t\}$, consider the search algorithm of stochastic approximation with two observations and randomization at the input:

$$\begin{cases} \mathbf{x}_{2((t-1)\div 2)+1} = A_t(\widehat{\boldsymbol{\theta}}_{t-1} - \beta^- \boldsymbol{\Delta}_{t\div 2}), \quad \mathbf{x}_{2(t\div 2)} = A_t(\widehat{\boldsymbol{\theta}}_{t-1} + \beta^+ \boldsymbol{\Delta}_{t\div 2}) \\ \widehat{\boldsymbol{\theta}}_{2((t-1)\div 2)+1} = \widehat{\boldsymbol{\theta}}_{2((t-1)\div 2)} \\ \widehat{\boldsymbol{\theta}}_{2(t\div 2)} = \widehat{\boldsymbol{\theta}}_{2((t-1)\div 2)+1} - \alpha \mathrm{A_t} \boldsymbol{\Delta}_T \dfrac{y_{2T} - y_{2T-1}}{\beta}, \end{cases} \tag{14}$$

where $\alpha > 0$ is a constant step size, $\beta^+ \geqslant 0$ and $\beta^- \geqslant 0$ are such that $\beta = \beta^+ + \beta^- > 0$.

2.4. Upper Bound of the Mean Squared Estimation Error

To evaluate the quality of estimates, we will use the following characteristic, similar to the one presented in [9].

Definition. A sequence of estimates $\{\widehat{\boldsymbol{\theta}}_t\}$ yields an asymptotically optimal weak upper bound $\bar{L} > 0$ of the mean squared residual if for every $\varepsilon > 0$ there exists $\bar{t}$ such that

$$\forall t > \bar{t} \ \sqrt{\mathrm{E}\|\widehat{\boldsymbol{\theta}}_t - \mathcal{A}(\lambda)\boldsymbol{\theta}_t\|^2} \leqslant \bar{L} + \varepsilon.$$

Next we formulate basic *assumptions* about perturbations and functions $\bar{f}_w(\mathbf{x})$, $\bar{F}_t(\mathbf{x})$.

1. *At the minimum points $\boldsymbol{\theta}_t$ of the functions $\bar{F}_t(\cdot)$ and gradient vectors of the functions $\bar{f}_{w_t}(A_t\mathbf{x})$ the following inequalities hold:*

$$\forall \mathbf{x} \in \mathbb{R}^d \ \ (\mathbf{x} - \boldsymbol{\theta}_t)^{\mathrm{T}} A_t^{\mathrm{T}} \mathrm{E}_{\mathcal{F}_{t-1}} \nabla \bar{f}_{w_t}(A_t\mathbf{x}) \geqslant \mu \|A_t(\mathbf{x} - \boldsymbol{\theta}_t)\|^2$$

for some constant $\mu > 0$.

2. *$\forall w \in \mathbb{W}$ gradient $\nabla \bar{f}_{w_t}(A_t\mathbf{x})$ satisfies the Lipschitz condition: $\forall \mathbf{x}', \mathbf{x}'' \in \mathbb{R}^d$*

$$\|\nabla \bar{f}_{w_t}(A_t\mathbf{x}') - \bar{f}_{w_t}(A_t\mathbf{x}'')\| \leqslant M\,\|A_t(\mathbf{x}' - \mathbf{x}'')\|$$

with a constant $M \geqslant \mu$.

3. *Gradient vector $\nabla \bar{f}_{w_t}(A_t\mathbf{x})$ is uniformly bounded at the minimum points $\boldsymbol{\theta}_t$:*

$$\|\mathrm{E}\nabla \bar{f}_{w_t}(A_t\boldsymbol{\theta}_t)\| \leqslant c_1, \quad \mathrm{E}\,\|\nabla \bar{f}_{w_t}(A_t\boldsymbol{\theta}_t)\|^2 \leqslant c_2,$$
$$\mathrm{E}\left(\nabla \bar{f}_{w_t}(A_t\boldsymbol{\theta}_t)\right)^{\mathrm{T}} \nabla \bar{f}_{w_{t-1}}(A_t\boldsymbol{\theta}_{t-1}) \leqslant c_2$$

($c_1 = c_2 = 0$, if the sequence of w_t is not random, i.e., $\bar{f}_{w_t}(\mathbf{x}) = \bar{F}_t(\mathbf{x})$).

4. *The drift is bounded: for $\boldsymbol{\eta}_t = A_t(\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1})$ $\|\boldsymbol{\eta}_t\| \leqslant \delta_{\boldsymbol{\theta}} < \infty$ or $\mathrm{E}\|\boldsymbol{\eta}_t\|^2 \leqslant \delta_{\boldsymbol{\theta}}^2$ and $\mathrm{E}\|\boldsymbol{\eta}_t\|\|\boldsymbol{\eta}_{t-1}\| \leqslant \delta_{\boldsymbol{\theta}}^2$ if the sequence $\{w_t\}$ is random.*

5. *The drift rate is bounded in such a way that $\forall \mathbf{x} \in \mathbb{R}^d$:*

$$\mathrm{E}_{\mathcal{F}_{t-2}}\left(\bar{f}_{w_t}(A_t\boldsymbol{\theta}_t) - \bar{f}_{w_{t-1}}(A_t\boldsymbol{\theta}_{t-1})\right)^2 \leqslant c_3\|A_t(\mathbf{x} - \boldsymbol{\theta}_{t-2})\| + c_4.$$

6. *Consecutive differences in observation interference are bounded:*

$$|v_{2t} - v_{2t-1}| \leqslant c_v < \infty \quad or \quad \mathrm{E}(v_{2t} - v_{2t-1})^2 \leqslant c_v^2,$$

if the sequence $\{v_t\}$ is random.

7. *For $T = 0, 1, \ldots,$ if v_t are random then the vector $\boldsymbol{\Delta}_T$ and interference difference $v_{2kT+2} - v_{2kT+1}, \ldots, v_{2k(T+1)} - v_{2k(T+1)-1}$ are independent; if w_t are random, then vector $\boldsymbol{\Delta}_T$ and $w_{2kT+1}, \ldots, w_{2k(T+1)}$ are independent.*

We denote

$$\gamma = 3d\left(M^2 d + \frac{c_3}{\beta}\right), \quad m = 2(\mu - \alpha\gamma),$$
$$b = 2\beta M d\sqrt{d}(1 + 6\alpha M d) + \delta_\theta(M + 2\mu + 6\alpha M^2 d^2),$$
$$\bar{l} = 2\alpha d\left(c_v^2 + 3\left(\frac{c_4}{\beta} + d\left(c_2 + M^2\left(\delta_\theta + 2\beta\sqrt{d}\right)^2\right)\right)\right) + 2\delta_\theta\left(4\beta M d\sqrt{d} + M\delta_\theta + c_1 + 3\mu\delta_\theta^2\right),$$
$$l = \bar{l} + 2bk\sqrt{k}\delta_{\boldsymbol{\theta}} + \frac{1 - \alpha m}{\alpha}\delta_{\boldsymbol{\theta}}^2.$$

We use the result of Theorem 1 from [9] and based on it formulate a theorem for the estimates formed by each sensor.

Theorem 1. *If conditions (12) and assumptions 1-7 hold, and constant α is sufficiently small:*

$$\alpha \in \begin{cases} (0; \mu/\gamma), & \text{if } \mu^2 < 2\gamma \\ \left(0; \dfrac{\mu - \sqrt{\mu^2 - 2\gamma}}{2\gamma}\right) \cup \left(\dfrac{\mu + \sqrt{\mu^2 - 2\gamma}}{2\gamma}; \mu/\gamma\right) & \text{otherwise}, \end{cases} \tag{15}$$

then the sequence of estimates $\{\widehat{\boldsymbol{\theta}}_{2kT}\}_{T=0}^{\infty}$, constructed with algorithm (14), has an asymptotically optimal weak upper bound of the mean squared residue

$$\bar{L} = \frac{\sqrt{k}\left(b + \sqrt{b^2 + ml}\,\right)}{m}. \tag{16}$$

Proof of Theorem 1 is given in the Appendix.

3. THE DISTRIBUTED MONITORING PROBLEM FOR PARAMETER CHANGES

3.1. Distributed Tracking

Suppose that there are n observers (sensors) in the considered system. For example, observers can be "tied" to robots, servers that process data, and so on. In distributed optimization, it is assumed that for any $w \in \mathbb{W}$, the function $\bar{f}_w(\boldsymbol{\theta})$ is separable with respect to either the function itself or a partition of the vector $\boldsymbol{\theta}$ into n subvectors, namely:

$$\bar{f}_w(\boldsymbol{\theta}) = \sum_{j=1}^{n} f_w^j(\boldsymbol{\theta}^j), \tag{17}$$

where $\boldsymbol{\theta}^j \in \mathbb{R}^d$ is a copy of the vector $\boldsymbol{\theta}$ for each $j = 1, \ldots, n$ or $\boldsymbol{\theta}^j \in \mathbb{R}^{d^j}$ is a subvector of vector $\boldsymbol{\theta} = \mathrm{col}(\boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^n)$ respectively. In the first case, one looks for a solution of optimization problem $\boldsymbol{\theta}^\star$ for each of the observers (sensors) j based on available local information, assuming that all functions $f_w^j(\cdot)$ have the same minimum points. In the second case, each observer j is associated with some part $\boldsymbol{\theta}^{j^\star}$ of the global optimal solution $\boldsymbol{\theta}^\star$. Further, when there are possible connections between observers, the local solutions obtained are aggregated in one way or another. For example, under fairly general conditions on the topology of the connections between observers, the results can be aggregated with the local voting protocol described in [13], which can operate under interference and delays in communication channels.

Problem (2) in the distributed form is reformulated as follows: *find the "drifting" minimum point $\boldsymbol{\theta}_t$ for the function*

$$\bar{F}_t(\boldsymbol{\theta}) = E_{\mathcal{F}_{t-1}} \sum_{j=1}^{n} f_{w_t}^j(\boldsymbol{\theta}^j) = \sum_{j=1}^{n} F_t^j(\boldsymbol{\theta}^j) \to \min_{\boldsymbol{\theta}}. \tag{18}$$

We denote by $\mathbb{D}^j$ the subsets of indices from $\mathbb{D}$ corresponding to the indices of non-zero components of the vector $\boldsymbol{\theta}^j$ in $\boldsymbol{\theta}$.

3.2. Distributed Estimation of Trajectories of Moving Objects

Let us return to the example of a distributed network shown in Section 2.2 where a network of n observers (sensors) have in their zone of visibility m objects whose state vectors are to be estimated.

We define the adjacency matrix $B_t = [b_t^{i,j}]$, where $b_t^{i,j} > 0$ if sensor j is watching object i, and $b_t^{i,j} = 0$ otherwise. Similarly, we introduce the interaction matrix $C_t = [c_t^{j,k}]$, where $c_t^{j,k} > 0$ if sensor j can communicate with sensor $k \in N$, and $c_t^{j,k} = 0$ otherwise. We denote by $N_t^j = \{j : c_t^{j,k} > 0\} \subset N$ the set of "neighbors" of sensor j; by $|N_t^j|$, the number of "neighbors" of sensor j. Denote by $M_t^j \subset M$ the set of sensor targets j which it observes at time t by itself or about which it can receive data from its neighbors.

We will consider two types of restrictions on the operation of the sensor network. The first restriction is that each sensor can only exchange data with a certain number of "neighbors," i.e., we assume that we are operating under constraints

$$|N_t^j| \leqslant n_{\max}^j. \tag{19}$$

In a real operating environment, these restrictions may result from the fact that some restrictions are imposed on the number of dedicated communication channels, or may be due to the inability to send data to a distance greater than a certain limit. The second restriction applies to the maximum number of targets allowed at time t,

$$|M_t^j| \leqslant m_{\max}^j, \tag{20}$$

information about which sensor j is able to receive from its "neighbors" or by independent observation. This value, in turn, can be associated with limited bandwidth of the communication channel. We also note that the formation of subsets M_t^j is achieved by varying the coefficients of the adjacency matrix B_t.

Suppose that matrices B_t and C_t satisfy conditions (19) and (20) and, in addition, for each observer j, $j \in N$, data is generated about the states of objects from some set $\mathbb{D}^j$ when partition conditions are fulfilled:

$$\bigcup_{u=1}^{k^j} \mathbb{I}_u^j = \mathbb{D}^j, \quad \mathbb{I}_{u'}^j \bigcap \mathbb{I}_{u''}^j = \emptyset \quad \text{for} \quad u' \neq u''. \tag{21}$$

We denote by A_t^j the corresponding matrices that sparsify vectors $\widehat{\boldsymbol{\theta}}_t$.

Taking into account the above notation, we can rewrite the functional (10) as (18) with

$$f_t^j(\widehat{\boldsymbol{\theta}}_t^j) = \frac{K}{2n} \sum_{i \in M_t^j} \left\| \varphi^{-1}\left(\mathbf{s}_t^j, \mathbf{z}_t^{i,j}\right) - \widehat{\mathbf{r}}_t^i \right\|^2 \Big/ \left(\sigma_t^{i,j}\right)^2, \tag{22}$$

since we can assume that $(\sigma_t^{i,j})^2 = \infty$ if at time moment t observer j does not receive any information about object i.

3.3. Multi-Agent Algorithm for Tracking Parameter Changes Using the Cyclic Approach

Algorithm (distributed cyclic estimation algorithm).

For each sensor j, $j \in N$, perform the following procedures.

1. *Initialization and choice of coefficients.* Set the counter to $T^j = 0$. Choose initial approximation $\widehat{\boldsymbol{\theta}}_0^j \in \mathbb{R}^d$ and sufficiently small $\alpha^j > 0$ and $\beta^j > 0$. Set the maximum allowed values for the number of "neighbors" $n_{\max}^j$ and tracking objects $m_{\max}^j$. Construct a sequence of matrices $\{A_t^j\}$ such that conditions (19)–(21) are satisfied.

2. *Iteration* $T^j \to T^j + 1$.

a. According to the Bernoulli distribution generate a random vector $\boldsymbol{\Delta}_{T^j}^j \in \mathbb{R}^d$ from independent components equal to ± 1 with probability $\frac{1}{2}$.

b. *Iteration over an observation cycle* for $u = 1, 2, \ldots, k^j$:

b-1) $t := 2T^j + 2u$;

b-2) construct an observation point $\mathbf{x}_{t-1}^j$ with lth component equal to $\widehat{\theta}_{2T^j}^{j,l} - \beta^j \Delta_{T^j}^{j,l}$, if $a_{t-1}^{j,l} > 0$ and equal to 0 if $a_{t-1}^{j,l} = 0$;

b-3) obtain the measurements $\mathbf{z}_{t-1}^{i,j}$, $i \in M_{t-1}^j$;

b-4) compute the empirical value of the quality functional $y_{t-1}^{j,-} = f_t^j(\mathbf{x}_{t-1}^j)$ by (22);

b-5) generate observation point $\mathbf{x}_t^j$ with lth components equal to $\widehat{\theta}_{2T^j}^{j,l} + \beta^j \Delta_{T^j}^{j,l}$, if $a_t^{j,l} > 0$, and equal to 0 if $a_t^{j,l} = 0$;

b-6) get the measurements $\mathbf{z}_t^{i,j}$, $i \in M_t^j$;

b-7) compute the empirical value of the quality functional $y_t^{j,+} = f_t^j(\mathbf{x}_t^j)$ by (22);

b-8) compute the pseudogradient

$$\widehat{\nabla}_t^j = A_t^j \mathbf{\Delta}_{T^j}^j \frac{y_t^{j,+} - y_{t-1}^{j,-}}{2\beta^j};$$

b-9) find the new estimate:

$$\widehat{\boldsymbol{\theta}}_t^j = \widehat{\boldsymbol{\theta}}_{t-1}^j - \alpha^j \widehat{\nabla}_t^j.$$

3. Go to step 2a.

We formulate the following theorem on the properties of the estimates produced by Algorithm.

Theorem 2. *If the drift is bounded: $\|\mathbf{r}_t^i - \mathbf{r}_{t-1}^i\| \leqslant \sigma_{\mathbf{r}}^i$, $i \in M$, conditions* (4) *hold for the observation model, conditions* (19)–(21) *hold for the matrix sequences $\{B_t\}$, $\{C_t\}$ and $\{A_t^j\}$, $j \in N$, and the constant α is sufficiently small:*

$$\alpha \in \begin{cases} (0; \mu^j/\gamma^j), & \textit{if} \quad (\mu^j)^2 < 2\gamma^j \\ \left(0; \dfrac{\mu^j - \sqrt{(\mu^j)^2 - 2\gamma^j}}{2\gamma^j}\right) \cup \left(\dfrac{\mu^j + \sqrt{(\mu^j)^2 - 2\gamma^j}}{2\gamma^j}; \mu^j/\gamma^j\right) & \textit{otherwise,} \end{cases}$$

then the sequences of estimates $\{\widehat{\boldsymbol{\theta}}_{2k^jT}^j\}_{T=0}^{\infty}$, constructed according to Algorithm, have asymptotically optimal weak upper bounds of the mean square residue

$$\bar{L}^j = \frac{\sqrt{k^j}\left(b^j + \sqrt{(b^j)^2 + m^j l^j}\right)}{m^j}, \tag{23}$$

where

$$\mu^j = \frac{K}{2n \max_{i,t}(\sigma_t^{i,j})^2}, \quad M^j = \frac{K}{2n \min_{i,t}(\sigma_t^{i,j})^2}, \quad \gamma^j = 3d^2(M^j)^2,$$

$$m^j = 2(\mu^j - \alpha\gamma^j), \quad \delta_{\boldsymbol{\theta}}^j = k^j \max_{i,t} \sum_{i \in M_t^j} \delta_{\mathbf{r}}^i,$$

$$b^j = 2\beta M^j d\sqrt{d}(1 + 6\alpha M^j d) + \delta_\theta^j \left(M^j + 2\mu^j + 6\alpha(M^j)^2 d^2\right),$$

$$\bar{l}^j = 6d\frac{\alpha}{\beta} \max_t \frac{K}{2n} \sum_{i \in M_t^j} \left(\frac{M_4}{(\sigma_t^{i,j})^4} + \frac{M_4}{(\sigma_{t-1}^{i,j})^4} - 2\right)$$

$$+ 6d^2 \left(\frac{K}{2n} \max_{i,t} \frac{\operatorname{trace}(\Xi_t^{i,j})}{(\sigma_t^{i,j})^2} + (M^j)^2(\delta_\theta^j + 2\beta\sqrt{d})^2\right)$$

$$+ 2\delta_\theta^j \left(4\beta M^j d\sqrt{d} + M^j \delta_\theta^j + 3\mu^j (\delta_\theta^j)^2\right),$$

$$l^j = \bar{l}^j + 2b^j k^j \sqrt{k^j} \delta_{\boldsymbol{\theta}}^j + \frac{1 - \alpha m^j}{\alpha} (\delta_{\boldsymbol{\theta}}^j)^2.$$

Proof of Theorem 2 is given in the Appendix.

4. MODELING

We apply the results obtained in Section 3 to the problem of distributed estimation of the trajectory of moving objects from Section 2.2. In our construction of the observation model we use functions (5)–(8).

Suppose that the six monitored objects are moving uniformly and rectilinearly across a square region with area 300×300 km^2 with the same speed equal to 2500 km/h.

In the area of interest (tracking area), there are six randomly located stationary sensors. The level of errors in the measurements obtained from each sensor is set at 5 % for distances and 0.5 degrees for angles. For these values, the variance of the measurement error of the trajectory of object i by sensor j will be

$$\Sigma_t^{i,j} = \begin{bmatrix} \sigma_\psi^2 & 0 \\ 0 & (z_t^{i,j,2}\sigma_\rho)^2 \end{bmatrix} = \begin{bmatrix} 0.5^2 & 0 \\ 0 & (0.05z_t^{i,j,2})^2 \end{bmatrix}.$$

Objects and sensors have the same rectangular coordinate system with axes x^1 and x^2. Objects start their movement at points with coordinates $\mathbf{r}_0^1 = [270,\ 295]^{\mathrm{T}}$; $\mathbf{r}_0^2 = [240,\ 290]^{\mathrm{T}}$; $\mathbf{r}_0^3 = [210,\ 285]^{\mathrm{T}}$; $\mathbf{r}_0^4 = [180,\ 280]^{\mathrm{T}}$; $\mathbf{r}_0^5 = [150,\ 275]$; $\mathbf{r}_0^6 = [120\ ,270]^{\mathrm{T}}$. During the simulation, we set parameters of the algorithm to $\alpha^j = 0.05$ and $\beta^j = 0.03$ for each sensor $j \in N$. The initial approximation $\widehat{\boldsymbol{\theta}}_0^j$ is sampled randomly from the interval $[299,\ 300]^{\mathrm{T}}$. Let $m_{\max}^j \in \{1, \dots, 6\}$ be the maximum possible number of targets that sensor j can monitor, and let $n_{\max}^j = 1$ be the maximum possible number of "neighbors" of sensor j.

Figure 1 presents an example of a sensor network of six devices that track six targets. For simplicity of visualization, we only show one target with $i = 1$ on the figure. The figure uses the following notation: dotted line—trajectory of the target; circles with dots—sensors; solid line with a square marker—trajectory estimate by the fourth from the left sensor. Figures 2–5 present simulation results for different values of $m_{\max}^j$, where err_t is the deviation of the estimate from the true value at time t. We can see that with a minimal number of possible tracking objects the rate of convergence of the estimate to the true value slows down significantly. In turn, the best convergence is demonstrated at the maximum possible value of $m_{\max}^j$. Based on simulation results, we can also say that when setting an allowed time interval for estimation, it is possible not to use all available sensors to track each target. This implies that it is possible to use the resources of the sensor network more rationally and further improve its tracking characteristics.

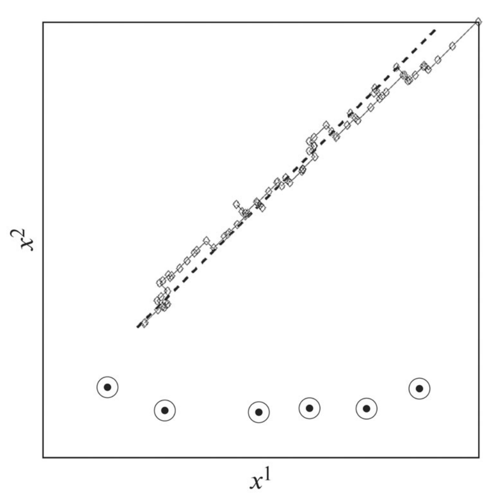

Fig. 1. A sample sensor network with visualized trajectory estimation for object $i = 1$ by sensor $j = 4$ with $m_{\max}^4 = 2$.

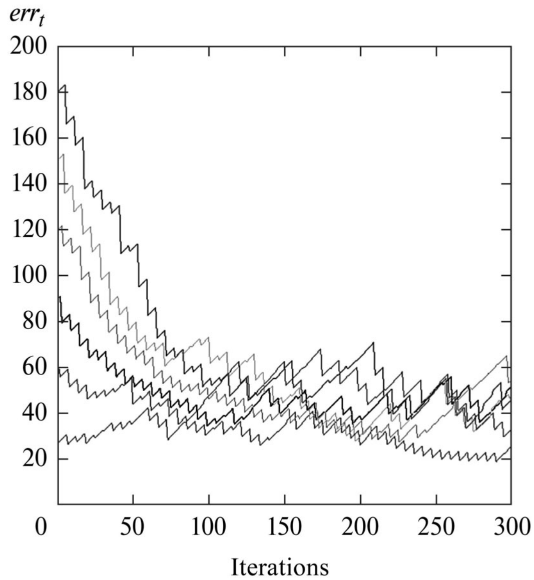

Fig. 2. Deviation of the estimate from the true value at $m^j_{\max} = 1$.

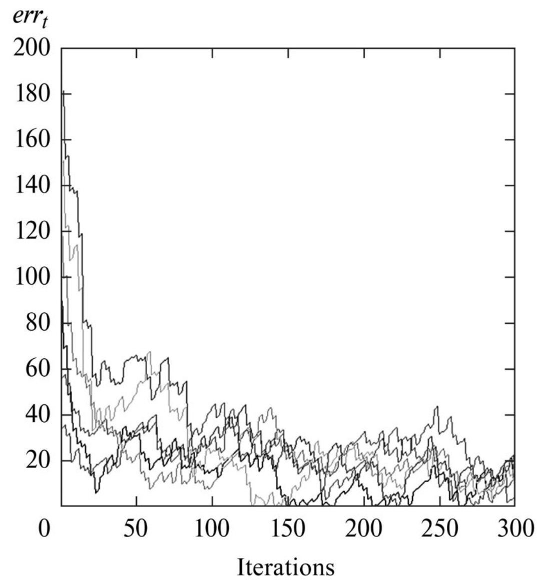

Fig. 3. Deviation of the estimate from the true value for $m^j_{\max} = 2$.

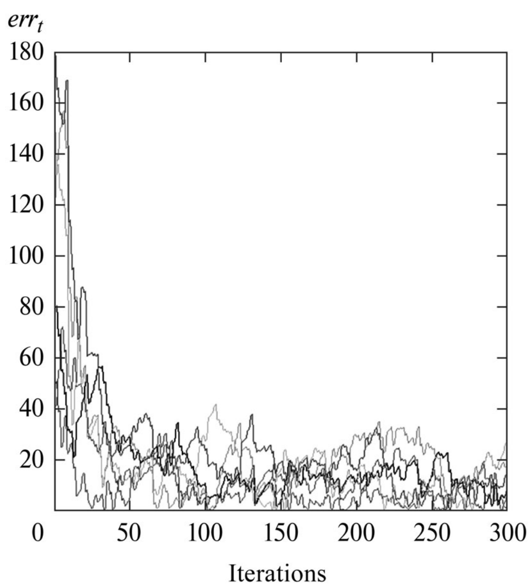

Fig. 4. Deviation of the estimate from the true value for $m^j_{\max} = 3$.

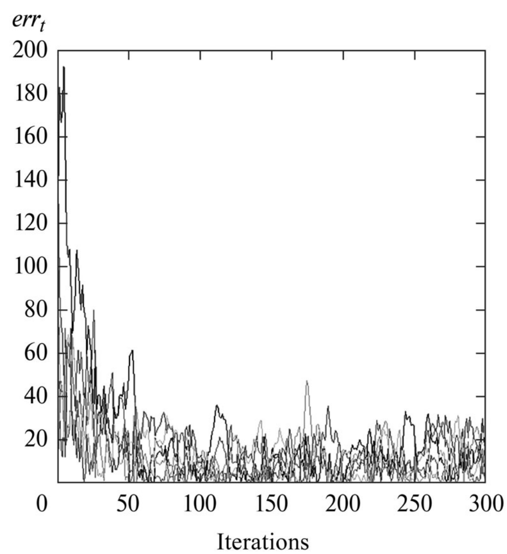

Fig. 5. Deviation from the true value for $m^j_{\max} = 6$.

5. CONCLUSION

In this work, we have studied and justified the application of cyclic stochastic approximation for solving optimization problems for a nonstationary functional. A distributed problem setting is formulated with respect to the optimization of the functional using the cyclic approach. We have considered the problem of tracking parameters in a distributed network of sensors, for example, tracking moving objects by stationary sensors. We have proposed a multiagent estimation algorithm for this problem. We have showed a theoretical result that proves that the sequence of estimates obtained by the proposed algorithm has an asymptotically optimal weak upper bound of the mean square residue of the form (23). The algorithm's operation has also been illustrated by an example of modeling the process of tracking moving objects. Further work may be related to optimizing the use of sensor network resources, namely finding a rational distribution of tracking objects among the sensors with the ability to predict the best tracking group based on the quality of received estimates.

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APPENDIX

Proof of Theorem 1. Consider the following auxiliary Lemma 1 from [34].

Lemma 1 [34]. *If* $e_n > 0$, $\alpha, m > 0$, $\alpha m < 1$, $b, \bar{l} \geqslant 0$ *and*

$$e_n \leqslant (1-\alpha\, m)\, e_{n-1} + 2\alpha\, b\, \sqrt{e_{n-1}} + \alpha\, \bar{l}, \quad n = 1, 2, \ldots, \tag{A.1}$$

then for every $\varepsilon > 0$ *there exists* N *such that* $\forall n > N$

$$e_n \leqslant \left(\frac{b + \sqrt{b^2 + m\bar{l}}}{m} \right)^2 + \varepsilon.$$

Proof Lemma 1 is presented in [8].

Let $T = 0, 1, \ldots$. Consider the cycle with index T. Similar to the proof of Theorem 1 from [9, (16), p. 1358], since assumptions 1–7 hold and taking into account the form of algorithm (14), we obtain for every $u = 1, 2, \ldots, k$ the following relation:

$$\begin{aligned} &\mathrm{E}_{\mathcal{F}_{2kT+2u-2}} \left\| A_{2kT+2u}(\widehat{\boldsymbol{\theta}}_{2kT+2u} - \boldsymbol{\theta}_{2kT+2u}) \right\|^2 \\ &\leqslant (1-\alpha m) \left\| A_{2kT+2u}(\widehat{\boldsymbol{\theta}}_{2kT} - \boldsymbol{\theta}_{2kT+2u-2}) \right\|^2 \\ &\quad + 2\alpha b \left\| A_{2kT+2u}(\widehat{\boldsymbol{\theta}}_{2kT} - \boldsymbol{\theta}_{2kT+2u-2}) \right\| + \alpha \bar{l}. \end{aligned}$$

Summing up the left and right parts of the last equality over $u \in \{1, \ldots, k\}$, averaging them with respect to the σ-algebra $\mathcal{F}_{2kT}$, and taking into account the form of the matrices A_{2kT+2u}, we obtain

$$\begin{aligned} \mathrm{E}_{\mathcal{F}_{2kT}} \nu_{T+1}^2 \leqslant \mathrm{E}_{\mathcal{F}_{2kT}} (1-\alpha m) &\left\| \widehat{\boldsymbol{\theta}}_{2kT} - \mathcal{A}(\lambda) \boldsymbol{\theta}_{2k(T+1)-2} \right\|^2 \\ + 2\alpha b \sqrt{k} &\left\| \widehat{\boldsymbol{\theta}}_{2kT} - \mathcal{A}(\lambda) \boldsymbol{\theta}_{2k(T+1)-2} \right\| + k\alpha \bar{l}, \end{aligned}$$

where we have denoted $\nu_{T+1} = \|\widehat{\boldsymbol{\theta}}_{2k(T+1)} - \mathcal{A}(\lambda)\boldsymbol{\theta}_{2k(T+1)}\|$.

From the triangle inequality and assumption 4, we obtain the estimates

$$\mathrm{E}_{\mathcal{F}_{2kT}} \left\| \widehat{\boldsymbol{\theta}}_{2kT} - \mathcal{A}(\lambda) \boldsymbol{\theta}_{2k(T+1)-2} \right\|^2 \leqslant \nu_T^2 + k^2 \delta_{\boldsymbol{\theta}}^2$$

and

$$\mathrm{E}_{\mathcal{F}_{2kT}} \left\| \widehat{\boldsymbol{\theta}}_{2kT} - \mathcal{A}(\lambda) \boldsymbol{\theta}_{2k(T+1)-2} \right\| \leqslant \nu_T + k^2 \delta_{\boldsymbol{\theta}}$$

and, taking them into account, we get that

$$\mathrm{E}_{\mathcal{F}_{2kT}} \nu_{T+1}^2 \leqslant (1-\alpha m)\nu_T^2 + 2\alpha b \sqrt{k} \nu_T + k\alpha \left(\bar{l} + 2bk\sqrt{k}\delta_{\boldsymbol{\theta}} + \frac{1-\alpha m}{\alpha} \delta_{\boldsymbol{\theta}}^2 \right).$$

If condition (2) is satisfied, in our notation the application of Lemma 1 to the last inequality completes the proof of Theorem 1.

Proof of Theorem 2. To prove Theorem 2, in our notation it suffices to verify assumptions 1–7 of Theorem 1 for the functions $F_t^j(A_t^j\mathbf{x})$ and $f_t^j(A_t^j\mathbf{x})$. To do this, first compute the gradient components of the function $F_t^j(A_t^j\mathbf{x})$:

$$\frac{\partial}{\partial x^{i,l}}\nabla F_t^j(A_t^j\mathbf{x}) = E_{\mathcal{F}_{t-1}}\frac{\partial}{\partial x^{i,l}}\nabla f_t^j(A_t^j\mathbf{x})$$
$$= E_{\mathcal{F}_{t-1}}\sum_{i\in M_t^j}\frac{K}{2n(\sigma_t^{i,j})^2}(x^{i,l}+\xi^{i,l}-r^{i,l})\times 1 = \sum_{i\in M_t^j}\frac{K}{2n(\sigma_t^{i,j})^2}x^{i,l}-r^{i,l}.$$

From the last formula it is clear that

assumption 1 holds with $\mu^j = \frac{K}{2n\max_{i,t}(\sigma_t^{i,j})^2}$,

assumption 2 holds with $M^j = \frac{K}{2n\min_{i,t}(\sigma_t^{i,j})^2}$,

assumption 3 holds with $c_1 = 0$ and $c_2^j = \frac{K}{2n}\max_{i,t}\frac{\mathrm{trace}(\Xi_t^{i,j})}{(\sigma_t^{i,j})^2}$.

assumption 4 on the drift holds for $\delta_{\boldsymbol{\theta}}^j = k^j\max_{i,t}\sum_{i\in M_t^j}\delta_{\mathbf{r}}^i$.

Let us verify that assumption 5 holds for $c_3 = 0$ and $c_4^j = \max_t\frac{K}{2n}\sum_{i\in M_t^j}\left(\frac{M_4}{(\sigma_t^{i,j})^4}+\frac{M_4}{(\sigma_{t-1}^{i,j})^4}-2\right)$. For the corresponding difference we have

$$\mathrm{E}_{\mathcal{F}_{t-2}}\left(\bar{f}_{w_t}^j(A_t\boldsymbol{\theta}_t)-\bar{f}_{w_{t-1}}^j(A_t\boldsymbol{\theta}_{t-1})\right)^2$$
$$= \frac{K}{2n}\mathrm{E}_{\mathcal{F}_{t-2}}\sum_{i\in M_t^j}\left(\left\|\frac{\boldsymbol{\xi}_t^{i,j}}{(\sigma_t^{i,j})}\|^2-\|\frac{\boldsymbol{\xi}_{t-1}^{i,j}}{(\sigma_{t-1}^{i,j})}\right\|^2\right)^2$$
$$\leqslant \frac{K}{2n}\sum_{i\in M_t^j}\left(\frac{M_4}{(\sigma_t^{i,j})^4}+\frac{M_4}{(\sigma_{t-1}^{i,j})^4}-2\right)\leqslant c_4^j.$$

Since in our problem $v_t = 0$ then $c_v = 0$ in assumption 6, and assumption 7 holds because the noise are independent in our observation model.

This completes the proof of Theorem 2.

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