Simultaneous Perturbation Stochastic Approximation for Tracking Under Unknown but Bounded Disturbances

Oleg Granichin, *Senior Member, IEEE*, and Natalia Amelina

***Abstract*—Multi-dimensional stochastic optimization plays an important role in analysis and control of many technical systems. To solve the challenging multidimensional problems of nonstationary optimization, it is suggested to use a stochastic approximation algorithm (like SPSA) with perturbed input and constant step-size which has simple form. We get a finite bound of residual between estimates and time-varying unknown parameters when observations are made under *an unknown but bounded noise*. Applications of the algorithm are considered for a random walk, an optimization of UAV's flight, and a load balancing problem.**

***Index Terms*—Arbitrary noise, randomized algorithm, SPSA, stochastic approximation, unknown but bounded disturbances.**

I. Introduction

Stochastic approximation was introduced by Robbins and Monro [1] and was further developed for optimization problems by Kiefer and Wolfowitz (KW) [2]. In [3] the stochastic approximation algorithm was extended to the multidimensional case. When $\theta \in \mathbb{R}^d$, the conventional KW-procedure, which is based on finite-difference approximations of the function gradient vector, uses $2d$ observations at each iteration to construct the sequence of estimates (two observations for approximations of each component of the gradient d-vector). Spall [4] suggested *a simultaneous perturbation stochastic approximation (SPSA)* algorithm with only two observations at each iteration which recursively generates estimates along random directions. He demonstrated that for a large d the probabilistic distribution of appropriately scaled estimation errors is approximately normal. The formula obtained for the asymptotic error variance and a similar characteristic of the KW procedure are used to compare overall performances of algorithms. It turned out that, all other things being equal, the SPSA algorithm has the same order of convergence rate as the KW-procedure, even though in the multidimensional case (even $d \to \infty$) appreciably fewer (by the factor of d) observations are used (see [5]).

The stochastic approximation method was originally introduced as a tool for statistical computations and was further developed within the separate field of control theory. Today this topic has a wide variety of applications in areas such as adaptive signal processing, adaptive resource allocation in communication networks, system identification and adaptive control.

Initially, stochastic approximation algorithms were proven in the case of stationary functionals minimization. In [6] for time-varying functionals, the Newton method and the gradient method are applied to problems of minimization but they are applicable only in the case of two times differentiable functionals and with known bounds of the Hessian matrix. Both methods require the availability of a direct measurement of a gradient at an arbitrary point. The stochastic setting is not discussed there. The books [7], [8] address the issue of applications of stochastic approximation with constant step-size to tracking and time-varying systems.

Distributed asynchronous stochastic approximation algorithms were studied in [9]. The stochastic approximation method with constant step-size has also been used in [10] for multi-agent systems under dynamic state changes (e.g., processing jobs and feeding new jobs in a computer network) in the presence of stochastic disturbances and noise. It allows one to achieve an approximate consensus which means a load balancing in the context of computer network processing.

In this technical note, the application of the SPSA algorithm is considered for non-constrained optimization in the context of the minimum tracking problem. In the case of once differentiable time-varying quality functionals and almost arbitrary external noise, the upper bound of a mean square estimation error is derived for estimates of SPSA type algorithms with constant step-size. It could have a sufficiently small level compared to a significant level of noise when the rate of parameter change is slow. The obtained result continues the line of research on SPSA for tracking [11]–[13]. The linear case was studied in [11]. In comparison with previous results [12], [13] for the nonlinear case we consider a new upper bound which corresponds to improved conditions and a generalized version of the algorithm. The consideration of a new algorithm was introduced and motivated for use in practice in [14] but without a theoretical study of the estimate's properties.

This paper is organized as follows: in Section II, we formulate a formal problem setting of a non-constrained time-varying mean-risk optimization and present the main assumptions. Section III introduces the excitation testing perturbation and estimation algorithm. In Section IV, we give the main result about the estimates with respect to the SPSA algorithm in the context of the minimum tracking problem. Section V shows some applications. And in the end, we make conclusions.

II. Mean-Risk Optimization and Assumptions

Many practical applications need to optimize one or another mean risk functional. Although the extremal values can sometimes be established analytically, engineering systems often deal with an unknown functional whose value or gradient can be calculated at the given points.

Let Ξ be a set, $\{f_\xi(\boldsymbol{\theta})\}_{\xi\in\Xi}$, be a family of differentiable functions: $f_\xi(\boldsymbol{\theta}) : \mathbb{R}^d \to \mathbb{R}$, and let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be a sequence of measurement points chosen by the experimenter (observation plan), where the values $y_1, y_2, \ldots$ of functions $f_\xi(\cdot)$ are accessible to observations at every time instant $t = 1, 2, \ldots$, with additive external noise v_t

$$y_t = f_{\xi_t}(\mathbf{x}_t) + v_t \tag{1}$$

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The authors are with the Faculty of Mathematics and Mechanics and the Research Laboratory for Analysis and Modeling of Social Processes, Saint Petersburg State University, 198504 St. Petersburg, Russia (e-mail: oleg_granichin@mail.ru; n.amelina@spbu.ru).

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where $\{\xi_t\}$ is a non-controllable sequence: $\xi_t \in \Xi$ (e.g., $\Xi = \mathbb{N}$ and $\xi_t = t$, or $\Xi \subset \mathbb{R}^p$ and $\{\xi_t\}$ is a sequence of some random elements).

Let $\mathcal{F}_{t-1}$ be the σ-algebra of all probabilistic events which happened up to time instant $t = 1, 2, \ldots$. Hereinafter $E_{\mathcal{F}_{t-1}}$ is a symbol of the conditional mathematical expectation with respect to the σ-algebra $\mathcal{F}_{t-1}$, E is a symbol of the mathematical expectation.

Non-Stationary Problem Formulation: The time-varying point of minimum $\boldsymbol{\theta}_t$ of the function

$$F_t(\boldsymbol{\theta}) = E_{\mathcal{F}_{t-1}} f_{\xi_t}(\boldsymbol{\theta}) \to \min_{\boldsymbol{\theta}} \tag{2}$$

needs to be estimated.

More precisely, using the observations $y_1, y_2, \ldots, y_t$ and inputs $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_t$, construct an estimate $\widehat{\theta}_t$ of an unknown vector $\boldsymbol{\theta}_t$ minimizing the time-varying *mean-risk functional* (2).

Minimization of the function $F_t(\boldsymbol{\theta})$ is usually studied with simpler observation models

$$y_t = F_t(\mathbf{x}_t) + v_t \quad \text{or} \quad y_t = f_{\xi_t}(\mathbf{x}_t).$$

The generalization used in the formulation (1) allows separation of observation disturbances with "good" (e.g., zero-mean and independent and identically distributed—i.i.d.) statistical properties $\{\xi_t\}$ and arbitrary additive external noise $\{v_t\}$. Of course, this separation is not needed when we can assume that $\{v_t\}$ is a random zero-mean and independent and identically distributed as well.

Let us formulate Assumptions about disturbances and functions $f_\xi(\mathbf{x})$, $F_t(\mathbf{x})$.

1) For $n = 1, 2, \ldots$, the successive differences $\bar{v}_n = v_{2n} - v_{2n-1}$ of observation noise are bounded: $|\bar{v}_n| \le c_v < \infty$, or $E\bar{v}_n^2 \le c_v^2$ if a sequence $\{v_t\}$ is random.
2) The drift is bounded: $\|\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}\| \le \delta_{\boldsymbol{\theta}} < \infty$, or $E\|\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}\|^2 \le \delta_{\boldsymbol{\theta}}^2$ and $E\|\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}\|\|\boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_{t-2}\| \le \delta_{\boldsymbol{\theta}}^2$ if a sequence $\{\xi_t\}$ is random.
3) The rate of drift is bounded in a such way that for any arbitrary point $\mathbf{x}$: $\|E_{\mathcal{F}_{2n-2}} \nabla\varphi_n(\mathbf{x})\| \le a_1\|\mathbf{x} - \boldsymbol{\theta}_{2n-2}\| + a_0$, $E_{\mathcal{F}_{2n-2}}\varphi_n(\mathbf{x})^2 \le a_2\|\mathbf{x} - \boldsymbol{\theta}_{2n-2}\|^2 + a_3$, where $\varphi_n(\mathbf{x}) = f_{\xi_{2n}}(\mathbf{x}) - f_{\xi_{2n-1}}(\mathbf{x})$.
4) Functions $F_t(\cdot)$ have unique minimum points $\boldsymbol{\theta}_t$ and
$$\forall \mathbf{x} \in \mathbb{R}^d \langle \mathbf{x} - \boldsymbol{\theta}_t, E_{\mathcal{F}_{t-1}} \nabla f_{\xi_t}(\mathbf{x})\rangle \ge \mu\|\mathbf{x} - \boldsymbol{\theta}_t\|^2$$
with a constant $\mu > 0$. Here and further $\langle\cdot,\cdot\rangle$ is a scalar product of two vectors.
5) The gradient ∇f_{ξ_t} is uniformly bounded in the mean-squared sense at the minimum points $\boldsymbol{\theta}_t$: $E\|\nabla f_{\xi_t}(\boldsymbol{\theta}_t)\|^2 \le g^2$, $E\langle \nabla f_{\xi_t}(\boldsymbol{\theta}_t), \nabla f_{\xi_{t-1}}(\boldsymbol{\theta}_{t-1})\rangle \le g^2$ ($g = 0$ if ξ_t is not a random parameter, i.e. $f_{\xi_t}(\mathbf{x}) = F_t(\mathbf{x})$).
6) $\forall \xi \in \Xi$ the gradient $\nabla f_\xi(\mathbf{x})$ satisfies the Lipschitz condition: $\forall \mathbf{x}', \mathbf{x}'' \in \mathbb{R}^d$
$$\|\nabla f_\xi(\mathbf{x}') - \nabla f_\xi(\mathbf{x}'')\| \le M\|\mathbf{x}' - \mathbf{x}''\|$$
with a constant $M \ge \mu$.

Examples:

1) Assumption 2 about the drift holds for the drift with model
$$\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1} + \zeta_{t-1},\ \boldsymbol{\theta}_t \in \mathbb{R}^d$$
when $\{\zeta_t\}$ is a sequence of random i.i.d. vectors which have symmetrical distribution on the ball: $\|\zeta_t\| \le \delta_{\boldsymbol{\theta}}$, $E\zeta_t = 0$, $E\|\zeta_t\|^2 = \sigma_\zeta^2$, $E\|\zeta_t\|^4 = M_\zeta^4$. If at time instant t we can measure the squared distance $\|\mathbf{x} - \boldsymbol{\theta}_t\|^2$ between a chosen point $\mathbf{x}$ and $\boldsymbol{\theta}_t$ with additive bounded non-random noise v_t: $\|v_t\| < 1$, then we have $\Xi = \mathbb{N}$ and $F_t(\mathbf{x}) = \|\mathbf{x} - \boldsymbol{\theta}_t\|^2$. Assumptions 3–6 hold with constants $a_1 = 0$, $a_0 = \sigma_\zeta^2$, $a_2 = 8\sigma_\zeta^2$, $a_3 = 8\sigma_\zeta^2\delta_{\boldsymbol{\theta}}^2 + M_\zeta^4$, $g = 0$, $\mu = M = 2$.
2) The next example is inspired by the application in a physical experiment [15]. If at time instant t we can measure with additive bounded non-random noise $v_t(\|v_t\| < 1)$ the squared distance $\|\mathbf{x} - \boldsymbol{\theta}_\star\|^2$ between a target $\boldsymbol{\theta}_\star$ and chosen point $\mathbf{x}$ which is corrupted by a random i.i.d. perturbation $\xi_t \in \Xi \subset \mathbb{R}^d$ distributed symmetrically around zero: $E\xi_t = 0$, $E\|\xi_t\|^2 = \sigma_\xi^2$, $E\|\xi_t\|^4 = M_\xi^4$, then we have $f_{\xi_t}(\mathbf{x}) = \|\mathbf{x} + \xi_t - \boldsymbol{\theta}_\star\|^2$. Assumptions 2–6 hold with constants $\delta_{\boldsymbol{\theta}} = 0$, $a_1 = a_0 = 0$, $a_2 = 8\sigma_\xi^2$, $a_3 = 2M_\xi^4 - 2\sigma_\xi^4$, $g = 0$, $\mu = M = 2$.

III. Excitation Testing Perturbation and Estimation Algorithms

Let $\boldsymbol{\Delta}_n$, $n = 1, 2, \ldots$ be an observed sequence of independent random vectors in $\mathbb{R}^d$, called the *simultaneous test perturbation*, with symmetrical distribution functions $\mathrm{P}_n(\cdot)$, and let $\mathbf{K}_n(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$, $n = 1, 2, \ldots$, be a set of vector functions (kernels).

Let us take a fixed nonrandom initial vector $\widehat{\boldsymbol{\theta}}_0 \in \mathbb{R}^d$, a positive step-size α, and choose sequences of such non-negativenumbers $\{\beta_n^+\}$ and $\{\beta_n^-\}$ that $\beta_n = \beta_n^+ + \beta_n^- > 0$.We consider the algorithm with two observations for constructing sequences ofpoints of observations $\{\mathbf{x}_t\}$ and estimates $\{\widehat{\boldsymbol{\theta}}_t\}$:

$$\begin{cases} \mathbf{x}_{2n} = \widehat{\boldsymbol{\theta}}_{2n-2} + \beta_n^+\boldsymbol{\Delta}_n,\ \mathbf{x}_{2n-1} = \widehat{\boldsymbol{\theta}}_{2n-2} - \beta_n^-\boldsymbol{\Delta}_n, \\ \widehat{\boldsymbol{\theta}}_{2n-1} = \widehat{\boldsymbol{\theta}}_{2n-2}, \\ \widehat{\boldsymbol{\theta}}_{2n} = \widehat{\boldsymbol{\theta}}_{2n-1} - \alpha\mathbf{K}_n(\boldsymbol{\Delta}_n)\frac{y_{2n} - y_{2n-1}}{\beta_n}. \end{cases} \tag{3}$$

Assume that the following conditions hold:

7) For any $n = 1, 2, \ldots$,
   a) $\boldsymbol{\Delta}_n$ and ξ_{2n-1}, ξ_{2n} (if they are random) do not depend on the σ-algebra $\mathcal{F}_{2n-2}$.
   b) If ξ_{2n-1}, ξ_{2n} are random, then random vectors $\boldsymbol{\Delta}_n$ and elements ξ_{2n-1}, ξ_{2n} are independent.
   c) If $\bar{v}_n$ is random, then $\bar{v}_n$ and vector $\boldsymbol{\Delta}_n$ are independent.
8) For $n = 1, 2, \ldots$, vectors Δ_n and $\mathbf{K}_n(\Delta_n)$ are bounded: $\|\Delta_n\| \le c_\Delta < \infty$, $\|\mathbf{K}_n(\Delta_n)\| \le \kappa < \infty$, and vector functions $\mathbf{K}_n(\cdot)$ along with simultaneous perturbation symmetrical distribution functions $\mathrm{P}_n(\cdot)$ satisfy the conditions
$$\int \mathbf{K}_n(\mathbf{x})\mathrm{P}_n(d\mathbf{x}) = 0, \qquad \int \langle\mathbf{K}_n(\mathbf{x}), \mathbf{x}\rangle\,\mathrm{P}_n(d\mathbf{x}) = \mathbf{I} \tag{4}$$
where $\mathbf{I}$ is the identity matrix.

The same assumptions were made in previous work [11]–[13], [16] when observations under almost arbitrary external noise and SPSA methods are considered.

For example, we can choose a realization of a sequence of independent Bernoulli random vectors from $\mathbb{R}^d$ with each component independently taking values ± 1 with probabilities $1/2$ as a sequence $\{\boldsymbol{\Delta}_n\}$ and $\mathbf{K}_n(\mathbf{x}) \equiv \mathbf{x}$ as kernel functions. The case $\beta_n^+ = \beta_n^-$ and decreasing to zero sequence α_n instead constant step-size α corresponds with the SPSA algorithm in [4]. The similar algorithm with randomly varying truncations and randomized difference was studied in [17] where the case $\beta_n^- = 0$ was additionally considered.

IV. Upper Bound of Residuals of Estimation

To analyze the quality of estimates we apply the following definition for the problem of minimum tracking for mean-risk functional (2):

Definition: A sequence of estimates $\{\widehat{\boldsymbol{\theta}}_{2n}\}$ has *an asymptotically efficient upper bound* $\bar{L} > 0$ of residuals of estimation if $\forall\varepsilon > 0\ \exists N$

such that $\forall n > N$

$$\sqrt{E\|\widehat{\boldsymbol{\theta}}_{2n} - \boldsymbol{\theta}_{2n}\|^2} \leq \bar{L} + \varepsilon.$$

Denote $\beta_{\max} = \max_n \beta_n$, $\bar{\beta} = \max_n(1/\beta_n)$, $\bar{\beta}^+ = \max_n(\beta_n^+/\beta_n)$, $\bar{\beta}^- = \max_n(\beta_n^-/\beta_n)$, $k = 2\mu - 6\alpha\kappa^2(a_2\bar{\beta}^2 + c_\Delta^2 M^2)$, $h = (2\delta_{\boldsymbol{\theta}}/\alpha k) - \delta_{\boldsymbol{\theta}} + ((\bar{\beta}^- a_1 + 6\alpha c_\Delta^2 M^2(c_\Delta + \bar{\beta}^-\sigma_\theta))/k)$, $c_1 = M((\bar{\beta}^+)^2 + 3(\bar{\beta}^-)^2)$, $c_2 = g + M^2(c_\Delta^2 + (\bar{\beta}^+)^2\sigma_\theta^2 + 2\beta^+\beta_{\max}\sigma_\theta(c_\Delta + \sigma_\theta))$, $\bar{l} = 2(\bar{\beta}^- a_0 + 3\kappa^2\alpha a_3\bar{\beta}^2 + \kappa c_\Delta^2(\beta_{\max}c_1 + \kappa\alpha(3c_2 + c_v^2)))$. The following theorem shows the asymptotically efficient upper bound of estimation residuals by algorithm (3).

Theorem 1: **If** Assumptions 1–8 hold, $\beta_{\max} + \bar{\beta} < \infty$, and the constant α is sufficiently small

$$k\alpha < 1 \text{ and } \alpha < \frac{\mu}{3\kappa^2\left(a_2\bar{\beta}^2 + c_\Delta^2 M^2\right)} \tag{5}$$

then the sequence of estimates provided by the algorithm (3) has an asymptotically efficient upper bound which equals to

$$\bar{L} = h + \sqrt{h^2 + \bar{l}/k}. \tag{6}$$

See the proof of Theorem 1 in Appendix.

Remarks: The observation noise v_t in Theorem 1 can be said to be almost arbitrary since it may either be nonrandom but bounded or it may also be a realization of some stochastic process with arbitrary internal dependencies. In particular, to prove the results of Theorem 1, there is no need to assume that v_t and $\mathcal{F}_{t-1}$ are not dependent.

The proof of Theorem 1 allows for consideration of the random sequences $\{\beta_n^+\}$ and $\{\beta_n^-\}$ whose values at time instant n are measurable under the corresponding σ-algebra $\mathcal{F}_{2n-2}$. This fact is sometimes useful from a practical point of view.

The result of the Theorem 1 shows that for the case without drift ($\delta_{\boldsymbol{\theta}} = a_0 = \ldots = a_3 = 0$) the asymptotic upper bound is $\bar{L} = (c_\Delta/\sqrt{\mu})(3\sqrt{\alpha}((c_v + g)\kappa + M) + \sqrt{\beta_{\max}}\sqrt{\kappa c_1}) + o(\sqrt{\alpha} + \sqrt{\beta_{\max}})$. This bound can be made infinitely small under any noise level c_v, simply by choosing α and $\beta_n^\pm$ to be sufficiently small. At the same time in the case of drift, the bigger drift norm $\delta_{\boldsymbol{\theta}}$ can be compensated by choosing a bigger step-size α and $\beta_n^\pm$. This leads to a tradeoff between making α smaller because of noisy observations and making α bigger due to the drift of optimal points.

V. Applications

Most stochastic approximation algorithm applications are concerned with adaptive systems based on the fact that SA algorithms have properties useful for uncertain environments. These important properties allow these algorithms to track the typical behavior of such systems. Furthermore, these algorithms are memory and computationally efficient, which makes them applicable to real time dynamic environments. Due to these properties, the algorithms are applicable in such new fields as soft computing, where they are used for "parameter tuning" (see, e.g. [7]). Notable among these are algorithms for neural network training and for reinforcement learning. They are used in popular learning paradigms for autonomous software agents, with applications in e-commerce, robotics and other fields. They are also widely applied in economic theory, providing a good model for collective phenomena when the algorithms are used to model the behavior of individual bounded rational agents.

A. Random Walk

A rather simple application of the SPSA algorithm to minimize the aforementioned nonstationary functional was considered in [13] in the context of coordinate estimation of a point moving in the multidimensional space. There the single possible measurement is the squared distance to it which is measured with noise (see Example 1 in Section II). By virtue of Theorem 1, the estimates made by algorithm (3) will stabilize along the drift trajectory, provided that the norm of the extremum drift is bounded. Simulation results in the hundred-dimensional space were presented in [13] for the case $c_v = 1$ and $\sigma_\theta = 0.01$. With such a dimension, the standard algorithms based on approximation of the gradient vector use, at each step, 101 or 200 observations; that is, the drift during one iteration is sufficiently appreciable. In the typical cases, the estimates of the SPSA algorithm (3), with only two measurements per iteration, demonstrate the sufficiently good behavior after 2000 iterations.

B. UAV Soaring

Extending the endurance of the flight of UAVs (Unmanned Air Vehicles) is currently an area of major research interest because these vehicles are very popular for aircraft missions that would be too dangerous or too boring for human pilots. The importance of this problem was highlighted in our previous works [14], [18]. This subsection presents a SPSA-based algorithm for quick and precise detection of the center of a thermal updraft where the vertical velocity of the air stream is the highest. It allows maximization of the flight duration of a single UAV and of UAV groups using the thermal model developed by Allen at NASA Dryden [19].

The main assumption of the experiment was that SPSA-like methods are effective approaches for updraft center detection. The method takes into account the unstable behavior of updraft dynamics and the drift of its center over time. This method is permitted for an effective treatment of the updraft center drift because of the tracking properties of SPSA shown above. It also helps in compensating the effect of horizontal wind considered as systematic (arbitrary) noise. The physical characteristics of considered UAV were described in [14]. The objective of our UAV is to conserve battery energy and soar as long as possible over the test area.

The following step-by-step summary shows how SPSA iteratively produces a sequence of updraft center estimates.

Algorithm:

1) *Initialization and coefficient selection*. Set a counter index $n = 0$. Select an initial guess $\widehat{\boldsymbol{\theta}}_0 \in \mathbb{R}^2$ and a fairly small non-negative coefficient $\alpha > 0$. The initial guess in our implementation of the algorithm is the point where a positive updraft was first measured.
2) *Iteration* $n \to n+1$. Set $n := n+1$.
3) *Generation of the simultaneous perturbation vector*. Use Monte Carlo to generate a 2-dimensional random perturbation vector $\boldsymbol{\Delta}_n$ whose components are independently generated from a zero mean probability distribution satisfying the preceding conditions. A common choice for each component of $\boldsymbol{\Delta}_n$ is to use a Bernoulli ± 1 distribution with probability of $1/2$ for each ± 1 outcome.
4) *Proceeding to the new waypoints*. Proceed to next two points $\mathbf{x}_{2n-1}$ and $\mathbf{x}_{2n}$. They are intersections of the UAV trajectory projection on the 2D plain and the line which goes through the point of the previous estimate $\widehat{\boldsymbol{\theta}}_{2n-2}$ in the direction of the vector $\boldsymbol{\Delta}_n$.
5) *Velocity function evaluations*. Obtain two measurements at the points $\mathbf{x}_{2n-1}$ and $\mathbf{x}_{2n}$ of the velocity function $y_{2n-1} = F(\mathbf{x}_{2n-1})$ and $y_{2n} = F(\mathbf{x}_{2n})$.
6) *Computing the values* $\beta_n^\pm$. Measure the distances from the point of the previous estimate $\widehat{\boldsymbol{\theta}}_{2n-2}$, and points $\mathbf{x}_{2n-1}$ and $\mathbf{x}_{2n}$ and

compute two values $\beta_n^{\pm}$ such that $\mathbf{x}_{2n-1} = \widehat{\boldsymbol{\theta}}_{2n-2} - \beta_n^- \boldsymbol{\Delta}_n$, $\mathbf{x}_{2n} = \widehat{\boldsymbol{\theta}}_{2n-2} + \beta_n^+ \boldsymbol{\Delta}_n$.

7) *Quasigradient calculation*. Calculate the quasigradient:

$$\widehat{g} = \boldsymbol{\Delta}_n \frac{y_{2n} - y_{2n-1}}{\beta_n^+ + \beta_n^-}.$$

8) *Updating center estimation*. Use the standard stochastic approximation form $\widehat{\boldsymbol{\theta}}_{2n} = \widehat{\boldsymbol{\theta}}_{2n-2} + \alpha \widehat{g}$ to update the current center estimation (sign plus is used because we would like to find the point of function F maximum).

9) *Iteration or termination*. Return to Step 2 or terminate the algorithm if there is little change in the estimations obtained on several successive iterations or if the maximum allowed number of iterations has been reached.

10) *Climbing in the updraft*. Circle around the estimated updraft center in order to climb.

This method provides a good approximation of the updraft center using a small number of measurements and no *a priori* knowledge on updraft location (see simulation results in [18] for the case $\beta_n^+ = \beta_n^-$). Step 6 shows the advantage of the new generalized scheme of algorithm (3) when we can choose different algorithm parameters $\beta_n^{\pm}$ based on the current "free" trajectory. In [18] authors consider the algorithm which calculates the way point of a second measurement and UAV needs to go to it.

C. Load Balancing

Let the system consist of m computing nodes (processors) and it must process the jobs package of a known size z. Suppose that the entire package z can be arbitrarily divided into m tasks x^j, $j = 1, \ldots, m$, (hereinafter the upper index j is not a power, but it is instead the number of the node, $\mathbf{x} = col(x^1, x^2, \ldots, x^m)$)

$$\|\mathbf{x}\|_1 = \sum_{j=1}^{m} x^j = z$$

for all nodes, and the computation time of the node j is defined by $t^j(x^j) = x^j/\theta^j$, where $\theta^j \in \mathbb{R}$ is a value which equals to the productivity (performance) of the node j.

The problem *is to minimize* the total time of processing the jobs package z:

$$T(\mathbf{x}) = \max_{j \in \{1,\ldots,m\}} t^j(x^j) \to \min_{\mathbf{x}}. \tag{7}$$

An ideal scheduling algorithm is one which keeps all the nodes busy executing essential tasks, and minimizes the internode communication required to determine the schedule and pass data between tasks. The scheduling problem is particularly challenging when the tasks are generated dynamically and unpredictably in the course of executing the algorithm. This is the case with many recursive divide-and-conquer algorithms, including backtrack search, game tree search and branch-and-bound computation.

When the productivities (performance) of nodes are known, then the best control strategy is a proportional distribution of tasks such that $x^1/\theta^1 = x^2/\theta^2 = \cdots = x^m/\theta^m$. The proof of this result is not hard and can be found (see, e.g., in [10]) This control strategy is called *load balancing* and defined as follows:

$$\mathbf{x} = \mathcal{U}(\boldsymbol{\theta}, p, z):\ x^j = \frac{\theta^j}{p} z,\ j = 1, 2, \ldots, m \tag{8}$$

where $p = \|\theta\|_1$. In the case when $p \neq \|\theta\|_1$ we assume that a discrepancy is added or subtracted for a randomly chosen node.

Let the system works iteratively by processing at the iteration i the jobs package of a known size $z_i > 0$. In practice, the productivity (performance) of nodes may be unknown $\theta \in \mathbb{R}^m$. Moreover, they may be distorted because of side jobs, that is, $\boldsymbol{\theta}_i = \boldsymbol{\theta} + \xi_i$, or change with time: $\boldsymbol{\theta}_i = \boldsymbol{\theta}_{i-1} + \xi_i$, where $\xi_i \in \mathbb{R}^m$ are vectors of independent random variables.

The usual way is to use estimates of the productivity (performance) $\widehat{\boldsymbol{\theta}}_i$ at each iteration i which are defined in a such a way that $\|\widehat{\boldsymbol{\theta}}_i - \boldsymbol{\theta}_i\|^2 \to \min$ in some reasonable sense, and to compute $\mathbf{x}_i$ at the iteration i as $\mathbf{x}_i = \mathcal{U}(\widehat{\boldsymbol{\theta}}_i, p, z_i)$.

One of the reasonable quality functions is

$$f_i(\widehat{\boldsymbol{\theta}}_i) = \frac{1}{2z_i^2} \sum_{j,k=1}^{m} \left(\bar{t}_i^j - \bar{t}_i^k\right)^2 \to \min_{\widehat{\boldsymbol{\theta}}_i} \tag{9}$$

where $\bar{t}_i^j = t^j(\mathcal{U}^j(\widehat{\boldsymbol{\theta}}_i), \|\widehat{\boldsymbol{\theta}}_i\|_1, z_i)/z_i$, $j = 1, 2, \ldots, m$. Function $f_i(\widehat{\boldsymbol{\theta}}_i)$ has a minimum point $\widehat{\boldsymbol{\theta}}_i = \theta_i$ which corresponds to the optimal control strategy minimized (7).

To track changes $\boldsymbol{\theta}_i$, it is advisable to use the SPSA algorithm.

Algorithm:

1) *Initialization and coefficient selection*. Set a counter index $n = 0$. Choose initial guess $\widehat{\boldsymbol{\theta}}_0 \in \mathbb{R}^m$ and fairly small step-sizes $\alpha > 0$ and $\beta > 0$.

1) *Iteration* $n \to n + 1$. Set $n := n + 1$.

a. Generate the random vector $\boldsymbol{\Delta}_n$ according to the Bernoulli distribution of i.i.d. components that are equal to ± 1 with probability $1/2$.

b-1. Obtain the next jobs package z_{2n-1}.

c-1. Compute the next inputs by the rule: $\mathbf{x}_{2n-1} = \mathcal{U}(\widehat{\boldsymbol{\theta}}_{2n-2} - \beta \boldsymbol{\Delta}_n, \|\widehat{\boldsymbol{\theta}}_{2n-2}\|, z_{2n-1})$.

d-1. Start the cluster with input $\mathbf{x}_{2n-1}$ and wait until all tasks are finished.

b-2. Obtain the next jobs package z_{2n}.

c-2. Compute the next inputs by the rule: $\mathbf{x}_{2n} = \mathcal{U}(\widehat{\boldsymbol{\theta}}_{2n-2} + \beta \boldsymbol{\Delta}_n, \|\widehat{\boldsymbol{\theta}}_{2n-2}\|, z_{2n})$

d-2. Start the cluster with input $\mathbf{x}_{2n}$ and wait until all tasks are finished.

e. Calculate the quasigradient: $\hat{g} = (1/2\beta) \boldsymbol{\Delta}_n (f_{2n}(\mathbf{x}_{2n}) - f_{2n-1}(\mathbf{x}_{2n-1}))$.

f. Get the new estimate:

$$\widehat{\boldsymbol{\theta}}_{2n} = \widehat{\boldsymbol{\theta}}_{2n-2} - \alpha \hat{g}. \tag{10}$$

3) Repeat Step 2.

Simulation: Now let us present the simulation results of the algorithm described above. We consider $m = 100$ computational nodes. At the initial time we chose their productivities $\theta^1, \theta^2, \ldots, \theta^{100}$ randomly by the uniform distribution over the interval (0.5, 1.5). At iteration i the productivity θ_i^j of a randomly chosen node j changes slowly by a random value from (-0.05, 0.05) and the system (computer network) receives the job package z_i which was simulated by a Poisson distribution with mean value 100.

Fig. 1 shows the typical behavior of normalized run-times $T(\mathbf{x}_i)/z_i$ of the processing of packages z_i when we use $x_0^j = 1$, $j = 1, 2, \ldots, m$, $\alpha = 0.001$ and $\beta = 0.01$ in the algorithm (10). As one can see, the performance of computer networks converges over time to an optimal value.

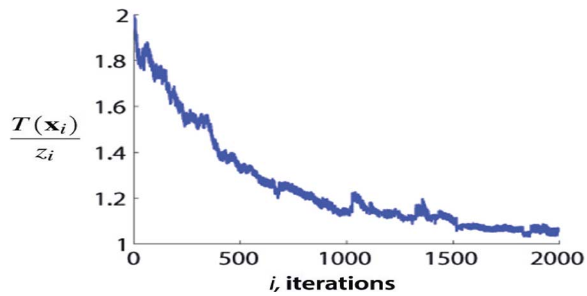

Fig. 1. Load balancing adaptation for 100 computing nodes.

VI. Conclusion

In the design of optimization or estimation algorithms, some useful statistical characteristics are usually attributed to noise, errors in measurements and model properties. They are used in demonstrating the validity of the algorithm. For example, noise is often assumed to be random and centered. Algorithms based on the ordinary least-squares method are typically used in engineering practice for simple averaging of observation data. If noise is assumed to be centered without any valid justification, such algorithms are unsatisfactory in practice and may even be harmful. Such is the state of affairs under "opponent" counteraction. In particular, if noise is defined by a deterministic unknown function (opponent suppresses signals) or measurement noise is a dependent sequence, averaging of observations does not yield any useful result.

To solve the challenging problems of nonstationary optimization under observations with *an unknown but bounded noise*, it was suggested to use a modified simultaneous perturbation stochastic approximation algorithm with a constant step-size. This has a simple form and provides a finite bound of residual between estimates and time-varying unknown parameters.

Appendix

The following Lemma 1 in [12] is instrumental to the proof of Theorem 1.

Lemma 1: **If** $e_n > 0, k, \alpha > 0, k\alpha > 0 < 1, h, \bar{l} \geq 0$,

$$e_n^2 \leq (1 - k\alpha)e_{n-1}^2 + 2hk\alpha e_{n-1} + \alpha\bar{l},\ n = 1, 2, \ldots$$

then $\forall \varepsilon > 0\ \exists N$ such that $\forall n > N\ e_n \leq h + \sqrt{h^2 + \bar{l}/k} + \varepsilon$.

Proof of Lemma 1: See [12].

Proof of Theorem 1: Denote $\nu_n = \|\widehat{\boldsymbol{\theta}}_{2n} - \boldsymbol{\theta}_{2n}\|$, $s_n = (\alpha/\beta_n)(y_{2n} - y_{2n-1})\mathbf{K}_n(\boldsymbol{\Delta}_\mathbf{n})$, $\bar{f}_n = f_{\xi_{2n}}(\mathbf{x}_{2n}) - f_{\xi_{2n-1}}(\mathbf{x}_{2n-1})$, $\tilde{\mathcal{F}}_{n-1} = \sigma\{\mathcal{F}_{n-1}, \mathbf{v}_{2n-1}, \mathbf{v}_{2n}, \xi_{2n-1}, \xi_{2n}\}$, $\mathbf{d}_t = \widehat{\boldsymbol{\theta}}_{2\lceil (t-1)/2 \rceil} - \boldsymbol{\theta}_t$, where $\lceil \cdot \rceil$ is a celling function. According to the observation model (1) and algorithm (3), we obtain

$$\nu_n^2 = \|\mathbf{d}_{2n}\|^2 + s_n^2 - 2\langle \mathbf{d}_{2n}, s_n \rangle. \tag{11}$$

By virtue of Assumptions 7,8 we have $E_{\tilde{\mathcal{F}}_{n-1}} \bar{v}_n \mathbf{K}_n(\boldsymbol{\Delta}_n) = E_{\tilde{\mathcal{F}}_{n-1}} \bar{v}_n E_{\tilde{\mathcal{F}}_{n-1}} \mathbf{K}_n(\boldsymbol{\Delta}_n) = E_{\tilde{\mathcal{F}}_{n-1}} \bar{v}_n \cdot 0 = 0$. Hence, taking the conditional expectation over σ-algebra $\tilde{\mathcal{F}}_{n-1}$ of both sides of the inequality (11) and using Assumption 8, we can bound $E_{\tilde{\mathcal{F}}_{n-1}} \nu_n^2$ as follows:

$$E_{\tilde{\mathcal{F}}_{n-1}} \nu_n^2 \leq \|\mathbf{d}_{2n}\|^2 - 2\left\langle \mathbf{d}_{2n}, \frac{\alpha}{\beta_n} E_{\tilde{\mathcal{F}}_{n-1}} \bar{f}_n \mathbf{K}_n(\boldsymbol{\Delta}_n) \right\rangle + 2\kappa^2 \frac{\alpha^2}{\beta_n^2} \left(\bar{v}_n^2 + E_{\tilde{\mathcal{F}}_{n-1}} \bar{f}_n^2 \right) \tag{12}$$

since $s_n = (\alpha/\beta_n)(\bar{f}_n + \bar{v}_n)\mathbf{K}_n(\boldsymbol{\Delta}_n)$.

By virtue of the representation of $f_{\xi_t}(\mathbf{x}_t)$ for $t^{\pm} = 2n - (1/2) \pm (1/2)$ as a Taylor series we have

$$f_{\xi_{t^\pm}}(\mathbf{x}_{t^\pm}) = f_{\xi_{t^\pm}}(\widehat{\boldsymbol{\theta}}_{2n-2}) \pm \left\langle \nabla^{\pm}_{\xi_{t^\pm}} \left(\rho^{\pm}_{\xi_{t^\pm}} \right), \beta_n^{\pm} \boldsymbol{\Delta}_n \right\rangle$$

where $\rho^{\pm}_{\xi_{t^\pm}} \in (0,1)$, $\nabla^{\pm}_{\xi_{t^\pm}}(\rho^{\pm}_{\xi_{t^\pm}}) = \nabla f_{\xi_{t^\pm}}(\widehat{\boldsymbol{\theta}}_{2n-2} \pm \rho^{\pm}_{\xi_{t^\pm}} \beta_n^{\pm} \boldsymbol{\Delta}_n)$.

Using the definition of $\varphi_n(\mathbf{x}_{2n-1})$ and its representation as a Taylor series: $\varphi_n(\mathbf{x}_{2n-1}) = \varphi_n(\widehat{\boldsymbol{\theta}}_{2n-2}) + \nabla_{\varphi_n}(\rho_\phi)$, where $\nabla_{\varphi_n}(\rho_\phi) = \nabla\varphi_n(\widehat{\boldsymbol{\theta}}_{2n-2} + \rho_\phi \beta_n^- \boldsymbol{\Delta}_n)$, $\rho_\phi \in (0,1)$, we derive for the difference $\bar{f}_n$:

$$\begin{aligned}\bar{f}_n &= f_{\xi_{2n}}(\mathbf{x}_{2n}) - f_{\xi_{2n}}(\mathbf{x}_{2n-1}) + \varphi_n(\mathbf{x}_{2n-1}) = \varphi_n(\widehat{\boldsymbol{\theta}}_{2n-2}) \\ &\quad + \left\langle \nabla_{\varphi_n}(\rho_\phi) - \nabla_{\varphi_n}(0), \beta_n^- \boldsymbol{\Delta}_n \right\rangle + \left\langle \nabla_{\varphi_n}(0), \beta_n^- \boldsymbol{\Delta}_n \right\rangle \\ &\quad + \sum_{t^\pm} \beta_n^{\pm} \left(\left\langle \nabla^{\pm}_{\xi_{2n}}\left(\rho^{\pm}_{\xi_{2n}}\right) - \nabla^{\pm}_{\xi_{2n}}(0), \boldsymbol{\Delta}_n \right\rangle + \left\langle \nabla^{\pm}_{\xi_{2n}}, \boldsymbol{\Delta}_n \right\rangle \right).\end{aligned}$$

Since $E_{\tilde{\mathcal{F}}_{n-1}} \varphi_n(\widehat{\boldsymbol{\theta}}_{2n-2})\mathbf{K}_n(\boldsymbol{\Delta}_n) = 0$ by virtue of Assumptions 7 and 8, for the second term in (12), applying Assumptions 6 and 8, we have

$$\begin{aligned}-2\left\langle \mathbf{d}_{2n}, \frac{\alpha}{\beta_n} E_{\tilde{\mathcal{F}}_{n-1}} \bar{f}_n \mathbf{K}_n(\boldsymbol{\Delta}_n) \right\rangle &\leq 2\frac{\alpha}{\beta_n} \beta_n^-(a_1 \nu_{n-1} + a_0) \\ + 2\frac{\alpha}{\beta_n} \left(\left(\beta_n^+\right)^2 + 3\left(\beta_n^-\right)^2 \right) M c_\Delta^2 \kappa &- 2\sum_{t^\pm} \left\langle \mathbf{d}_{2n}, \nabla f_{\xi_{2n}}(\widehat{\boldsymbol{\theta}}_{2n-2}) \beta_n^{\pm} \right\rangle.\end{aligned}$$

Taking the conditional expectation over σ-algebra $\mathcal{F}_{n-1}$ and using Assumption 3, we get

$$\begin{aligned}-E_{\mathcal{F}_{n-1}} 2\left\langle \mathbf{d}_{2n}, \frac{\alpha}{\beta_n} \bar{f}_n \mathbf{K}_n(\boldsymbol{\Delta}_n) \right\rangle &\leq -2\alpha E_{\mathcal{F}_{n-1}} \mu \|\mathbf{d}_{2n-1}\|^2 \\ &\quad + 2\alpha\left(\bar{\beta}^-(a_1 \nu_{n-1} + a_0) + \beta_n c_\Delta^2 \kappa c_1 \right).\end{aligned} \tag{13}$$

Consider the squared difference $\bar{f}_n^2$. Using representations $\bar{f}_n = f_n^+ + f_n^- + \varphi_n(\widehat{\boldsymbol{\theta}}_{2n-2})$, $f_n^{\pm} = \beta_n^{\pm}(\langle \nabla_n^{\pm}(\rho_n^{\pm}) - \nabla f_{\xi_{t^\pm}}(\boldsymbol{\theta}_{t^\pm}), \boldsymbol{\Delta}_n \rangle + \langle \nabla f_{\xi_{t^\pm}}(\boldsymbol{\theta}_{t^\pm}), \boldsymbol{\Delta}_n \rangle)$, the symmetrical property of Δ_n distribution (Assumption 8) and Assumptions 6 make it possible to derive for the last term in (12)

$$\begin{aligned}E_{\tilde{\mathcal{F}}_{n-1}} \bar{f}_n^2 &\leq 3\varphi_n(\widehat{\boldsymbol{\theta}}_{2n-2})^2 + 3c_\Delta^2 \left(\sum_{t^\pm} \beta_n^{\pm} \nabla f_{\xi_{t^\pm}}(\boldsymbol{\theta}_{t^\pm}) \right)^2 \\ &\quad + 3c_\Delta^2 M^2 \left(\sum_{t^\pm} \beta_n^{\pm} \left(\|\mathbf{d}_{t^\pm}\| + \beta_n^{\pm} c_\Delta \right) \right)^2.\end{aligned}$$

Taking the conditional expectation over σ-algebra $\mathcal{F}_{n-1}$ and using Assumptions 2, 3, 5, 6, we obtain

$$\begin{aligned}&E_{\mathcal{F}_{n-1}} \bar{f}_n^2 \leq 3\left(a_2 \nu_{n-1}^2 + a_3 + c_\Delta^2 \beta_n^2 g \right) \\ &+ 3c_\Delta^2 M^2 \Big(\beta_n^2 c_\Delta^2 + \left(\beta_n^+ \sigma_\theta\right)^2 + 2\beta_n^2 \beta_n^+ \sigma_\theta (c_\Delta + \sigma_\theta) \\ &\qquad + \beta_n^2 E_{\mathcal{F}_{n-1}} \|\mathbf{d}_{2n-1}\|^2 + 2\left(\beta_n^2 c_\Delta + \beta_n \beta_n^+ \sigma_\theta \right) \nu_{n-1} \Big).\end{aligned} \tag{14}$$

According to the first part of Assumption 2, we get

$$E_{\mathcal{F}_{n-1}} \|\mathbf{d}_{2n}\|^2 \leq E_{\mathcal{F}_{n-1}} \|\mathbf{d}_{2n-1}\|^2 + 2\delta_{\boldsymbol{\theta}} \nu_{n-1} + 3\delta_{\boldsymbol{\theta}}^2. \tag{15}$$

Summing up the findings bounds (13)–(15) and taking the conditional expectation over σ-algebra $\mathcal{F}_{n-1}$, we derive the following from the (12) by virtue of Assumption 1

$$
\begin{aligned}
E_{\mathcal{F}_{n-1}}\nu_n^2 &\le (1-k\alpha)E_{\mathcal{F}_{n-1}}\|\mathbf{d}_{2n-1}\|^2 \\
&+ 3\delta_{\boldsymbol{\theta}}^2 + 2\alpha\left(\frac{\delta_{\boldsymbol{\theta}}}{\alpha} + \bar{\beta}^- a_1 + 6\alpha c_\Delta^2 M^2(c_\Delta + \bar{\beta}^-\sigma_\theta)\right)\nu_{n-1} \\
&+ \alpha\bar{l} \le (1-k\alpha)\nu_{n-1}^2 + 2hk\alpha\nu_{n-1} + \alpha\bar{l}. \qquad (16)
\end{aligned}
$$

Taking the unconditional expectation of both sides of (16), we see by virtue (5) that all conditions of Lemma 1 hold for $e_n = \sqrt{E\nu_n^2}$. This completes the proof of Theorem 1.

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