2012 13th ACIS International Conference on Software Engineering, Artificial Intelligence, Networking and Parallel/Distributed Computing

Multi-Agent Stochastic Systems with Switched Topology and Noise

Konstantin Amelin, Natalia Amelina, Oleg Granichin, and Olga Granichina

Abstract—**In this paper the multi-agent stochastic systems are considered under randomly switched agents communications topology and noisy information about the current states of neighbouring agents. We study the consensus problem for such systems when controls actions are formed by the local voting protocol (stochastic approximation type algorithm). This protocol is applied to the load balancing distributed computer system and to the system of Unnamed Aerial Vehicles (the UAVs system).**

Obtained results are important for the control properties analysis of production or logistic networks, multiprocessor or multicomputer networks, *etc*.

Keywords: multi-agent stochastic systems, consensus problem, approximate consensus.

I. INTRODUCTION

The multi-agent stochastic systems are widely used and studied in various applications in the fields of production, logistic, multiprocessor or multicomputer networks, *etc*. Often communication links between the agents may change over time and information about the states of neighbouring agents may be received with noise.

In [1] the author study the consensus problem for such systems and the properties of controls obtained by the local voting protocol (stochastic approximation type algorithm) with step-size tending to zero. In the case of an external dynamic changing of agents states (e. g., getting new task) the stochastic approximation type algorithms with decreasing to zero step-size are not applicable. In [2], [3], [4], [5] authors study the performance of stochastic approximation type algorithms with a constant step-size in the case of non-stationary mean-risk quality functional. The similar approaches we apply to considered multi-agent stochastic systems and a local voting protocol with a constant positive step-size. In addition we describe couple of possible applications of this protocol.

The rest of this paper is organized as follows. In the next section, we describe a basic knowledge from a graph theory. Section III deals with the consensus problem, main definitions, some preliminary results for non-stochastic case. Main theoretical result is presented in Section IV. At the end, we consider some applications of theoretical results and we discuss our future plan to apply obtained results to the system of Unnamed Aerial Vehicles (the UAVs system).

Konstantin Amelin, Natalia Amelina, Oleg Granichin, and Olga Granichina are with Faculty of Mathematics and Mechanics, Saint-Petersburg State University, 198504, Universitetskii pr. 28, St. Petersburg, Russia konstantinamelin@gmail.com, ngranichina@gmail.com, oleg_granichin@mail.ru, olga_granichina@mail.ru

The work was supported by Russian Federal Program "Cadres" (contract 16.740.11.0042) and by RFBR (project 11-08-01218).

II. CONCEPTS OF GRAPH THEORY

We explain some notation used in this article. The node index is used as a superscript but not an exponent in different variables.

Consider a dynamic network as a set of agents (nodes) $N = \{1,2,\ldots,n\}$.

A *directed graph (digraph)* (N,E) consists of N and a set of directed edges E. Denote the *neighbour set* of node i as $N^i = \{j : (j,i) \in E\}$.

Associate with each edge $(j,i) \in E$ a weight $a^{i,j} > 0$. Denote an *adjacency or connectivity matrix* $A = [a^{i,j}]$ of the graph denoted hereinafter $\mathscr{G}_A$. Define the weighted *in-degree* of node i as the i-th row sum of A: $d^i(A) = \sum_{j=1}^n a^{i,j}$ and $D(A) = \mathrm{diag}\{d^1(A), d^2(A), \ldots, d^n(A)\}$ is a corresponding diagonal matrix. The symbol $\mathscr{L}(A) = D(A) - A$ stands for *Laplacian* of graph $\mathscr{G}_A$.

A directed path from i_1 to i_s is a sequence of nodes $i_1,\ldots,i_s$, $s \geq 2$, such that $(i_k, i_{k+1}) \in E, k \in \{1,2,\ldots,s-1\}$. Node i is said to be *connected* to node j if there is a directed path from i to j. *The distance* from i to j is the length of the shortest path from i to j. Graph is said to be *strongly connected* if i and j are connected for all distinct nodes $i,j \in N$.

A *directed tree* is a digraph where each node i, except the root, has exactly one parent node j so that $(j,i) \in E$. We call $\overline{\mathscr{G}_A} = (\overline{N}, \overline{E})$ a *subgraph* of $\mathscr{G}_A$ if $\overline{N} \subset N$ and $\overline{E} \subset E \cap \overline{N} \times \overline{N}$. The digraph $\mathscr{G}_A$ is said to contain a *spanning tree* if there exists a directed tree $\mathscr{G}_{tr} = (N, E_{tr})$ as a subgraph of $\mathscr{G}_A$.

The following fact from graph theory will be important.

Lemma *1:* [6], [7] Laplacian $\mathscr{L}(A)$ of the graph $\mathscr{G}_A$ has rank $n-1$ if and only if the graph $\mathscr{G}_A$ has a spanning tree.

We note an important corollary:

Collorary *1:* If the graph $\mathscr{G}_A$ is strongly connected, then Laplacian $\mathscr{L}(A)$ has rank $n-1$.

The symbol $d_{\max}(A)$ denotes a maximal in-degree of the graph $\mathscr{G}_A$. In correspondence with the Gershgorin Theorem [8], we can deduce another important property of Laplacian: *all eigenvalues of the matrix $\mathscr{L}(A)$ have nonnegative real part and belong to the circle with center on the real axis at the point $d_{\max}(A)$ and with radius which equals to $d_{\max}(A)$.*

Let $\lambda_1,\ldots,\lambda_n$ denote eigenvalues of the matrix $\mathscr{L}(A)$. We arrange them in ascending order of real parts: $0 \leq Re(\lambda_1) \leq Re(\lambda_2) \leq \ldots \leq Re(\lambda_n)$. If the graph has a spanning tree then $\lambda_1 = 0$ is a simple eigenvalue and all other eigenvalues of $\mathscr{L}$ are in the open right half of the complex plane.

The second eigenvalue λ_2 of matrix $\mathscr{L}$ is important for analysis in many applications. It is usually called Fiedler eigenvalue. For undirected graph it was shown in [9] that:

$$Re(\lambda_2) \leq \frac{n}{n-1} \min_{i \in N} d^i(A),$$

978-0-7695-4761-9/12 $26.00 © 2012 IEEE
DOI 10.1109/SNPD.2012.12

IEEE computer society

and for the connected undirected graph G_A

$$Re(\lambda_2) \geq \frac{1}{\mathrm{diam}G_A \cdot \mathrm{vol}G_A},$$

where $\mathrm{diam}G_A$ is the longest distance between two nodes and $\mathrm{vol}G_A = \sum_{i\in N} d^i(A)$.

For all vectors ℓ_2-norm will be used, i. e. a square root of the all its elements squares sum.

III. Consensus problem in multi-agent systems

Consider a dynamic network of a nodes (agents) set that cooperate to achieve a goal that one can not achieve alone.

The concepts of graph theory will be used to describe the network topology. Let the dynamic network topology be modelled by a sequence of digraphs $\{(N,E_t)\}_{t\geq 0}$, where $E_t \subset E$ changes with time, and corresponding adjacency matrices are denoted as A_t. The maximal set of communication links is $E_{\max} = \{(j,i) : \sup_{t\geq 0} a_t^{i,j} > 0\}$.

We assume that a time-varying state variable $x_t^i \in \mathbb{R}$ corresponds to each node $i \in N$ of the graph at time $t \in [0,T]$. Its dynamics is described for the discreet time case by the equation

$$x_{t+1}^i = x_t^i + f^i(x_t^i, u_t^i),\ t = 0,1,2\ldots,T \tag{1}$$

or for the continuous time case

$$\dot{x}_t^i = f^i(x_t^i, u_t^i),\ t \in [0,T], \tag{2}$$

with some functions $f^i(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, depending on states in the previous time x_t^i and control actions $u_t^i \in \mathbb{R}$.

To form its control strategy each node $i \in N$ uses its own state

$$y_t^{i,i} = x_t^i + w_t^{i,i}, \tag{3}$$

(possibly noisy) and if $N_t^i \neq \emptyset$, noisy measurements of its neighbours states

$$y_t^{i,j} = x_t^j + w_t^{i,j},\ j \in N_t^i, \tag{4}$$

where $w_t^{i,i}, w_t^{i,j}$ is the noise.

We consider the multi-agent system consisting of dynamic agents $i \in N$ with inputs u_t^i, outputs $y_t^{i,j}$ and states x_t^i.

If $(j,i) \in E_t$ then node i receives information from node j for the purposes of feedback control.

Definition *1:* A feedback on observations

$$u_t^i = K_t^i(y_t^{i,j_1},\ldots,y_t^{i,j_{m_i}}), \tag{5}$$

where $\{j_1,\ldots,j_{m_i}\} \in \{i\}\bigcup \overline{N}_t^i$, $\overline{N}_t^i \subseteq N_t^i$ is called a *protocol* (*control algorithm*) with topology (N,E_t).

Nodes i and j is said to *agree* in a network at time t if and only if $x_t^i = x_t^j$.

Definition *2:* n nodes of a network is said to reach a *consensus* at time t if $x_t^i = x_t^j\ \ \forall i,j \in N, i \neq j$.

Definition *3:* n nodes is said to achieve *asymptotic consensus* if there exists a variable $x^\star$: $x^\star = \lim_{t\to\infty} x_t^i$ for all $i \in N$.

Consider the *local voting protocol*:

$$u_t^i = \alpha_t \sum_{j\in\bar{N}_t^i} b_t^{i,j}(y_t^{i,j} - y_t^{i,i}), \tag{6}$$

where $\alpha_t > 0$ are step-sizes of control protocol, $b_t^{i,j} > 0\ \forall j \in \bar{N}_t^i$. We set $b_t^{i,j} = 0$ for other pairs i,j and denote $B_t = [b_t^{i,j}]$ as the matrix of control protocol.

Under some general assumption in [1] the author proved a necessary and sufficient condition for the asymptotic mean square consensus when step-size α_t in (6) tends to zero and second term in (1) has a simple form: $f^i(x_t^i, u_t^i) = u_t^i$. We will consider more general case of functions $f^i(x_t^i, u_t^i)$ and step-size α_t nondecreasing to zero.

At first, consider the particular case of dynamic systems on graphs when the second term in (1) has a simple form: $f^i(x_t^i, u_t^i) = u_t^i$, for all nodes i and all observations are made without noise: $y_t^{i,j} = x_t^j$, $j \in \{i\} \cup N_t^i$.

Denote $\bar{x}_t = [x_t^1;\ldots;x_t^n]$ and $\bar{u}_t = [u_t^1;\ldots;u_t^n]$ column vectors obtained by the vertical concatenation of n corresponding variables. Control protocol (6) can be rewritten in a matrix form:

$$\bar{u}_t = (\alpha_t B_t - D(\alpha_t B_t))\bar{x}_t = -\mathscr{L}(\alpha_t B_t)\bar{x}_t \tag{7}$$

the dynamics (1) for the discrete time case is described by:

$$\bar{x}_{t+1} = \bar{x}_t + \bar{u}_t,\ t = 0,1,2,\ldots,T, \tag{8}$$

and for the continuous-time case is:

$$\dot{\bar{x}}_t = \bar{u}_t,\ t \in [0,T]. \tag{9}$$

Then the closed-loop system for the discrete time case takes the form:

$$\bar{x}_{t+1} = (I - \mathscr{L}(\alpha_t B_t))\bar{x}_t,\ t = 0,1,2,\ldots,T, \tag{10}$$

where I is matrix of size $n \times n$ of ones and zeros on the diagonal, and for the continuous time case is

$$\dot{\bar{x}}_t = -\mathscr{L}(\alpha_t B_t)\bar{x}_t,\ t \in [0,T]. \tag{11}$$

We will show that the control protocol (6) with $\alpha_t = \alpha$ and $B_t = A$ provides consensus asymptotically for both discrete and continuous-time models.

Indeed, for the discrete case the equation (10) turns into

$$\bar{x}_{t+1} = (I - \mathscr{L}(\alpha A))\bar{x}_t \equiv P\bar{x}_t, \tag{12}$$

where the Perron matrix $P = I - \mathscr{L}(\alpha A)$ has one simple eigenvalue equal to one and all others are inside the unit circle if

$$\alpha < \frac{1}{d_{\max}}. \tag{13}$$

Since the sum of row elements of Laplacian $\mathscr{L}$ equals to zero, the sum of row elements of matrix P equals to one, i. e. vector $\underline{1}$ consisting of units is a right eigenvector of P corresponding to the unit eigenvalue. The unit eigenvalue is simple if the graph has a spanning tree. All other eigenvalues are inside the unit circle. Let $\bar{z}_1 = [z^1,\ldots,z^n]$ denote the left eigenvector of matrix P which is orthogonal to $\underline{1}$. Consequently, if the graph has a spanning tree then in the limit of $t \to \infty$ we got

$$\bar{x}_t \to \underline{1}(\bar{z}_1^{\mathrm{T}}\bar{x}_0), \tag{14}$$

i. e. an asymptotic consensus is reached. The consensus value $x^\star$ equals to the normalized linear combination of initial states with weights equal to elements of the left eigenvector of matrix P

$$x^\star = \frac{\bar{z}_1^{\mathrm{T}} \bar{x}_0}{\bar{z}_1^{\mathrm{T}} \underline{1}} = \frac{\sum_{i=1}^n z^i x_0^i}{\sum_{i=1}^n z^i}. \tag{15}$$

This value depends on the graph topology and, consequently, on connection links between nodes.

Lemma *2:* If the graph $\mathscr{G}_A$ has a spanning tree and control protocol (6) parameters $B_t = A$ and $\alpha_t = \alpha$ are such that the condition (13) is satisfied then the control protocol (6) provides asymptotic consensus for the discrete system (8) and its value $x^\star$ is given by (15).

If the graph is balanced then rows sums of the Laplacian $\mathscr{L}$ equal to sums of corresponding columns, and this property is transferred to the matrix P. Then $\bar{z}_1 = c\underline{1}$, and consensus value equals to the initial values average

$$x^\star = \frac{1}{n}\sum_{i=1}^n x_0^i$$

and does not depend on the topology of the graph.

For the continuous-time case we have

$$\dot{\bar{x}} = -\mathscr{L}\bar{x}. \tag{16}$$

Let $\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n$ and $\bar{r}_1 = \frac{1}{\sqrt{n}}\underline{1}, \bar{r}_2, \ldots, \bar{r}_n$ be left and right orthonormal eigenvectors of the matrix $\mathscr{L}$ corresponding to its ordered eigenvalues $\lambda_1, \ldots, \lambda_n$. If the graph has a spanning tree then $\lambda_1 = 0$ is a simple eigenvalue and all other eigenvalues of $\mathscr{L}$ are in the open right half of complex plane. Thus, the system (16) is partially stable with one pole at the origin and the rest are in the open left half plane.

For the first left eigenvector $\bar{z}_1 = [\bar{z}^1, \ldots, \bar{z}^n]$ of matrix $\mathscr{L}$ we have

$$\frac{d}{dt}(\bar{z}_1^{\mathrm{T}} \bar{x}_t) = \bar{z}_1^{\mathrm{T}} \dot{\bar{x}}_t = -\bar{z}_1^{\mathrm{T}} \mathscr{L} \bar{x}_t = 0,$$

i. e. $\bar{x} \equiv \bar{z}_1^T \bar{x}_t = \sum_{i=1}^n z_1^i x_t^i$ is invariant, that is constant and independent of the states of nodes. Thus, $\sum_{i=1}^n \bar{z}_1^i x_0^i = \sum_{i=1}^n \bar{z}_1^i x_t^i, \forall t$.

We apply the modal expansion and rewrite the state vector in terms of eigenvalues and eigenvectors of the matrix $\mathscr{L}$. If all the eigenvalues of $\mathscr{L}$ are simple (in fact, it is only important that λ_1 is simple), then

$$\bar{x}_t = e^{-\mathscr{L}t}\bar{x}_0 = \sum_{j=1}^n \bar{r}_j e^{-\lambda_j t} \bar{z}_j^{\mathrm{T}} \bar{x}_0 = \sum_{j=2}^n (\bar{z}_j^{\mathrm{T}} \bar{x}_0) e^{-\lambda_j t} \bar{r}_j + \frac{\bar{x}}{\sqrt{n}}\underline{1}. \tag{17}$$

In the limit of $t \to \infty$ we get $x_t \to \frac{\bar{x}}{\sqrt{n}}\underline{1}$ or $x_t^i \to x^\star = \frac{\bar{x}}{\sqrt{n}}, \forall i \in N$, i. e. an asymptotic consensus is reached.

Lemma *3:* If the graph $\mathscr{G}_A$ has a spanning tree then the control protocol (6) with $\alpha_t = \alpha$ and $B_t = A$ provides an asymptotic consensus for the continuous-time system (9) and its value $x^\star$ is given by

$$x^\star = \frac{1}{\sqrt{n}}\sum_{i=1}^n \bar{z}_1^i x_0^i \tag{18}$$

with vector of initial data $\bar{x}_0$ and the orthonormal first left eigenvector $\bar{z}_1$ of the matrix $\mathscr{L}$.

Consider the problem of reaching an approximate ε-consensus ($\varepsilon > 0$).

Definition *4:* n nodes is said to achieve *ε-consensus* at time t if there exists a variable $x^\star$ such that $||x_t^i - x^\star||^2 \le \varepsilon$ for all $i \in N$.

Definition *5:* $T(\varepsilon)$ is called *time to ε-consensus*, if n nodes achieve ε-consensus for all $t \ge T(\varepsilon)$.

From (17) by evaluating the square of the norm of the first term we can obtain

$$||\bar{x}_t - x^\star\underline{1}||^2 = ||\sum_{j=2}^n (\bar{z}_j^{\mathrm{T}} \bar{x}_0) e^{-\lambda_j t} \bar{r}_j||^2 =$$

$$= ||\sum_{j=2}^n (\bar{z}_j^{\mathrm{T}} (\bar{x}_0 - x^\star)) e^{-\lambda_j t} \bar{r}_j||^2 \le (n-1) e^{-2Re(\lambda_2)t} ||\bar{x}_0 - x^\star\underline{1}||^2.$$

From here we have the expression for the time to ε-consensus in system (16)

$$T(\varepsilon) = \frac{1}{2Re(\lambda_2)} \ln\left(\frac{(n-1)||x_0 - x^\star\underline{1}||^2}{\varepsilon}\right). \tag{19}$$

Lemma *4:* If the graph $\mathscr{G}_A$ has a spanning tree then control protocol (6) with $\alpha_t = \alpha$ and $B_t = A$ provides ε-consensus for the continuous-time system (9) for any $t \ge T(\varepsilon)$, where $T(\varepsilon)$ was defined by (19), and the consensus value $x^\star$ is given by the formula (18).

A similar estimate for the time to ε-consensus can be obtained for the discrete system (12).

IV. Main Result

Let $(\Omega, \mathscr{F}, P)$ be the underlying probability space. The symbol E stands for mathematical expectation and the symbol E_x denotes conditional expectation under the condition x.

In the theoretical results obtained below, we assume that the following conditions are satisfied.

A1. Functions $f^i(x,u)$ $\forall i \in N$ are Lipschitz in x and u: $|f^i(x,u) - f^i(x',u')| \le L_1(L_x|x-x'| + |u-u'|)$, the growth rate is bounded: $|f^i(x,u)|^2 \le L_2(L_c + L_x|x|^2 + |u|^2)$, and for any fixed x the function $f^i(x,\cdot)$ is such that $\mathrm{E}_x f^i(x,u) = f^i(x, \mathrm{E}_x u)$;

A2. a) The noises $w_t^{i,j}$ $\forall i \in N, j \in \{i\} \cup N^i$ are centered, independent and have bounded variance: $\mathrm{E}(w_t^{i,j})^2 \le \sigma_w^2$.

b) The appearance of edges (j,i) $\forall i \in N, j \in N^i$ in the graph $\mathscr{G}_{A_t}$ is independent random event with probability $p_a^{i,j}$ (i. e. matrices A_t are independent identically distributed random matrices).

c) Weights of control protocol $b_t^{i,j}$ $\forall i \in N, j \in N^i$ are bounded random variables: $\underline{b} \le b_t^{i,j} \le \bar{b}$ with probability 1, and there exist $b^{i,j} = \mathrm{E}b_t^{i,j}$.

Moreover, all of these random variables and matrices are independent of each other.

A3. The graph $\mathscr{G}_{A_{\max}}$ has a spanning tree where elements of matrix $A_{\max}$ size of $n \times n$ are $a_{\max}^{i,j} = p_a^{i,j} b^{i,j}$, $i \in N$, $j \in N$, $a_{\max}^{i,i} = 0$, $i \in N$.

Definition *6:* n nodes is said to achieve *mean square ε-consensus* at time $t \ge 0$ if $\mathrm{E}||x_t^i||^2 < \infty$, $i \in N$, and there exists a random variable $x^\star$ such that $\mathrm{E}||x_t^i - x^\star||^2 \le \varepsilon$ for all $i \in N$.

Consider the discreet time case and rewrite the dynamics of the nodes in vector-matrix form:

$$\bar{x}_{t+1} = \bar{x}_t + F(\alpha_t, \bar{x}_t, \bar{w}_t), \tag{20}$$

here $F(\alpha_t,\bar{x}_t,\bar{w}_t)$ is the vector of size n: $F(\alpha_t,\bar{x}_t,\bar{w}_t) =$

$$= \begin{pmatrix} \cdots \\ f^i(x_t^i, \alpha_t \sum_{j\in \bar{N}_t^i} b_t^{i,j}((x_t^j - x_t^i) + (w_t^{i,j} - w_t^{i,i}))) \\ \cdots \end{pmatrix}. \quad (21)$$

The *method of continuous models* [10], [11], (also called DE approach [12], or Derevitskii-Fradkov-Ljung (DFL)-cheme [13]) consists on the approximate replacement of inial stochastic difference equation (20) by ordinary differential quation

$$\frac{d\bar{x}}{d\tau} = R(\alpha,\bar{x}), \quad (22)$$

here

$$R(\alpha,\bar{x}) = R\begin{pmatrix} & x^1 \\ \alpha, & \vdots \\ & x^n \end{pmatrix} = \begin{pmatrix} \cdots \\ \frac{1}{\alpha} f^i(x^i, \alpha s^i(\bar{x})) \\ \cdots \end{pmatrix}, \quad (23)$$

$$(\bar{x}) = \sum_{j\in N_{\max}^i} a_{\max}^{i,j}(x^j - x^i) = -d^i(A_{\max})x^i + \sum_{j=1}^{n} a_{\max}^{i,j} x^j,\ i \in N.$$

For a finite time interval trajectories $\{\bar{x}_t\}$ from (20)-(21) nd $\{\bar{x}(\tau_t)\}$ from (22)-(23) closeness conditions follow from 11]. Here and further $\tau_t = \alpha_0 + \alpha_1 + \ldots + \alpha_{t-1}$, (in particular, $= T\alpha$ if $\forall t\ \alpha_t = \alpha = const$).

Theorem *1:* [14] Let the conditions **A1**, **A2a–c** be satisfied, $i \in N$ function $f^i(x,u)$ is smooth in u, $f^i(x,0) = 0$ for any x nd $0 < \alpha_t \le \bar{\alpha}$, **then** there exists $\tilde{\alpha}$ such that for $\bar{\alpha} < \tilde{\alpha}$ the ollowing inequality holds:

$$\mathrm{E} \max_{0\le\tau_t\le\tau_{\max}} ||\bar{x}_t - \bar{x}(\tau_t)||^2 \le C_1 e^{C_2\tau_{\max}} \bar{\alpha}, \quad (24)$$

here $C_1 > 0$, $C_2 > 0$ are some constants.

We assume that in the continuous model (22)-(23) the -consensus is reached over time, i. e. all components of e vector $\bar{x}(\tau)$ become close to some common value x^* consensus value) for all $i \in N$.

Theorem *2:* [14] Let the conditions **A1**, **A2a–c** be satisfied, $i \in N$ functions $f^i(x,u)$ are smooth by u, $f^i(x,0) = 0$ for ny x, $0 < \alpha_t \le \bar{\alpha}$, for the continuous model (22)-(23) $\frac{\varepsilon}{4}$-onsensus is achieved for time $\mathscr{T}(\frac{\varepsilon}{4})$, consensus protocol arameters $\{\alpha_t\}$ are chosen so that $\tau_{\max} = \sum_{t=0}^{T} \alpha_t > \mathscr{T}(\frac{\varepsilon}{4})$ nd for constants C_1, C_2 the following inequality holds

$$C_1 e^{C_2\tau_{\max}} \max_{\alpha_t:\tau_t\le\tau_{\max}} \alpha_t \le \frac{\varepsilon}{4},$$

hen mean square ε-consensus is achieved in the stochastic iscrete system (20)-(21) at any time $t: \mathscr{T}(\frac{\varepsilon}{4}) \le t \le \tau_{\max}$.

Consider an important particular case $\forall i \in N\ f^i(x,u) = u$. The upper bound was obtained:

$$\mathscr{T}\left(\frac{\varepsilon}{4}\right) = \frac{1}{2Re(\lambda_2)} \ln\left(\frac{4(n-1)||\bar{x}_0 - x^*\underline{1}||^2}{\varepsilon}\right), \quad (25)$$

or the time to $\frac{\varepsilon}{4}$-consensus in the continuous model (22)-(23).

From Theorem 2 we can get the important consequence.

Theorem *3:* [14] Let the conditions **A2a–c**, **A3** be satisfied, $f^i(x,u) = u$ for any $i \in N$, **then** at any time $t: \mathscr{T}(\frac{\varepsilon}{4}) \le t \le \tau_{\max}$ in the stochastic discrete system (20)-(21) n nodes achieve mean square ε-consensus. for any arbitrarily small positive number $\varepsilon > 0$ and for any $\tau_{\max} > \mathscr{T}(\frac{\varepsilon}{4})$ denoted in (25) when selecting sufficiently small α_t

$$\max_{\alpha_t:\tau_t\le\tau_{\max}} \alpha_t \le \frac{\varepsilon}{4C_1 e^{C_2\tau_{\max}}}.$$

Here $C_1, C_2, \bar{\alpha}$ are some constants and λ_2 is the closest to the imaginary axis eigenvalue of matrix $\mathscr{L}$ with nonzero real part.

Consider the case of an infinite time interval, i. e. $T = \infty$.

Definition *7:* n nodes is said to achieve *asymptotic mean square ε-consensus* if $\mathrm{E}||x_t^i||^2 < \infty$, $t = 0,1,\ldots,\ i \in N$, and there exists a random variable x^* such that $\overline{\lim}_{t\to\infty} \mathrm{E}||x_t^i - x^*||^2 \le \varepsilon$ for $i \in N$.

In [10] the mean square model (22) accuracy estimates were obtained for both cases of infinite and finite time intervals. For example, in [10] for independent $\bar{w}_t$ and in [11] for $\bar{w}_t$ satisfying the strong mixing conditions the following fact was obtained.

Lemma *5:* If $\bar{w}_t$ are independent (or $\bar{w}_t$ satisfy the strong mixing conditions), Lipschitz:

$$||R(\alpha,\bar{z}) - R(\alpha,\bar{z}')|| \le \bar{L}_1 ||\bar{z} - \bar{z}'||, \quad (26)$$

$$||R(\alpha,\bar{z}) - R(\alpha',\bar{z})|| \le \bar{L}_1(1 + ||\bar{z}||)|\alpha - \alpha'|, \quad (27)$$

and growth conditions:

$$\mathrm{E}||\frac{1}{\alpha_t} F_t(\alpha_t,\bar{z},\bar{w}_t) - R(\alpha,\bar{z})||^2 \le \bar{L}_2(1 + ||z||^2), \quad (28)$$

satisfy then the following inequality holds:

$$\mathrm{E} \max_{0\le\tau_t\le\mathscr{T}} ||\bar{x}_t - \bar{x}(\tau_t)||^2 \le C_1 e^{C_2\mathscr{T}} \bar{\alpha}, \quad (29)$$

where $C_1 > 0$, $C_2 > 0$ are some constants, $\bar{\alpha} = \max_{1\le t\le T} \alpha_t$, $\tau_T \le \mathscr{T}$.

If the continuous model (22) is exponentially stable then it was shown in [10], [11] that the approximation accuracy for an infinite time interval has an order of α^γ for some $0 < \alpha < 1$.

Lemma *6:* [10], [11] If the conditions of Lemma 5 are satisfied and the continuous model (22) is exponentially stable then there exists $\bar{\alpha} > 0$ such that for $0 \le \alpha_k \le \alpha < \bar{\alpha}$ the following inequalities hold

$$\mathrm{E}||\bar{x}_t - \bar{x}(\tau_t)||^2 \le C_3\alpha^\gamma,\ t = 1,2,\ldots, \quad (30)$$

where values $C_3 > 0$, $\gamma > 0$ do not depend on α.

Theorem 2 does not give the information about the asymptotic behavior of the system. Conditions for achieving asymptotic mean-square ε-consensus are given in the next theorem.

Theorem *4:* [14] Let the conditions **A1**, **A2a–b** be satisfied, $\forall i \in N$ functions $f^i(x,u)$ are smooth by u, $f^i(x,0) = 0$ for any x, $0 < \alpha_t \le \bar{\alpha}$ and the continuous model (22)-(23) is exponentially stable, **then** n nodes achieve asymptotic mean square ε-consensus with $\varepsilon = C_3\bar{\alpha}^\mu$ for some independent from $\bar{\alpha}$ constants C_3 and $\mu: 0 < \mu < 1$ from Lemma 6.

V. CONCLUSION

The theoretical results can be applied in transport and logistics networks, production networks, computer networks, and other networks. Consider some practical applications.

In [15], [16] load balancing algorithms are presented for a decentralized computer network with incomplete information about current nodes states and changing set of communication links. A load balancing problem is reformulated as a consensus problem in a noisy model with a switching topology. The local voting protocol was used to provide uniform loading of a network. Evaluating performance of the system was presented for simulation results.

The simulation was carried out for the system shown in Fig. 1 consisting of 6 computing blocks.

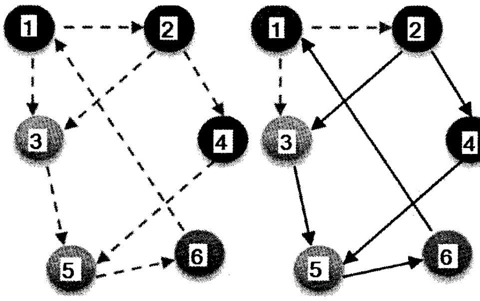

Fig. 1. Network topology.

To estimate convergence to consensus under the using of the local voting protocol (6), we introduce *error of estimation* $Err = \sum_i \sqrt{\frac{(x_t^i - x^*)^2}{n}}$. Fig. 2 shows plots of the estimation error for different constant step-sizes α. One can see that if we increase the step-size α then the states of nodes reach a consensus faster but up to a point [17].

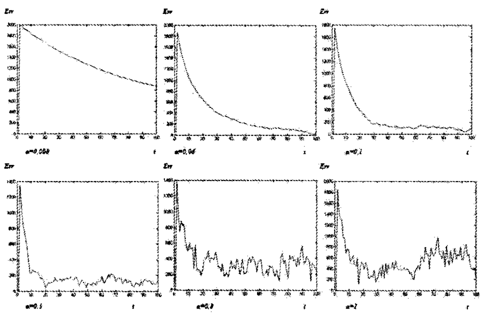

Fig. 2. Dependence on the step-size α.

In [18] there was an attempt to compare different protocols for trucks loading. And in [19] the formation control problem is considered. It also can be presented as consensus problem in network.

In the future work we plan to use above theoretical results in our practical project: multi-agents group of UAVs (Fig. 3) [21].

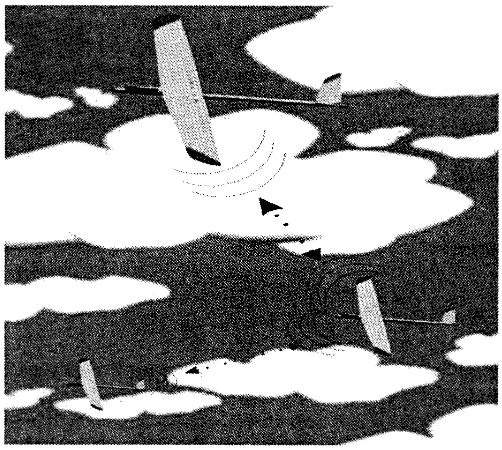

Fig. 3. Group of UAVs.

We get a new three levels of the UAV-agents control system (Fig. 4). Upper layer is also software base station. But it creates global tasks for group of UAV. It is goals, initial conditions, adjustment or modification of the initial problems. Also the base station receives and processes data from the group of UAVs. Base station is the computer (server) with different communication modules (modems, radio receivers, *etc*).

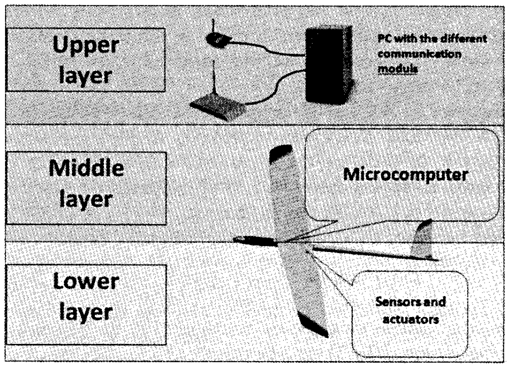

Fig. 4. Three levels of the UAVs control system.

Communication with the base station carried out due to separate channel or via GPRS over GSM modem. A GSM modem can be easily integrated with a microcomputer but data packets should be compressed.

Connection between microcomputers of different UAVs carried out due to FM radio with a frequency of 2.4 GHz and the communication protocol 802.11 n (Wi-Fi) which uses technology that connects the two nearest channels into one. Thus microcomputers in the UAVs will be able simultaneously receive and send information to each other. Communication with the base station carried out due to individual channel or via GPRS over GSM modem [20].

Due to the small UAVs weight the takeoff is carried out with human hands or with a catapult. Landing is carried out either through the built-parachute, or due to "takeover of control" of the operator to manual control.

Middle control layer carries out by autopilot of UAV-agent. Autopilot is a set of devices with a microcontroller with the real-time system. The main task of the autopilot is to control the actuators (servos, engine, additional equipment) based on given flight program and data from sensors (inertial, infrared sensors, pressure and velocity sensors, *etc*). Interoperability between the main microcomputer and autopilot is organized by SIP.

Lower layer controls the actuators and processes sensor data to achieve the goal. But microcomputer generates the program for autopilot rather than the base station.

For our UAV-agent we use the model of lung glider "PAPRIKA". It is 1.2 m in length, 2 m wing span, 2-2.1 kg max take of weight, 600 g payload, 40–120 km/h velocity and 200 km range. On the middle layer we use the microcomputer Gumstix. It is $17mm \times 58mm \times 4.2mm$ sizes, Linux operating system, ARM Cortex-A8 processor with 600 Mhz clock frequency, 256 MB RAM and 256 MB NAND Flash. Microcomputer is the main on board device in the control system of UAV-agent [20].

One of the important task in the development of UAVs control programs is an optimization of flight algorithms. In [22] there is a description of multi-agent technology possible effective application to accumulate energy and increase the flight range by using the thermal updrafts which are formed in the lower atmosphere due to disruption of warm air from the surface when it is heated by sunlight.

The authors would like to thank the SPRINT laboratory of SPbSU and Intel Corp. for supporting the project.

References

[1] *Huang M.* Stochastic approximation for consensus with general time-varying weight matrices // Proc. of IEEE Conf. on Decision and Control (CDC-49), Atlanta, USA, 2010, pp. 7449–7454.

[2] *Granichin O., Gurevich L., and Vakhitov A.* Discrete-time minimum tracking based on stochastic approximation algorithm with randomized differences // Proc. of the Combined 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference, December 16-18, Shanghai, P.R. China, 2009, pp. 5763–5767.

[3] *Borkar V.S.* Stochastic Approximation. A Dynamical Systems Viewpoint — Cambridge University Press, 2008.

[4] *Granichin O.N.* Stochastic optimization and system programming // Stochastic Optimization in Informatics, vol. 6, pp. 3-44, 2010.

[5] *Vakhitov A.T., Granichin O.N., and Panshenskov M..* Methods of data transfer speed estimation in the data Grid based on linear regression // Neurocomputers: design and application, 2009, no. 11, pp. 45–52.

[6] *Agaev R.P., and Chebotarev P.Yu.* The matrix of maximal outgoing forests of a digraph and its applications // Automation and Remote Control, vol. 61, no. 9, 2000, pp. 1424–1450.

[7] *Ren W., and Beard R.W.* Consensus seeking in multiagent systems under dynamically changing interaction topologies // IEEE Trans. Automat. Control, 2005, vol. 50, no. 5, pp. 655–661.

[8] *Gerschgorin S.* Uber die abgrenzung der eigenwerte einer matrix // Bulletin de l'Academie des Sciences de l'URSS. Classe des sciences mathematiques et na, no. 6, 1931, pp. 749–754.

[9] *Wu Chai Wah* Synchronization in complex networks of nonlinear dynamical systems — World Scientific Publishing Company, 2007. — 168 p.

[10] *Derevitskii D.P., and Fradkov A.L.* Two models for analysis the dynamics of adaptation algorithms // Automation and Remote Control, no. 1, 1974, pp. 59–67.

[11] *Derevitskii D.P., and Fradkov A.L.* — Applied Theory of Discrete Adaptive Control Systems. — Moscow: Nauka (in Russian). 1981.

[12] *Ljung L.* Analysis of recursive stochastic algorithms // IEEE Trans. Aut. Control, 1977, no. 4, pp. 551–575.

[13] *Gerencser L.* A representation theorem for the error of recursive estimators // SIAM J. Control Optim, 2006, vol. 44, no. 6, pp. 2123–2188.

[14] *Amelina N.O., and Fradkov A.L.* The method of averaged models for consensus problem // Stochastic Optimization in Informatics, vol. 8, 2012, pp. 3–39.

[15] *Amelina N.O.* Workload node balancing in decentralized network system with imperfect information // Neurocomputers: design, application, no. 6, 2011, pp. 56–63.

[16] *Amelina N.O.* Scheduling networks with variable topology in the presence of noise and delays in measurements // Vestnik Sankt-Petersburgskogo University. Mathematics, vol. 45, no. 2, 2012, pp. 56–60.

[17] *Amelina N.O., and Fradkov A.L.* Consensus in decentralized stochastic dynamic networks with incomplete information and delays in the measurements // Automation and Remote Control, 2012 (to be appeared).

[18] *Amelina N.O., Lada A.N., Mayorov I.V., et al* Cargo transportation models analysis using multi-agent adaptive real-time truck scheduling system // Control Sciences, no. 6, 2011, pp. 31–37.

[19] *Amelina N.O.* The consensus problem for autonomous group of UAVs // Stochastic Optimization in Informatics, vol. 6, 2010, pp. 127–132.

[20] *Amelin K.S.* Small UAV for the autonomous group // Stochastic Optimization in Informatics, vol. 6, 2010, pp. 117–126.

[21] *Amelin K.S., and Granichin O.N.* Multi-agent network control for group of UAVs // Neurocomputers: design and application, no. 6, 2011, pp. 64–72.

[22] *Antal C., Granichin O., and Levi S.* Adaptive autonomous soaring of multiple UAVs using SPSA // In Proc. of the 49th IEEE Conference on Decision and Control. December 15-17, Atlanta, GA, USA, 2010, pp. 3656–3661.