RANDOMIZED ALGORITHMS OF OPTIMIZATION AND THEIR IMPLEMENTATION ON QUANTUM COMPUTERS

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1 The scalar “useful" signal detecting

$$y_n = \varphi_n \theta^\star + v_n, \quad n = 1, 2, \ldots$$

$\{\varphi_n\}$ — random known useful signal
with known mean valur M_φ and $\sigma_\varphi^2 > 0$
$\{\theta^\star = 1\}$ — *signal is present*
$\{\theta^\star = 0\}$ — *signal is absent*
$\{v_n\}$ — bounded measurement disturbances

Least–squares method

$$\hat{\theta}_n = \frac{\sum_{k=1}^n \varphi_k y_k}{\sum_{k=1}^n \varphi_k^2} \to \theta^\star + \frac{M_\varphi M_v}{\sigma_\varphi^2} \quad \text{a.s.}$$

$\delta = \dfrac{M_\varphi M_v}{\sigma_\varphi^2} + \dfrac{1}{2}$ — level of decisionmaking
If $\hat{\theta}_n < \delta$ then *signal is absent*
Otherwise — *signal is present*

Usual assumptions:
$M_v = 0$ and $\{v_n\}$ is a random i.i.d.

Let M_v be unknown and
$\{v_n\}$ doesn't depend on $\{\varphi_n\}$
or $\{v_n\}$ is a nonrandom *unknown but bounded*

Membership set approach
If $|v_n| \leq C_v,\ n = 1, 2, \ldots$ then at the time moment N the membership set is

$$\Theta_N = \cap_{n=1}^{N} \{\theta \in \mathrm{R} :\ |y_n - \varphi_n \theta| \leq C_v\}.$$

$$\theta^\star \in \Theta_N$$

Does the sequence $\{\Theta_N\}$ converges to $\theta^\star$???

- How to solve the problem?
- What is an appropriate result of measurements?
- Is it necessary to have many independent measurements with zero-mean?

Denote $\Delta_n = \varphi_n - M_\varphi$

$$\Delta_n y_n = \Delta_n^2 \theta^\star + \Delta_n M_\varphi \theta^\star + \Delta_n v_n$$

$$\frac{\sum_{k=1}^n \Delta_k y_k}{\sum_{k=1}^n \Delta_k^2} = \theta^\star + \frac{\sum_{k=1}^n \Delta_k (M_\varphi \theta^\star + v_k)}{\sum_{k=1}^n \Delta_k^2}$$

Randomized least–squares method

$$\hat{\theta}_n = \frac{\sum_{k=1}^n \Delta_k y_k}{\sum_{k=1}^n \Delta_k^2} \to \theta^\star \quad \text{a.s.}$$

Level of decisionmaking is $\delta = 1/2$.

Randomized algorithms:

$$\hat{\theta}_n = \hat{\theta}_{n-1} - \Delta_n Y_n$$

Δ_n is a *trial simultaneous perturbation*

Examples

$\{\varphi_n\}$ is a random i.i.d.: $\varphi_n \in [0.5, 1.5]$

$\{v_n\}$ is a nonrandom *unknown but bounded* sequence: $|v_n| \leq 2$

Let $n = 1, \ldots, 100, 350, \ldots, 500$

$\{\theta_n^\star = 0\}$ — *signal is absent*

$n = 101, \ldots, 349$

$\{\theta_n^\star = 1\}$ — *signal is present*

$$\sum_{n=1}^{250} v_n = 1, \qquad \sum_{n=250}^{500} v_n = -1$$

$$\hat{\theta}_n = \hat{\theta}_{n-1} - 0.1\varphi_n(\varphi_n\hat{\theta}_{n-1} - y_n)$$

$$\hat{\theta}_n = \hat{\theta}_{n-1} - 0.1\Delta_n(\varphi_n\hat{\theta}_{n-1} - y_n)$$

2 Problem statement and main assumption

Let $F(w,\theta):\ \mathbb{R}^p \times \mathbb{R}^d \to \mathbb{R}^1$ be diff. on θ, $x_1, x_2 \ldots$ is an observation plan,

$$y_n = F(w_n, x_n) + v_n,$$

$\{w_n\}$ is an uncontrollable i.i.d. $w_n \in \mathbb{R}^p$ with unknown distribution $\mathrm{P}_w(\cdot)$.

It is required to find the unknown vector $\theta^\star$ which minimizes a function

$$f(\theta) = \int_{\mathbb{R}^p} F(w,\theta)\mathrm{P}_w(dw)$$

by using the observations $y_1, y_2 \ldots$.

More simple observation model:

$$y_n = f(x_n) + v_n.$$

Assumptions:

(A.1) $f(\cdot)$ has a unique root in R^d at $\theta^\star$

$$\langle x-\theta^\star, \nabla f(x)\rangle \geq \mu\|x-\theta^\star)\|^2, \quad \forall x \in \mathrm{R}^d$$

with some constant $\mu > 0$.

(A.2) Lipschitz condition for the gradient

$$\|\nabla f(x) - \nabla f(\theta)\| \leq A\|x-\theta\|, \ \forall x, \theta \in \mathrm{R}^d$$

with some constant $A > \mu$.

(A.3) Function $f(\cdot) \in C^\ell$ is ℓ–times continuously differentiable and for all its partial derivatives up to the order ℓ the Holder condition of order ρ $(0 < \rho \leq 1)$, holds on R^d so that

$$|f(x) - \sum_{|\bar{l}|\leq\ell} \frac{1}{\bar{l}!} D^{\bar{l}} f(\theta)(x-\theta)^{\bar{l}}| \leq M\|x-\theta\|^\gamma,$$

where $\gamma = \ell + \rho \geq 2$, M — some constant,
$\bar{l} \in \mathbb{N}^d$ is a multi-index,
If $\gamma = 2$ then $M = A/2$.

3 SPSA algorithms and trial simultaneous perturbation

Let $\{\Delta_n\}$ be *trial simultaneous perturbation.* $\Delta_n \in \mathrm{R}^d$, its distribution function is $\mathrm{P}_n(\cdot)$.

One measurement form of SPSA method

$$\begin{cases} x_n = \hat{\theta}_{n-1} + \beta_n \Delta_n \\ y_n = F(w_n, x_n) + v_n \\ \hat{\theta}_n = \hat{\theta}_{n-1} - \frac{\alpha_n}{\beta_n} \mathcal{K}_n(\Delta_n) y_n \end{cases} \tag{1}$$

“Smoothed" versions of the Kiefer-Wolfowitz procedure

$$\begin{cases} x_{2n} = \hat{\theta}_{n-1} + \beta_n \Delta_n \\ x_{2n-1} = \hat{\theta}_{n-1} - \beta_n \Delta_n \\ \hat{\theta}_n = \hat{\theta}_{n-1} - \frac{\alpha_n}{2\beta_n} \mathcal{K}_n(\Delta_n)(y_{2n} - y_{2n-1}) \end{cases} \tag{2}$$

$$\begin{cases} x_{2n} = \hat{\theta}_{n-1} + \beta_n \Delta_n \\ \\ x_{2n-1} = \hat{\theta}_{n-1} \\ \\ \hat{\theta}_n = \hat{\theta}_{n-1} - \frac{\alpha_n}{\beta_n} \mathcal{K}_n(\Delta_n)(y_{2n} - y_{2n-1}) \end{cases} \tag{3}$$

Here $\mathcal{K}_n(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ are smoothed kernels with the finite support:

$$\int \mathcal{K}_n(x) x^{\mathrm{T}} \mathrm{P}_n(dx) = \mathrm{I}$$

$$\int \mathcal{K}_n(x) \mathrm{P}_n(dx) = 0 \tag{4}$$

$$\sup_n \int \|\mathcal{K}_n(x)\|^2 \mathrm{P}_n(dx) < \infty$$

$$\mathcal{K}_n(\Delta_n) = \Delta_n$$

$$\mathcal{K}_n(\Delta_n) = \begin{pmatrix} \frac{1}{\Delta_n^{(1)}} \\ \frac{1}{\Delta_n^{(2)}} \\ \vdots \\ \frac{1}{\Delta_n^{(d)}} \end{pmatrix}$$

Let $\{\Theta_n\}$ is a sequence of convex closed sets: $\theta_\star \in \Theta_n \subset \mathrm{R}^d$ for sufficiently large $n \geq 1$

$$\begin{cases} x_n = \hat{\theta}_{n-1} + \beta_n \Delta_n \\ \\ y_n = F(w_n, x_n) + v_n \\ \\ \hat{\theta}_n = \mathcal{P}_{\Theta_n}(\hat{\theta}_{n-1} - \frac{\alpha_n}{\beta_n}\mathcal{K}_n(\Delta_n) y_n) \end{cases} \tag{5}$$

$\mathcal{P}_{\Theta_n}$ is a projector on Θ_n.

Denote $\mathbb{W} = \mathrm{supp}(\mathrm{P}_w(\cdot))$,
$\mathcal{F}_{n-1}$ — σ-algebra $\{\hat{\theta}_0, \hat{\theta}_1, \ldots, \hat{\theta}_{n-1}\}$,

$$\bar{v}_n = v_{2n} - v_{2n-1},\ \bar{w}_n = \begin{pmatrix} w_{2n} \\ w_{2n-1} \end{pmatrix},\ d_n = 1$$

for (2) or (3)

$$\bar{v}_n = v_n,\ \bar{w}_n = w_n,\ d_n = \mathrm{diam}\Theta_n \quad \text{for} \quad (5)$$

<u>**T.**</u> **1** *If the condition* (A.1) *is held for* $f(\theta)$*;*
(A.2) *for* $F(w,\cdot)$ $\forall w \in \mathbb{W}$*;*
(4) *for* $\mathcal{K}_n(\cdot)$ *and* $\mathrm{P}_n(\cdot)$*;*
$F(\cdot,\theta)$, $\nabla_\theta F(\cdot,\theta)$ *uniformly bounded on* $\mathbb{W}$*;*
$\bar{v}_1,\ldots,\bar{v}_n,\bar{w}_1,\ldots,\bar{w}_{n-1}$ *don't depend on* $\bar{w}_n,\Delta_n$, $\bar{w}_n$ *doesn't depend on* Δ_n*;*

$$\mathrm{E}\{\bar{v}_n^2\} \le \sigma_n^2.$$

If $\sum_n \alpha_n = \infty$ *and*

$$\alpha_n \to 0,\ \beta_n \to 0,\ \alpha_n^2\beta_n^{-2}(1+d_n^2+\sigma_n^2) \to 0$$

then

$$\mathrm{E}\{\|\hat{\theta}_n - \theta^\star\|^2\} \to 0 \ \text{ as } \ n \to \infty.$$

Moreover, if $\sum_n \alpha_n\beta_n^2 < \infty$ *and*

$$\sum_n \alpha_n^2\beta_n^{-2}(1+\mathrm{E}\{\bar{v}_n^2|\mathcal{F}_{n-1}\}) < \infty \ \text{ a.s.}$$

then $\hat{\theta}_n \to \theta^\star$ *as* $n \to \infty$ *a.s.*

Newton method

$$\hat{\theta}_n = \hat{\theta}_{n-1} - [\nabla^2 f(\hat{\theta}_{n-1})]^{-1} \nabla f(\hat{\theta}_{n-1})$$

$[\nabla^2 f(\hat{\theta}_{n-1})]^{-1}$ is replaced by α_n.
Let $\nabla f(\hat{\theta}_{n-1})$ be unknown.

$$\nabla f(\hat{\theta}_{n-1})^{(i)} \approx$$

$$\approx \frac{f(\hat{\theta}_{n-1} + \beta_n e_i) - f(\hat{\theta}_{n-1} - \beta_n e_i)}{2\beta_n}$$

$2d$ measurements per iteration

$\{\Delta_n\}$ — i.i.d. random vectors:

$$\mathrm{cov}\{\Delta_n \Delta_j^{\mathrm{T}}\} = \delta_{nj}\,\mathrm{I}, \quad \mathrm{E}\{\Delta_n v_n\} = 0$$

One measurement form of SPSA method

$$\begin{cases} y_n = F(w_n, \hat{\theta}_{n-1} + \beta_n \Delta_n) + v_n, \\ \hat{\theta}_n = \hat{\theta}_{n-1} - \frac{\alpha_n}{\beta_n} \Delta_n y_n \end{cases}$$

Denote $D_n = (\hat{\theta}_n - \theta)^2$

$$\mathrm{E}\{D_n|\hat{\theta}_i, i<n\} \le D_{n-1} - \frac{\alpha_n}{\beta_n}(\hat{\theta}_{n-1}-\theta)\mathrm{E}\{\Delta_n y_n|\hat{\theta}_i, i<n\}+$$

$$+\frac{\alpha_n^2}{\beta_n^2}\mathrm{E}\{\Delta_n^2 y_n^2|\hat{\theta}_i, i<n\} = D_{n-1}-$$

$$-\frac{\alpha_n}{\beta_n}(\hat{\theta}_{n-1}-\theta)\mathrm{E}\{\Delta_n f(\theta_{n-1}+\beta_n\Delta_n)|\hat{\theta}_i, i<n\}+$$

$$+\frac{\alpha_n^2}{\beta_n^2}\mathrm{E}\{y_n^2|\hat{\theta}_i, i<n\}.$$

Taylor series

$$f(\hat{\theta}_{n-1}+\beta_n\Delta_n) = f(\hat{\theta}_{n-1})+\beta_n\Delta_n\nabla f(\hat{\theta}_{n-1})+$$

$$+\beta_n^2\zeta_n,\ \zeta_n \in [\hat{\theta}_{n-1}, \hat{\theta}_{n-1}+\beta_n\Delta_n]$$

Since $\mathrm{E}\{\Delta_n\} = 0$

$$\mathrm{E}\{D_n|\hat{\theta}_i, i<n\} \le (1+\frac{1}{2}\alpha_n\beta_n)D_{n-1}-$$

$$-\alpha_n(\hat{\theta}_{n-1}-\theta)\nabla f(\hat{\theta}_{n-1}) + \xi_n,$$

$$\xi_n = \mathrm{E}\{\alpha_n\beta_n\zeta_n^2/2 + \alpha_n^2\beta_n^{-2}y_n^2|\hat{\theta}_i, i<n\}$$

$\Longrightarrow \{D_n\}$ is almost semimartingal

$$\mathrm{E}\{D_n|\hat{\theta}_i, i<n\} \le (1-\gamma_n)D_{n-1} + \xi_n,$$

$$\gamma_n = \mu\alpha_n - \alpha_n\beta_n/2$$

4 Rate of convergence

Let $\mathrm{P}_n(\cdot) = \mathrm{P}_\Delta(\cdot)$ and

$$\mathcal{K}_n(x) = \mathcal{K}(x) = (\mathcal{K}^{(1)}(x), \ldots, \mathcal{K}^{(r)}(x))^{\mathrm{T}},$$

$$\mathcal{K}^{(i)}(x) = K_0(x^{(i)}) \prod_{j \neq i} K_1(x^{(j)}) \qquad (6)$$

T. 2 *Let be* $\alpha_n = \alpha\, n^{-1}$, $\beta_n = \beta\, n^{-\frac{1}{2\gamma}}$.
If the following conditions are fulfilled:
(A.1,3) *under* $\gamma \geq 2$, $\alpha\beta > \frac{\gamma-1}{2\mu\gamma}$ *for* $f(\theta)$;
(A.2) *for* $F(w, \cdot)$ $\forall w \in \mathbb{W}$;
$F(\cdot, \theta)$, $\nabla_\theta F(\cdot, \theta)$ *uniformly bounded on* $\mathbb{W}$;

$$\int u K_0(u) \mathrm{P}_\Delta(du) = 1, \int u^k K_0(u) \mathrm{P}_\Delta(du) = 0, \ k = 0, 2, \ldots, \ell, \qquad (7)$$

$$\int K_1(u) \mathrm{P}_\Delta(du) = 1, \int u^k K_1(u) \mathrm{P}_\Delta(du) = 0, \ k = 1, \ldots, \ell - 1;$$

$\bar{w}_n, \Delta_n$ *don't depend on* $\bar{v}_1, \ldots, \bar{v}_n, \bar{w}_1, \ldots, \bar{w}_{n-1}$,
Δ_n *doesn't depend on* $\bar{w}_n$;
$\mathrm{E}\{(v_{2n} - v_{2n-1})^2/2\} \leq \sigma_2^2$, $(\mathrm{E}\{v_n^2\} \leq \sigma_1^2)$;

$$d_n n^{-1+\frac{1}{2\gamma}} \to 0 \ \text{ as } \ n \to \infty$$

then

$$\mathrm{E}\{\|\hat{\theta}_n - \theta^\star\|^2\} = \mathcal{O}(n^{-\frac{\gamma-1}{\gamma}}) \ \text{ as } \ n \to \infty.$$

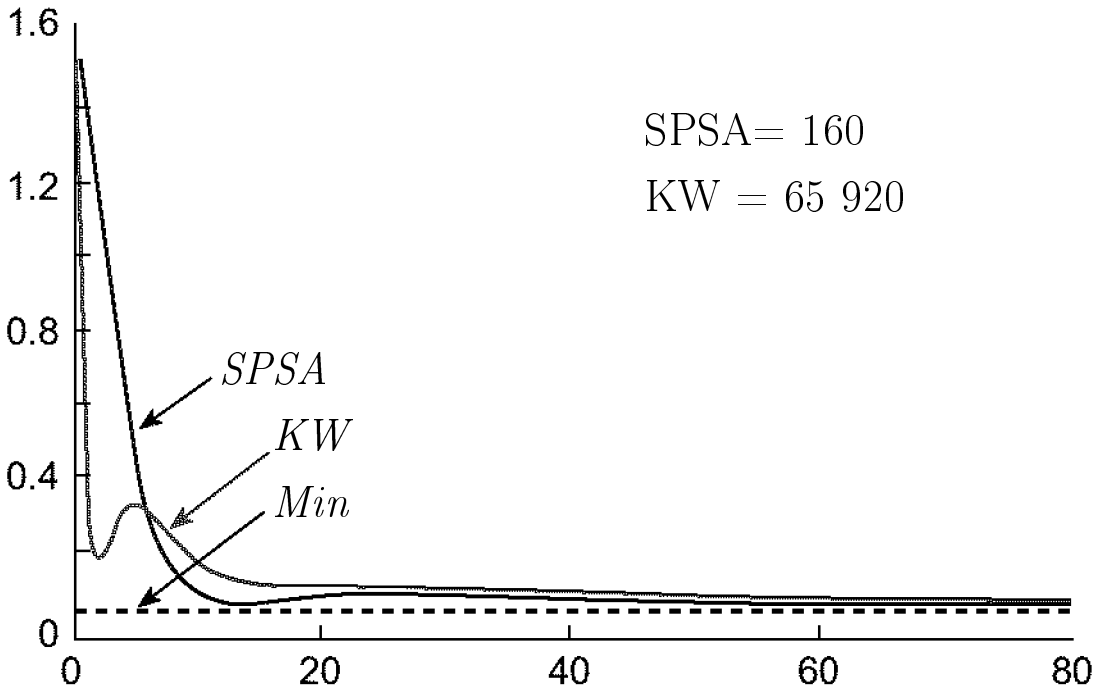
If $\{\Delta_n\}$ is uniformly distributed on $[-1/2, 1/2]$ then for $\ell = 1, 2$ (i.e. $2 \le \gamma \le 3$) we have

$K_0(u) = 12u, K_1(u) = 1,$

for $\ell = 3, 4$ (i.e. $3 < \gamma \le 5$)

$K_0(u) = 5u(15 - 84u^2),\ K_1(u) = \frac{9}{4} - 15u^2.$

Note, $K_0(u) = 0$ and $K_1(u) = 0$ as $|u| > 1/2$.



$d = 412$

5 Linear regression

$$y_n = \varphi_n^{\mathrm{T}}\theta_n^\star + v_n,\ \theta_n^\star = \theta^\star + w_n \tag{8}$$

$y_n \in \mathbb{R}^1$ — output (observation)
$\varphi_n \in \mathbb{R}^r$ — input (regressor)
$v_n \in \mathbb{R}^1,\ w_n \in \mathbb{R}^r$ — noise (disturbance)

It is required to find the unknown vector $\theta^\star$ by using the observations $y_i,\ \varphi_i,\ i \le n$.

(A) $\{\varphi_n\}$ — i.i.d., φ_n not depends with $\tilde{\mathcal{F}}_{n-1}$,
$\mathrm{E}\varphi_n$ is known: $\|\mathrm{E}\varphi_n\| \le M_\varphi < \infty$,
$\Delta_n = \varphi_n - \mathrm{E}\varphi_n$ has a symmetric distribution,

$$\mathrm{E}\Delta_n\Delta_n^T = \mathrm{B_n} > 0, \|\mathrm{B}_n\| \le \sigma_\Delta^2$$

(B) w_n doesn't depend on $\hat{\mathcal{F}}_{n-1}$ and $\mathrm{E}w_n = 0$

$(i)\quad \mathrm{E}\{\, v_n^2|\mathcal{F}_{n-1}\} \le \sigma_v^2\ \mathrm{E}\{\|w_n\|^2\} \le \sigma_w^2,$

$(ii)\quad \mathrm{E}\{v_n^2\} \le \sigma_v^2,\ \mathrm{E}\{w_n w_n^{\mathrm{T}}\} \le \mathrm{Q}_w$

$\mathcal{F}_n$ — σ-algebra
$\{\varphi_1, \ldots, \varphi_n, w_1, \ldots, w_n, v_1, \ldots, v_n\}$
$\hat{\mathcal{F}}_{n-1}\ \{\varphi_1, \ldots, \varphi_{n-1}, w_1, \ldots, w_{n-1}, v_1, \ldots, v_n\}$
$\tilde{\mathcal{F}}_{n-1}\ \{\varphi_1, \ldots, \varphi_{n-1}, w_1, \ldots, w_n, v_1, \ldots, v_n\}$
$\mathcal{F}_{n-1} \subset \hat{\mathcal{F}}_{n-1} \subset \tilde{\mathcal{F}}_{n-1} \subset \mathcal{F}_n\,.$

$$\hat{\theta}_n = \hat{\theta}_{n-1} - \alpha_n \Gamma \Delta_n (\varphi_n^{\mathrm{T}} \hat{\theta}_{n-1} - y_n) \qquad (9)$$

<u>**T.**</u> **3** *Let Assumption* (A) *be fulfilled and*

$$\alpha_n \to 0, \ \sum_{n=1}^{\infty} \alpha_n \ = \ \infty.$$

If (Bi) *and* $\sum \alpha_n^2 < \infty$
then

$$\hat{\theta}_n \to \theta^\star \ \text{ a.s.}$$

If (Bii) **then**

$$\mathrm{E}(\hat{\theta}_n - \theta^\star)(\hat{\theta}_n - \theta^\star)^{\mathrm{T}} \to 0$$

If $\Gamma = \mathrm{B}^{-1}$ then (9) = (2) and

$$F(w, x) = \frac{1}{2}(x - \theta^\star - w)^{\mathrm{T}}(x - \theta^\star - w)$$

<u>T.</u> 4 *If* (A) *and* (Bii) *are fulfilled,* $\alpha_n = n^{-1}$, $-\Gamma\mathrm{B} + \frac{1}{2}\mathrm{I}$ *is a Hurwitz matrix*

then

$$\mathrm{E}\{(\hat{\theta}_n - \theta^\star)(\hat{\theta}_n - \theta^\star)^{\mathrm{T}}\} \leq n^{-1}\,\mathrm{S} \;+\; o(n^{-1}),$$

where S *is a solution of matrix equation*

$$\Gamma\,\mathrm{B}\,\mathrm{S} \;+\; \mathrm{S}\,\mathrm{B}\,\Gamma \;-\; \mathrm{S} \;=\; \Gamma\mathrm{R}\Gamma \qquad (12)$$

$$\mathrm{R} \;=\; (\sigma_v^2(1 + M_\varphi^2\rho) + M_\varphi^2\mathrm{Tr}[\mathrm{Q}_w])\mathrm{B} + \\ + \mathrm{E}\{\Delta_n\Delta_n^{\mathrm{T}}\mathrm{Q}_w\Delta_n\Delta_n^{\mathrm{T}}\}$$

$\rho > 0$ *is any small positive constant*

If $\Gamma \;=\; \mathrm{B}^{-1}$, $\mathrm{Tr}[\mathrm{Q}_w] = 0$, $M_\varphi = 0$ then for

$$\hat{\theta}_n \;=\; \hat{\theta}_{n-1} \;-\; (n\mathrm{B})^{-1}\varphi_n(\varphi_n^{\mathrm{T}}\hat{\theta}_{n-1} - y_n)$$

we have

$$\mathrm{E}\{(\hat{\theta}_n - \theta^\star)(\hat{\theta}_n - \theta^\star)^{\mathrm{T}}\} \leq n^{-1}\sigma_v^2\mathrm{B}^{-1} + o(n^{-1})$$

$$\begin{cases} \hat{\theta}_n = \hat{\theta}_{n-1} - \Gamma_n \Delta_n (\varphi_n^{\mathrm{T}} \hat{\theta}_{n-1} - y_n) \\ \Gamma_n = \Gamma_{n-1} - \frac{\Gamma_{n-1} \Delta_n \Delta_n^{\mathrm{T}} \Gamma_{n-1}}{1 + \Delta_n^{\mathrm{T}} \Gamma_{n-1} \Delta_n} \end{cases} \tag{14}$$

<u>T.</u> 5 *Let* (A) *be fulfilled.*

If (Bi)

then

$$\hat{\theta}_n \to \theta^\star.$$

If $|v_n| \leq C_v,\ \|w_n\| \leq C_w,\ \|\Delta_n\| \leq C_\Delta$

then

$$\mathrm{E}\{(\hat{\theta}_n - \theta^\star)(\hat{\theta}_n - \theta^\star)^{\mathrm{T}}\} \to 0.$$

6 Filtering

$$y_n = \varphi_n^{\mathrm{T}} \theta_n^\star + v_n \tag{15}$$

$$\theta_{n+1}^\star = \mathrm{A}\theta_n^\star + w_{n+1} \tag{16}$$

$\|\mathrm{A}\| < 1$, w_n doesn't depend on $\mathcal{F}_{n-1}$
$\{w_n\}$ and $\{v_n\}$ satisfy conditions (Bii)

$$\mathrm{E}\|\hat{\theta}_{n+1} - \theta_{n+1}^\star\|^2 \to \min$$

$\{\varphi_n\}$ — random and (A)

Randomized least mean squares algorithm

$$\hat{\theta}_{n+1} = \mathrm{A}\hat{\theta}_n - \alpha \mathrm{A}\Gamma\Delta_n(\varphi_n^{\mathrm{T}}\hat{\theta}_n - y_n) \tag{17}$$

$$\Delta_n = \varphi_n - \mathrm{E}\{\varphi_n\}$$

$$\alpha > 0 \text{ step} - \text{size}, \ \ \Gamma > 0$$

Substitute (15) and (16) into (17):

$$\hat{\theta}_{n+1} - \theta^\star_{n+1} = \mathrm{A}(\mathrm{I} - \alpha\Gamma\Delta_n\Delta_n^{\mathrm{T}})(\hat{\theta}_n - \theta^\star_n) -$$
$$-\alpha\mathrm{A}\Gamma\Delta_n(\mathrm{E}\{\varphi_n\}^{\mathrm{T}}(\hat{\theta}_n - \theta^\star_n) - v_n) - w_{n+1}.$$

Successively take the conditional expectations

$$\mathrm{E}\{\|\hat{\theta}_{n+1} - \theta^\star_{n+1}\|^2|\tilde{\mathcal{F}}_{n-1}\} \le (1 - 2\alpha\lambda_{\min}(\mathrm{B}\Gamma) +$$
$$+\alpha^2\|\Gamma\|^2 M_4^4)\|\mathrm{A}\|^2\|\hat{\theta}_n - \theta^\star_n\|^2 +$$
$$+\alpha^2(\mathrm{E}\{\varphi_n\}^{\mathrm{T}}(\hat{\theta}_n - \theta^\star_n) - v_n)^2\|\Gamma\|^2\mathrm{Tr}[\mathrm{B}] + \mathrm{Tr}[\mathrm{Q}_w].$$

Taking the unconditional expectation we obtain with any $\rho > 0$

$$\mathrm{E}\{\|\hat{\theta}_{n+1} - \theta^\star_{n+1}\|^2\} \le b(\alpha,\rho)\mathrm{E}\{\|\hat{\theta}_n - \theta^\star_n\|^2\} +$$
$$+\alpha^2(1 + M_\varphi\rho)\|\Gamma\|^2\mathrm{Tr}[\mathrm{B}]\sigma_v^2 + \mathrm{Tr}[\mathrm{Q}_w],$$

where

$$b(\alpha,\rho) = (1 - 2\alpha\lambda_{\min}(\mathrm{B}\Gamma) + \alpha^2\|\Gamma\|^2 M_4)\|\mathrm{A}\|^2 + \tag{18}$$
$$+\alpha^2(M_\varphi + \frac{1}{\rho})M_\varphi\|\Gamma\|^2\mathrm{Tr}[\mathrm{B}].$$

<u>T.</u> 6 *If* (A) *and* (Bii) *are fulfilled*
then $\forall\rho > 0$ *and small* α*:* $b(\alpha,\rho) < 1$

$$\mathrm{E}\{\|\hat{\theta}_{n+1} - \theta^\star_{n+1}\|^2\} \le$$
$$\le \frac{\mathrm{Tr}[\mathrm{Q}_w] + \alpha^2(1 + M_\varphi\rho)\|\Gamma\|^2\mathrm{Tr}[\mathrm{B}]\sigma_v^2}{1 - b(\alpha,\rho)} +$$
$$+b(\alpha,\rho)^n\mathrm{E}\{\|\hat{\theta}_0 - \theta^\star_0\|^2\}$$

$\Gamma = \mathrm{B}^{-1}, \|\mathrm{A}\|^{-2} = 1 + \mathcal{O}(\alpha^3)$

Denote

$$r(\rho) = \frac{M_4 + (M_\varphi + 1/\rho) M_\varphi \mathrm{Tr}[\mathrm{B}]}{2\lambda^2_{\min}(\mathrm{B})}$$

For sufficiently small α

$$\mathrm{E}\{\|\hat{\theta}_{n+1} - \theta^\star_{n+1}\|^2\} \leq D(\alpha, \rho) + \mathcal{O}(\alpha^2),$$

where

$$D(\alpha, \rho) = \frac{1}{2}\mathrm{Tr}[\mathrm{Q}_w](\frac{1}{\alpha} + r(\rho) + \tag{19}$$

$$+ \left(r(\rho)^2 + \frac{(1 + M_\varphi \rho)\mathrm{Tr}[\mathrm{B}]\sigma_v^2}{\mathrm{Tr}[\mathrm{Q}_w]\lambda^2_{\min}(\mathrm{B})} \right) \alpha)$$

If $M_\varphi = 0$ then we get

$$\alpha^\star = 2\lambda^2_{\min}(\mathrm{B}) \sqrt{\frac{\mathrm{Tr}[\mathrm{Q}_w]}{M_4^2 \mathrm{Tr}[\mathrm{Q}_w] + 2\lambda^2_{\min}(\mathrm{B})\mathrm{Tr}[\mathrm{B}]\sigma_v^2}}$$

Let be $d = 1$ and $\sigma_w^2 = \mathrm{Tr}[\mathrm{Q}_w] << \sigma_v^2$

$$\mathrm{Prob}\{\varphi_n = \bar{\varphi}\} = \mathrm{Prob}\{\varphi_n = -\bar{\varphi}\} = 1/2$$

We have

$$\alpha^\star \mathrm{A}\Gamma \approx \sigma_w / |\bar{\varphi}| \sigma_v$$

$\{\varphi_n\}$ are uniformly distributed on $[0.5, 1.5]$
$\{v_n\}$: $|v_n| \leq 2$

$$\theta^\star_{n+1} = 0.9999\theta^\star_n + w_{n+1}, \theta^\star_1 = 0$$

$\{w_n\}$ are uniformly distributed on $[-\frac{1}{3}, \frac{1}{3}]$

$$\tilde{D}(\{\hat{\theta}_n\}) = \frac{1}{199}\sum_{n=1}^{199} \|\hat{\theta}_{n+1} - \theta^\star_{n+1}\|^2$$

$$\alpha^\star\Gamma = 0.2371, D(\alpha^\star, \rho^\star) = 1.3699, \rho^\star = 0.269$$

$$\Delta_n = \varphi_n - 0.1$$

$$\hat{\theta}_{n+1} = 0.9999(\hat{\theta}_n - 0.2371\Delta_n(\varphi_n\hat{\theta}_n - y_n)),$$

$$\hat{\theta}_{n+1} = 0.9999(\hat{\theta}_n - 0.2371\varphi_n(\varphi_n\hat{\theta}_n - y_n)),$$

$$\hat{\theta}_{n+1} = 0.9999\hat{\theta}_n - k_n\varphi_n(\varphi_n\hat{\theta}_n - y_n),$$

$$k_n = \frac{0.9999\gamma_{n-1}}{\frac{16}{3} + \gamma_{n-1}\phi_n^2},$$

$$\gamma_n = \gamma_{n-1}0.9999^2 - \frac{\phi_n^2\gamma_{n-1}^2}{\frac{16}{3} + \gamma_{n-1}\varphi_n^2} + \frac{2}{81}, \ \gamma_0 = 0.$$

7 Quantum computer and randomized algorithms

The Quantum Circuit Model

Classical computer treats bits $\{0, 1\}$.
Quantum computer treats the quantum bits.
Qubit is a two–state microscopic system (an atom or nuclear spin or polarized photon).
It is a unit vector in $\mathbb{C}^2$ with inner product.
Denote base vectors of this space $|0\rangle$, $|1\rangle$.

The quantum state space of k qubits systems is the complex projective space $\mathbb{C}^{2^k}$.

$$|b_1 b_2 \ldots b_k\rangle = |b_1\rangle \otimes |b_2\rangle \otimes \ldots \otimes |b_k\rangle$$

$$W = \sum_s \psi_s |s\rangle,$$

W — a superposition of $|s\rangle$,

$$\psi_s \in \mathbb{C}, \ \sum_s |\psi_s|^2 = 1,$$

$|\psi_s|$ — probabilistic amplitudes.

The projection on some V is an outcome

$$\langle V, W\rangle$$

Quantum parallelism

A black box: $\{0,1\} \to \{0,1\}$.

Computation takes 24 hours.

$$f(x) : \mathbb{Z}^2 \to \mathbb{Z}^2$$

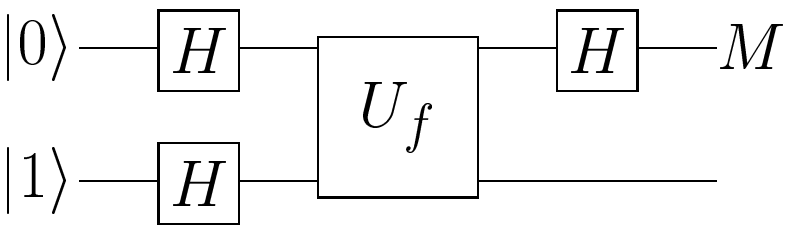
What time is needed to $M = f(0) \oplus f(1)$?

A quantum invertible black box

$$U_f : \ |x\rangle|y\rangle \to |x\rangle|y \oplus f(x)\rangle$$

Hadamard transform

$$H : |x\rangle \to \frac{1}{\sqrt{2}} \sum_y (-1)^{xy} |y\rangle, \ \ H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



Quantum Fourier transform

$$QFT : \ |x\rangle \to \frac{1}{\sqrt{N}} \sum_y e^{2\pi xy/N} |y\rangle =$$

$$= \frac{1}{\sqrt{2^n}} \left(|0\rangle + e^{2\pi(.x_0)}|1\rangle\right) \left(|0\rangle + e^{2\pi(.x_1 x_0)}|1\rangle\right) \cdots \left(|0\rangle + e^{2\pi(.x_{n-1} x_{n-2} \ldots x_0)}|1\rangle\right)$$

Efficient quantum algorithms

1. Quantum Fourier transform

Factoring is an intractable problem:

- The solution can be easily verified.
- But the solution is hard to find.

If p and q are large prime numbers then

$$n = pq$$

can be computed quickly $\log_2 p \cdot \log_2 q$.
But given n, it is hard to find p and q.

The "number field sieve" algorithm requires

$$time \approx \exp\{(64/9)^{\frac{1}{3}}(\ln n)^{\frac{1}{3}}(\ln \ln n)^{\frac{2}{3}}\}$$

P. Shor $time \approx \mathcal{O}[(\ln n)^3]$

2. Quantum database search

Grover's algorithm $time \approx \frac{\pi}{4}\sqrt{N}$.

3. Quantum bit teleportation

4. Approximation of the gradient of a multivariable function (SPSA iteration)?

Approximation of a Function Gradient

$$U_f : \mathbb{C}^{2^k} \to \mathbb{C}^{2^k}, \quad k = p \times d$$

$$\forall x \in \mathrm{R} \;\; s_x = \overline{b_x^{(1)} \dots b_x^{(p)}}$$

It maps

$$|s_{x^{(1)}} \dots s_{x^{(d)}}\rangle \to |s_{f(X)} 00 \dots 0\rangle = U_f |s_{x^{(1)}} \dots s_{x^{(d)}}\rangle$$

$\mathcal{I}$ — *input*
W — *worker*
Δ — *simultaneous perturbation*

Sum U_{+R}

Turn of the first qubits $U_{R_{1,p+1,\dots,(d-1)p+1}}$

Shift by j qubits U_{S_j}

1. Send zero to $\mathcal{I}, W, \Delta$.
2. $\mathcal{I} := |s_x\rangle, \; \Delta := U_{R_{1,p+1,\dots,(d-1)p+1}} \Delta$.
3. $W := U_f U_{+\mathcal{I}} U_{S_j} U_{+\Delta} W$,
4. To measure the outcome of of calculations

$$\hat{g}^{(i)}(X) = \langle U_{S_{-(i-1)p}} \Delta, W \rangle, \; i = 1, 2, \dots, d$$

Creating the Intelligent System

$$\{T_1, T_2, \ldots, T_m\} \leftrightarrow \{D_1, D_2, \ldots, D_m\}$$
$$D_i = D_i(\theta^{(i)}),\ \theta^{(i)} \in \Theta^{(i)}$$

- $\forall P_i$ the system has D_i which is able to optimize the solving of T_i by an appropriate choosing $\theta^{(i)}$
- information from external world must be delivered to all such devices simultaneously

Performance indexes of data w processing

$$f_i = f_i(w, \theta^{(i)}),\ i = 1, 2, \ldots, m$$

$$\theta = \begin{pmatrix} \theta^{(1)} \\ \theta^{(2)} \\ \vdots \\ \theta^{(m)} \end{pmatrix},\ \theta \in \Theta = \Theta^{(1)} \otimes \cdots \otimes \Theta^{(m)}$$

$$F(w, \theta) = -\sum_{i=1}^{m} f_i(w, \theta^{(i)})$$

Informational resonance

- only one device has resonance,
- several devices have resonance,
- none of the devices has resonance.

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