Rare event probability estimation

(Brescia, May 2004)

Abstract — **The paper is devoted to the estimation of the rare event probability.**

Index Terms— **rare event, simulation, splitting method, probability estimation, sampling per mode.**

1 INTRODUCTION

2 PROBLEM FORMULATION AND NOTATIONS

Consider the initial set $S \subset \mathbb{X}$ and target set $A \subset \mathbb{X}$. Suppose that we can simulate trajectories which begin from S, and for the each trajectory the probability to reach the set A is P (e.g. $P = 1.21 \cdot 10^{-8}$).

For the given level of error ϵ (e.g. $\epsilon = 10^{-9}$) the objective is to find an estimate $\hat{P}$ of P, i.e.

$$\mathrm{P}\{|\hat{P} - P| \geq \epsilon\} \leq \epsilon.$$

Consider the standard Monte-Carlo method. Let's we have N trajectories which start from S with some initial distribution P_0. For each of them we define the random values $\hat{P}_i,\ i = 1, \ldots, N$ which are i.i.d. and equal to 1 with probability P or 0 with probability $1 - P$ depending on trajectory achievement of set A or not. We can use the estimate

$$\hat{P} := \frac{1}{N} \sum_{i=1}^{N} \hat{P}_i.$$

By Hoeffding's inequality we need to simulate

$$N \geq \frac{\ln 2 - \ln \epsilon}{2\epsilon^2}$$

trajectories then the level of estimation error of $\hat{P}$ is less then ϵ, e.g.

$$\text{if } \epsilon = 10^{-3} \text{ then } N = 3\,800\,500,$$

$$\text{if } \epsilon = 10^{-4} \text{ then } N = 4.9517 \cdot 10^{8},$$

$$\text{if } \epsilon = 10^{-9} \text{ then } N = 1.0708 \cdot 10^{19}.$$

In the last case there is no real possibility to simulate this huge amount of trajectories. But our objective is the same: we need to estimate P with the given level of error ϵ. To solve the problem we need to made additional assumptions.

Let $\{\Omega, \mathcal{F}, \mathrm{P}\}$ be a probability space, Ω be a set of all possible trajectories ω beginning from S, $\mathcal{F}$ be a sigma-algebra of all Borel sets of Ω,
$B_0 = \{S\} = \{b_{0,k},\ k \in I_X\},\ I_X = 1,\ b_{0,1} = S,\ M \geq 0,\ B_{M+1} = A = \{b_{M+1,1}\}$
and for $j = 1, \ldots, M$

$$B_j = \{b_{j,k} : k \in I_j,\ |I_j| < \infty,\ b_{j,k} \in \mathbb{X},\ b_{j,k} \cap_{k' \in I_j, k' \neq k} b_{j,k'} = \emptyset,\ b_{j,k} \cap \cup_{l \in I_{j+1}} b_{j+1,l} = \emptyset,$$

$\forall\ \omega$ which achieve some point $\delta_{j+1,k} \in \cup_{l \in I_{j+1}} b_{j+1,l}$ cross before some point $\delta_{j,k} \in \cup_{l \in I_j} b_{j,l}\}$.
Note, for simplification one can consider the case when $b_{j,k} = \{\delta_{j,k}\},\ \forall j, k$.

Let's $j \leq M + 1$. Denote

$$s_{k_1 \ldots k_j} = \{\omega :\ \forall i = 1, \ldots, j\ \exists \delta_{i,k_i} \in \omega \cap b_{i,k_i}\},$$

$$p_{k_1 \ldots k_j} = \begin{cases} 0,\ \text{if } \mathrm{P}(s_{k_1 \ldots k_j}) = 0, \\ \mathrm{P}(\cup_{k_{j+1} \in I_{j+1}} \cdots \cup_{k_{M+1} \in I_{M+1}} s_{k_1 \ldots k_j, k_{j+1} \ldots k_{M+1}} | s_{k_1 \ldots k_j}),\ \text{if } \mathrm{P}(s_{k_1 \ldots k_j}) > 0 \end{cases}$$

$$S_j = \cup_{k_1 \in I_1} \cdots \cup_{k_j \in I_j} s_{k_1 \ldots k_j}.$$

and $\forall j > 1$

$$P_{j/j-1} = \max_{k_1 \in I_1} \ldots \max_{k_{j-1} \in I_{j-1}} \sum_{k_j \in I_j,\, P(s_{k_1 \ldots k_{j-1}}) > 0} P(s_{k_1 \ldots k_j}) / P(s_{k_1 \ldots k_{j-1}}),$$

$$P_{1/0} = \sum_{k_1 \in I_1,\, P(s_{k_1}) > 0} P(s_{k_1}),$$

$$P_{j,\min} = \min\{P(s_{k_1 \ldots k_j}) / P(s_{k_1 \ldots k_{j-1}}) :\ P(s_{k_1 \ldots k_j}) > 0,\ k_1 \in I_1, \ldots, k_j \in I_j\},$$

$$P_{1,\min} = \min\{P(s_{k_1}) :\ P(s_{k_1}) > 0,\ k_1 \in I_1\},$$

$$\bar{P}_j = P_{j,\min} / \max\{P(s_{k_1 \ldots k_j}) / P(s_{k_1 \ldots k_{j-1}}) :\ k_1 \in I_1, \ldots, k_j \in I_j\},$$

$$p_{1,\max} = P_{1,\min} / \max\{P(s_{k_1}) :\ k_1 \in I_1\},$$

$$p_{j,\min} = \min\{p_{k_1 \ldots k_j} :\ k_1 \in I_1, \ldots, k_j \in I_j,\ p_{k_1 \ldots k_j} > 0\}\},$$

$$\bar{p}_j = p_{j,\min} / \max\{p_{k_1 \ldots k_j} :\ k_1 \in I_1, \ldots, k_j \in I_j\}.$$

Note that

$$p_{k_1 \ldots k_j} = \mathrm{P}\{\omega :\ \omega \in s_{k_1 \ldots k_j} \text{ and trajectory } \omega \text{ achive set } A\}$$

$$P = \sum_{k_1 \in I_1} \mathrm{P}(s_{k_1}) p_{k_1} = \sum_{k_1 \in I_1} \mathrm{P}(s_{k_1}) \sum_{k_2 \in I_2} (\mathrm{P}(s_{k_1 k_2}) / \mathrm{P}(s_{k_1})) p_{k_1 k_2} = \ldots.$$

MAIN ASSUMPTION

(A) if $P(s_{k_1 \ldots k_j}) > 0\ j > 1,\ k_1 \in I_1,\ \ldots, k_j \in I_j,\ \Delta',\ \Delta'' \subset s_{k_1 \ldots k_j} :\ \mathrm{P}(\Delta') = \mathrm{P}(\Delta'') > 0$

then $\forall \bar{\Delta} \subset s_{k_1 \ldots k_{j-1}} :\ \mathrm{P}(\bar{\Delta}) > 0$

$$\mathrm{P}(\Delta' | \bar{\Delta}) = \mathrm{P}(\Delta'' | \bar{\Delta}).$$

3 AGORITHM AND UNBIASEDNESS

ALGORITHM OF SIMULATION

1. Let's start $N > 0$ trajectories from S with uniform distribution, $M \geq 0,\ R_j > 0,\ R_j \in \mathbb{N},\ j = 1, \dots, M.$
2. $j = 1.$
3. To kill all trajectories which $\notin S_j$.
4. Each of all rest trajectories ω_t cross the set B_j in some point $\delta_{j,k_t} \in b_{j,k_t}$. For all ω_t splitting $R_j - 1$ times the trajectory ω_t uniformly on b_{j,k_t}. We have got the new $R_j - 1$ trajectories for each of all rest trajectories.
5. If $j < M$ then $j = j + 1$ and GOTO step 3.

ALGORITHM OF ESTIMATION

$$\hat{P} = \frac{N_A}{NR_1 \cdot R_M}$$

where N_A is equal to all number of trajectories which achieve the set A.

Note that by the algorithm of simulation

$$\hat{P} = \frac{1}{N} \sum_{i=1}^{N} \hat{P}_i$$

where $\hat{P}_i$ for each of $N > 0$ starting trajectories i is a fraction of the number of its sub-trajectories (include someself) which achieve the set A to the all number of its sub-trajectories. Note, if $\hat{P}_i > 0$ then

$$\hat{P}_i = \frac{1}{R_1} \sum_{l=1}^{R_1} \hat{P}_{il},$$

where $\hat{P}_{il}$ are equal to 1 or 0 depending on the achievement l-th sub-trajectory of set A or not.

As done in [1] $\hat{P}$ is unbiased since

$$E(\hat{P}) = E\left(\frac{N_A}{NR_1 \cdot R_M}\right) = \frac{1}{NR_1 \cdot R_M} \sum_{k_0=1}^{N} \sum_{k_1=1}^{R_1} \cdot \sum_{k_M=1}^{R_M} E(\mathbf{1}_{k_0}\mathbf{1}_{k_0k_1} \dots \mathbf{1}_{k_0 \dots k_M}) = P.$$

4 ESTIMATION ERROR, CASE $M = 1$

First consider the special case $M = 1$.

In this case we have

$$B_1 = \{b_k,\ k \in I_1\},$$

$$P = \sum_{k \in I_1} \mathrm{P}(s_k) p_k \quad \text{and} \quad P_1 = P_{1/0} = \sum_{k \in I_1} \mathrm{P}(s_k).$$

Note that

$$\mathrm{P}(s_k) = \mathrm{P}\{\omega :\ \omega \text{ cross } B_1 \text{ first time in the point } \delta_k \in b_k\}.$$

Let $\bar{\Omega}_0 = \{\omega_t,\ t = 1, \ldots, N\}$ be a set of our N initial trajectories, $T_k,\ k \in I_1$ be the set of indexes t of trajectories which cross B_1 first time in the point $\delta_k \in b_k$. Denote $N_k = |T_k|,\ k \in I_1$. N_k is a random variable.

Lemma 1: Let's $\beta > 0$.

$$\mathrm{P}\{|N_k - N\mathrm{P}(s_k)| \geq \beta N\} \leq 2e^{-2N\beta^2}, \quad \forall\, k \in I_1.$$

Proof: Let's $v_{tk},\ t = 1, \ldots, N,\ k \in I_1$, are random values which are equal 1 when $t \in T_k$ or 0 when $t \notin T_k$. $\forall k\ \ \{v_{tk}\}_{t=1}^N$ i.i.d. and $\mathrm{P}\{v_{tk} = 1\} = \mathrm{P}(s_k)$. Hence by Hoeffding's inequality we have

$$\mathrm{P}\{|\frac{1}{N}\sum_{t=1}^{N} v_{tk} - \mathrm{P}(s_k)| \geq \beta\} \leq 2e^{-2N\beta^2}$$

but by definition

$$\sum_{t=1}^{N} v_{tk} = \sum_{t \in T_k} v_{tk} = N_k.$$

Lemma 2: Let's $\beta > 0,\ \gamma > 0,\ \bar{P} > 0,\ \bar{T}$ is a random subset of $\{1, 2, \ldots, N\}$, $\hat{P}_{il},\ i \in \bar{T}, l = 1, \ldots, R_1$ are conditionally on $\bar{T}$ i.i.d. random values which are equal to 1 with probability p or 0 with probability $1 - p$,

$$\bar{S} = \sum_{i \in \bar{T}} \sum_{l=1}^{R_1} \hat{P}_{il}.$$

A: If $\gamma \bar{P} \geq 3\beta$ then

$$\mathrm{P}\left\{ \frac{|\frac{1}{N\bar{P}R_1}\bar{S} - p|}{p} \geq \gamma \,\Big|\, |\frac{|\bar{T}|}{N} - \bar{P}| \leq \beta \right\} \leq 2e^{-2R_1 N \frac{p^2\gamma\bar{P}}{1+\bar{P}/\beta}}.$$

B: If $\gamma \bar{P} \geq 4\beta$

$$\mathrm{P}\left\{ \frac{|\frac{1}{N\bar{P}R_1}\bar{S} - p|}{p} \geq \gamma \,\Big|\, |\frac{|\bar{T}|}{N} - \bar{P}| \leq \beta \right\} \leq 2e^{-R_1 N \frac{p^2\gamma^2\bar{P}^2}{\frac{1}{4}\gamma\bar{P}+\bar{P}}} \leq 2e^{-4R_1 N \frac{p^2\gamma^2\bar{P}}{\gamma+4}}.$$

Proof:

$$\mathrm{P}\{|\frac{1}{N\bar{P}R_1}\bar{S}-p|\geq\gamma p\Big|\,|\frac{|\bar{T}|}{N}-\bar{P}|\leq\beta\}=\sum_{T_k:|\frac{|T_k|}{N}-\bar{P}|\leq\beta}\mathrm{P}\{T_k\}\mathrm{P}\{|\frac{1}{N\bar{P}R_1}\sum_{i\in T_k}\sum_{l=1}^{R_1}\hat{P}_{il}-p|\geq\gamma p\Big|\,T_k\}.$$

Note that

$$\sum_{T_k:||T_k|-N\bar{P}|\leq\beta N}\mathrm{P}\{T_k\}\leq 1.$$

Since

$$\mathrm{P}\{|\frac{1}{N\bar{P}R_1}\bar{S}-p|\geq\gamma p\Big|\,\bar{T}\}=\mathrm{P}\{\frac{1}{N\bar{P}R_1}\bar{S}-p\geq\gamma p\Big|\,\bar{T}\}+\mathrm{P}\{\frac{1}{N\bar{P}R_1}\bar{S}-p\leq-\gamma p\Big|\,\bar{T}\}.$$

we consider this two items separately.
First item:

$$\mathrm{P}\{\frac{1}{N\bar{P}R_1}\bar{S}-p\geq\gamma p\Big|\,\bar{T}\}=\mathrm{P}\{\frac{1}{R_1}\bar{S}-pN\bar{P}\geq p\gamma N\bar{P}\Big|\,\bar{T}\}=$$

$$=\mathrm{P}\{\frac{1}{R_1}\bar{S}-p|\bar{T}|\geq p\gamma N\bar{P}+p(N\bar{P}-|\bar{T}|)\Big|\,\bar{T}\}\leq$$

$$\mathrm{P}\{\frac{1}{|\bar{T}|R_1}\bar{S}-p\geq p\frac{\gamma N\bar{P}+(N\bar{P}-|\bar{T}|)}{|\bar{T}|}\Big|\,\bar{T}\}\leq$$

by Hoeffding's inequality

$$\leq e^{-2|\bar{T}|R_1(p\frac{\gamma N\bar{P}+(N\bar{P}-|\bar{T}|)}{|\bar{T}|})^2}\leq e^{-2R_1p^2\gamma N\bar{P}\frac{\gamma N\bar{P}-2|N\bar{P}-|\bar{T}||}{|\bar{T}|}}.$$

Second item:

$$\mathrm{P}\{\frac{1}{N\bar{P}R_1}\bar{S}-p\leq-\gamma p\Big|\,\bar{T}\}=\mathrm{P}\{\frac{1}{R_1}\bar{S}-pN\bar{P}\leq-p\gamma N\bar{P}\Big|\,\bar{T}\}=$$

$$=\mathrm{P}\{\frac{1}{R_1}\bar{S}-p|\bar{T}|\leq-p\gamma N\bar{P}+p(N\bar{P}-|\bar{T}|)\Big|\,\bar{T}\}\leq$$

$$\mathrm{P}\{\frac{1}{|\bar{T}|R_1}\bar{S}-p\leq p\frac{-\gamma N\bar{P}+(N\bar{P}-|\bar{T}|)}{|\bar{T}|}\Big|\,\bar{T}\}\leq$$

by Hoeffding's inequality and condition of Theorem

$$\leq e^{-2|\bar{T}|R_1(p\frac{-\gamma N\bar{P}+(N\bar{P}-|\bar{T}|)}{|\bar{T}|})^2}\leq e^{-2R_1p^2\gamma N\bar{P}\frac{\gamma N\bar{P}-2|N\bar{P}-|\bar{T}||}{|\bar{T}|}}.$$

Thus we have

$$A:\mathrm{P}\{|\frac{1}{N\bar{P}R_1}\bar{S}-p|\geq\gamma p\Big|\,|\frac{|\bar{T}|}{N}-\bar{P}|\leq\beta\}\leq 2e^{-2R_1p^2\gamma N\bar{P}\frac{\beta N}{N\bar{P}+\beta N}}\leq 2e^{-2R_1N\frac{p^2\gamma\bar{P}}{1+\bar{P}/\beta}},$$

$$B: \mathrm{P}\{|\frac{1}{N\bar{P}R_1}\bar{S}-p| \geq \gamma p \Big| \, |\frac{|\bar{T}|}{N}-\bar{P}| \leq \beta\} \leq 2e^{-2R_1 p^2 \gamma N\bar{P}\frac{\frac{1}{2}\gamma N\bar{P}}{N\bar{P}+\beta N}} \leq 2e^{-R_1 N\frac{p^2\gamma^2\bar{P}^2}{\frac{1}{4}\gamma\bar{P}+\bar{P}}} \leq 2e^{-4R_1 N\frac{p^2\gamma^2\bar{P}}{\gamma+4}}.$$

Theorem 1: If Assumption **(A)** holds and

$$A: \; N \geq 9\frac{\ln 2 - \frac{1}{2}\ln\epsilon}{\alpha^2\bar{p}_1^2 P_{1,\min}^2} \quad \text{and} \quad NR_1 \geq \frac{(\ln 2 - \frac{1}{2}\ln\epsilon)(\alpha\bar{p}_1 + 3\bar{P}_1^{-1})}{\alpha^2\bar{p}_1^2 P_{1,\min} p_{1,\min}^2}$$

or

$$B: \; N \geq 8\frac{2\ln 2 - \ln\epsilon}{\alpha^2\bar{p}_1^2 P_{1,\min}^2} \quad \text{and} \quad NR_1 \geq \frac{(2\ln 2 - \ln\epsilon)(\alpha\bar{p}_1 + 4)}{4\alpha^2\bar{p}_1^2 P_{1,\min} p_{1,\min}^2}$$

then

$$\mathrm{P}\{|\frac{\hat{P}-P}{P}| \geq \alpha\} \leq \epsilon.$$

Remind that

$$P_{1,\min} = \min\{P(s_{k_1}): \; P(s_{k_1}) > 0, \; k_1 \in I_1\}, \; \bar{P}_1 = P_{1,\min}/\max\{P(s_{k_1}): \; k_1 \in I_1\},$$

$$p_{1,\min} = \min\{p_{k_1}: \; k_1 \in I_1, \; p_{k_1} > 0\}\}, \; \bar{p}_1 = p_{1,\min}/\max\{p_{k_1}: \; k_1 \in I_1\}.$$

Proof:

$$\mathrm{P}\{|\frac{\hat{P}-P}{P}| \geq \alpha\} = \mathrm{P}\{|\frac{1}{N}\sum_{i=1}^{N}\hat{P}_i - P| \geq P\alpha\} = \mathrm{P}\{|\frac{1}{N}\sum_{i=1}^{N}\hat{P}_i - \sum_{k\in I_1}\mathrm{P}(s_k)p_k| \geq P\alpha\} =$$

$$= \mathrm{P}\{|\sum_{k\in I_1}\mathrm{P}(s_k)\left(\frac{1}{N\mathrm{P}(s_k)}\sum_{i\in T_k}\hat{P}_i - p_k\right)| \geq P\alpha\} \leq \mathrm{P}\{P_1\max_{k\in I_1}|\frac{1}{N\mathrm{P}(s_k)}\sum_{i\in T_k}\hat{P}_i - p_k| \geq P\alpha\} \leq$$

$$\leq \mathrm{P}\{\max_{k\in I_1}\frac{1}{p_{\max}}|\frac{1}{N\mathrm{P}(s_k)}\sum_{i\in T_k}\hat{P}_i - p_k| \geq \frac{P}{P_1 p_{\max}}\alpha\} \leq \mathrm{P}\{\max_{k\in I_1}\frac{1}{p_k}|\frac{1}{N\mathrm{P}(s_k)}\sum_{i\in T_k}\hat{P}_i - p_k| \geq \alpha\bar{p}_1\}.$$

Here we use notation $p_{\max} = \max\{p_{k_1}: \; k_1 \in I_1\}$.

Let's denote $\beta = \frac{1}{3}\alpha\bar{p}_1 P_{1,\min}$, for the case A or $\beta = \frac{1}{4}\alpha\bar{p}_1 P_{1,\min}$ for B. Define the random variable

$$k_m = \operatorname{argmax}\left\{\max_{k\in I_1}|\frac{1}{N\mathrm{P}(s_k)}\sum_{i\in T_k}\hat{P}_i - p_k|\right\}$$

and two random sets

$$U = \{|\frac{N_{k_m}}{N} - \mathrm{P}(s_{k_m})| < \beta\}, \; \bar{U} = \{|\frac{N_{k_m}}{N} - \mathrm{P}(s_{k_m})| \geq \beta\}.$$

We have

$$\mathrm{P}\{\frac{|\hat{P}-P|}{P} \geq \alpha\} \leq \mathrm{P}\{\bar{U}\}\mathrm{P}\{\max_{k\in I_1}\frac{1}{p_k}|\frac{1}{N\mathrm{P}(s_k)}\sum_{i\in T_k}\hat{P}_i - p_k| \geq \alpha\bar{p}_1\Big|\bar{U}\}+$$

$$+\mathrm{P}\{U\}\mathrm{P}\{\max_{k\in I_1}|\frac{1}{\mathrm{NP}(s_k)}\sum_{i\in T_k}\hat{P}_i - p_k| \geq \alpha\bar{p}_1\Big|U\}) \leq \mathrm{P}\{\bar{U}\} + \mathrm{P}\{\max_{k\in I_1}|\frac{1}{\mathrm{NP}(s_k)}\sum_{i\in T_k}\hat{P}_i - p_k| \geq \alpha\bar{p}_1\Big|U\}.$$

By virtue Lemma 1 and the condition on N we get

$$(\star) \qquad \mathrm{P}\{|\frac{\hat{P}-P}{P}| \geq \alpha\} \leq \frac{\epsilon}{2} + \mathrm{P}\{\max_{k\in I_1}\frac{1}{p_k}|\frac{1}{\mathrm{NP}(s_k)R_1}\sum_{i\in T_k}\sum_{l=1}^{R_1}\hat{P}_{il} - p_k| \geq \alpha\bar{p}_1\Big|U\}.$$

By Lemma 2 and the condition on R_1 we have

$$A : \mathrm{P}\{|\hat{P}-P| \geq \alpha P\} \leq \frac{\epsilon}{2} + 2e^{-2NR_1\frac{\alpha^2\bar{p}_1^2 p_{1,\min}^2 P_{1,\min}}{\alpha\bar{p}_1+3\bar{P}_1}} \leq \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon,$$

$$B : \mathrm{P}\{|\hat{P}-P| \geq \alpha P\} \leq \frac{\epsilon}{2} + 2e^{-4NR_1\frac{\alpha^2\bar{p}_1^2 p_{1,\min}^2 P_{1,\min}}{\alpha\bar{p}_1+4}} \leq \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

5 ?ESTIMATION ERROR, CASE $M > 1$

Theorem 2: If Assumption **(A)** holds and $p_{j,\min} > 0$, $j = 1,\ldots,M$

$$N \geq 2\frac{\ln 2 - \ln\epsilon}{p_{1,\min}^2},$$

$$NR_1R_2\cdots R_j \geq 2^{j+1}\frac{(j+1)\ln 2 - \ln\epsilon}{p_{1,\min}p_{2,\min}\cdots p_{j,\min}p_{j+1,\min}^2},\ 1 \leq j < M$$

and

$$NR_1R_2\cdots R_M \geq 2^{M+1}\frac{((M+1)\ln 2 - \ln\epsilon)P_{M/M-1}^2 P_{M-1/M-2}^2\cdots P_{1/0}^2}{p_{1,\min}p_{2,\min}\cdots p_{M,\min}\alpha^2}$$

then

$$\mathrm{P}\{|\hat{P}-P| \geq \alpha\} \leq \epsilon.$$

Proof: The relative error of estimator $\hat{P}$ is derived by induction whose principle is the following: if in a simulation with M tresholds, the retrials generated in the first level are not taken into account except one, we have a simulation with $M-1$ tresholds.

By Theorem 1 the result of Theorem 2 holds in the case M=1.

To go from K to $K+1$, assume that the result of Theorem 2 holds in the case M=K. Thus we have to prove it for $K+1$ tresholds.

Let's go back to the proof of Theorem 1. Main path of that proof till point $(\star)$ holds in our new case. We continue the proof from $(\star)$. Consider the last item in $(\star)$

$$(+) \qquad \mathrm{P}\{\max_{k\in I_1}\frac{1}{p_k}|\frac{1}{\mathrm{NP}(s_k)R_1}\sum_{i\in T_k}\sum_{l=1}^{R_1}\hat{P}_{il} - p_k| \geq \alpha\bar{p}_1\Big|U\}.$$

We need to prove that it is not great then $\epsilon/2$. In the conditions of Theorem 2 we have

$$N_k R_1 \geq p_{1,\min} N R_1/2 \geq 2 \frac{\ln 2 - \ln \frac{\epsilon}{2}}{p_{2,\min}^2},$$

$$N_k R_1 R_2 \cdots R_j \geq p_{1,\min} N R_1 R_2 \cdots R_j/2 \geq 2^j \frac{j \ln 2 - \ln \frac{\epsilon}{2}}{p_{2,\min} \cdots p_{j,\min} p_{j+1,\min}^2}, \; 2 \leq j < M$$

and

$$N_k R_1 R_2 \cdots R_j \geq p_{1,\min} N R_1 R_2 \cdots R_j/2 \geq 2^M \frac{(M \ln 2 - \ln \frac{\epsilon}{2}) P_{M/M-1}^2 P_{M-1/M-2}^2 \cdots P_{2/1}^2}{p_{2,\min} \cdots p_{M,\min} \frac{\alpha^2}{P_{1/0}^2}}.$$

Thus we can to apply Theorem 1 to the case

$$M = K, \; \epsilon = \frac{\epsilon}{2}, \; \alpha = \frac{\alpha}{P_{1/0}}, \; N = N_k R_1, \; R_j = R_{j+1}, \; j = 1, \ldots, K-1$$

and for (+) we get that it is not great then $\epsilon/2$. The proof by induction is completed.

6 NUMERICAL EXAMPLE

6.1 Monte Carlo simulation

To find the probability that from uniformly distributed random variables we can get the value beginning from 0.123 with accuracy $\epsilon = 10^{-3}$. We simulate 3800500 sample on Pentium 800MHz during 1 minute

$$\hat{P} = 9.7750 \cdot 10^{-4}.$$

6.2 One level splitting

We can randomly chose the value from $[0, 1]$ with uniform distribution.

Rare event is $A = \{$ the value beginning from 0.1234 or 0.9876 $\}$, $P = 0.0002$. Let's $\epsilon = 10^{-4}$.

Consider $B_1 = \{$ the value beginning from 0.12 or 0.98 $\}$. It is easy to get that $P_{1,\min} = 0.01$, $\bar{P}_1 = 1$, $\bar{p}_1 = 1$ and $p_{1,\min} = 0.01$. From Theorem 1 conditions (A) for $\alpha = 0.5$ we can find

$$N = 1\,907\,400, \quad \text{and} \quad R_1 = 39.$$

The duration of simulation was 1 minute 30 seconds and the result was

$$\hat{P} = 2.0034 \cdot 10^{-4}$$

and by the result of Theorem 1

$$\frac{2}{3}\hat{P} \leq P \leq 2\hat{P}.$$

There were made 3 350 868

case (B):

$$N = 3\ 390\ 900, \quad \text{and} \quad R_1 = 16.$$

The duration of simulation was 2 minutes and the result was

$$\hat{P} = 1.9875 \cdot 10^{-4}.$$

There were made 3 350 868 samples which (?)approximately equal to $N + 0.01NR_1 (=$ 3 867 800).

(?)Note that for Monte Carlo simulation we need to use $N = 4.9517 \cdot 10^8$ and the duration of simulation would be approximately 130 minutes.

6.3 Multilevel splitting

a).

b). Consider the rare event $A = \{$ the value beginning from 0.123456789 or 0.987654321 $\}$, $\epsilon = 10^{-9}$.

Consider $M = 4,\ B_1 = \{$ the value beginning from 0.12 or 0.98 $\}$,
$B_2 = \{$ the value beginning from 0.1234 or 0.9876 $\}$,
$B_3 = \{$ the value beginning from 0.123456 or 0.987654 $\}$,
$B_4 = \{$ the value beginning from 0.12345678 or 0.98765432 $\}$.
It is easy to get that $p_{1,\min} = p_{2,\min} = p_{3,\min} = 0.01,\ p_{4,\min} = 0.1$ and
$P_1 = 0.02,\ P_{2/1} = P_{3/2} = P_{4/3} = 0.01$.

From Theorem 2 conditions for $\alpha = 10^{-9}$ we can find

$$N = (?)198070,\ R_1 = 207,\ R_2 = 206,\ R_3 = 206 \text{ and } R_4 = 9.$$

The duration of simulation was 6 minutes and the result was

$$\hat{P} = 1.9422 \cdot 10^{-9}.$$

There were made 13 889 000 samples. Note that for Monte Carlo simulation we need to use $1.0708 \cdot 10^{19}$.

6.4 Brown motion

6.5 Brown motion with switching

6.6 Diffusion process

6.7 Diffusion with switching

7 HYBRID SYSTEMS

Let's $\mathbb{X} = \cup_{i \in I_X} \mathbb{X}_i :\ \mathbb{X}_i \cap \mathbb{X}_j = \emptyset,\ i \neq j$, e.g. in ATM problem with switching we need to consider $\mathbb{X} = \mathbb{R}^3 \times \mathbb{M}$ where $\mathbb{M}$ is a finite set of modes.

Suppose we have a probabilitistic measure $\mu(\cdot)$ on $\mathbb{X}$. Now we can not use the uniform distribution for the simulation and we need to generalize notations and main assumption from section II.

We define for $j = 1, \ldots, M+1$

$$\bar{s}^{(i_0 \ldots i_{j-1})}_{(k_1 \ldots k_j)} = \{\omega \in s_{k_1 \ldots k_j} :\ \forall l = 1, \ldots, j \text{ first of } \{\delta_{l,k_l} \in \omega \cap b_{l,k_l}\} \in \omega \cap b_{l,k_l} \cap \mathbb{X}_{i_{l-1}}\},$$

$$\bar{p}^{(i_0 \ldots i_{j-1})}_{(k_1 \ldots k_j)} = \begin{cases} 0, \text{ if } \mathrm{P}(\bar{s}^{(i_0 \ldots i_{j-1})}_{(k_1 \ldots k_j)}) = 0, \\ \mathrm{P}(\cup_{k_{j+1} \in I_{j+1}} \cdots \cup_{k_{M+1} \in I_{M+1}} \bar{s}_{k_1 \ldots k_j, k^{(i_0 \ldots i_{j-1})}_{j+1} \ldots k_{M+1}} | s^{(i_0 \ldots i_{j-1})}_{k_1 \ldots k_j}), \text{ if } \mathrm{P}(\bar{s}^{(i_0 \ldots i_{j-1})}_{(k_1 \ldots k_j)}) > 0 \end{cases}$$

and $\forall j > 1$

$$P_{j/j-1} = \max_{k_1 \in I_1} \ldots \max_{k_{j-1} \in I_{j-1}} \sum_{k_j \in I_j,\, P(s_{k_1 \ldots k_{j-1}}) > 0} P(s_{k_1 \ldots k_j}) / P(s_{k_1 \ldots k_{j-1}}),$$

$$P_{1/0} = 1,$$

$$p_{j,\min} = \min\{P(s_{k_1 \ldots k_j}) / P(s_{k_1 \ldots k_{j-1}}) :\ P(s_{k_1 \ldots k_j}) > 0,\ k_1 \in I_1, \ldots, k_j \in I_j\},$$

$$p_{1,\min} = \min\{P(s_{k_1}) :\ P(s_{k_1}) > 0,\ k_1 \in I_1\}.$$

Note that

$$P = \sum_{k_1 \in I_1} \mathrm{P}(s_{k_1}) p_{k_1} = \sum_{k_1 \in I_1} \mathrm{P}(s_{k_1}) \sum_{k_2 \in I_2} (\mathrm{P}(s_{k_1 k_2}) / \mathrm{P}(s_{k_1})) p_{k_1 k_2} = \ldots.$$

MAIN ASSUMPTION

(A') if $P(s_{k_1 \ldots k_j}) > 0$ $j > 1$, $k_1 \in I_1,\ \ldots, k_j \in I_j$, $\Delta',\ \Delta'' \subset s_{k_1 \ldots k_j} :\ \mathrm{P}(\Delta') = \mathrm{P}(\Delta'') > 0$

then $\forall \bar{\Delta} \subset s_{k_1 \ldots k_{j-1}} :\ \mathrm{P}(\bar{\Delta}) > 0$

$$\mathrm{P}(\Delta' | \bar{\Delta}) = \mathrm{P}(\Delta'' | \bar{\Delta}).$$

ALGORITHM OF SIMULATION

1. $\forall i \in I_X : \mu(S \cap \mathbb{X}_i) > 0$ let's start $N^{(i)}$ trajectories from $S \cap \mathbb{X}_i$ with uniform distribution,
   $M \geq 0,\ R^{(i)}_j > 0,\ R^{(i)}_j \in \mathbb{N},\ j = 1, \ldots, M,\ i \in I_X$.
2. $j = 1$.
3. To kill all trajectories which $\notin S_j$.
4. Each of all rest trajectories ω_t cross the set B_j in some point $\delta_{j,k_t} \in b_{j,k_t}$. For all ω_t splitting $R_j - 1$ times the trajectory ω_t uniformly on b_{j,k_t}. We have got the new $R_j - 1$ trajectories for each of all rest trajectories.

5. If $j < M$ then $j = j + 1$ and GOTO step 3.

ALGORITHM OF ESTIMATION

$$\hat{P} = \frac{N_A}{NR_1 \cdot R_M}$$

where N_A is equal to all number of trajectories which achieve the set A.

Note that by the algorithm of simulation

$$\hat{P} = \frac{1}{N} \sum_{i=1}^{N} \hat{P}_i$$

where $\hat{P}_i$ for each of $N > 0$ starting trajectories i is a fraction of the number of its sub-trajectories (include parent) which achieve the set A to the all number of its sub-trajectories. Note, if $\hat{P}_i > 0$ then

$$\hat{P}_i = \frac{1}{R_1} \sum_{l=1}^{R_1} \hat{P}_{il},$$

where $\hat{P}_{il}$ are equal to 1 or 0 depending on the achievement l-th sub-trajectory of set A or not.

$\hat{P}$ is unbiased since

$$?E(\hat{P}) = E\left(\frac{N_A}{NR_1 \cdot R_M}\right) = \frac{1}{NR_1 \cdot R_M} \sum_{k_0=1}^{N} \sum_{k_1=1}^{R_1} \cdot \sum_{k_M=1}^{R_M} E(\mathbf{1}_{k_0} \mathbf{1}_{k_0 k_1} \ldots \mathbf{1}_{k_0 \ldots k_M}) = P.$$

First consider the case $M = 1$.

In this case we have

$$B_1 = \{b_k,\ k \in I_1\},$$

$$P = \sum_{k \in I_1} \mathrm{P}(s_k) p_k \quad \text{and} \quad P_1 = P_{1/0} = \sum_{k \in I_1} \mathrm{P}(s_k).$$

Note that

$$\mathrm{P}(s_k) = \mathrm{P}\{\omega : \ \omega \text{ cross } B_1 \text{ first time in the point } \delta_k \in b_k\}.$$

Let $\bar{\Omega}_0 = \{\omega_t,\ t = 1, \ldots, N\}$ be a set of our N initial trajectories, $T_k,\ k \in I_1$ be the set of indexes t of trajectories which cross B_1 first time in the point $\delta_k \in b_k$. Denote $N_k = |T_k|,\ k \in I_1$. N_k is a random variable.

Theorem 3: If Assumption **(A')** holds and $p_{1,\min} > 0$,

$$N \geq 2 \frac{\ln 2 - \ln \epsilon}{p_{1,\min}^2} \quad \text{and} \quad NR_1 \geq \frac{(2 \ln 2 - \ln \epsilon) P_1^2}{p_{1,\min} \alpha^2}$$

then

$$\mathrm{P}\{|\hat{P}-P|\geq\alpha\}\leq\epsilon.$$

Note that by Lemma 1 $\forall k\in I_1$ we have $\mathrm{P}\{N_k<Np_{1,\min}/2\}\leq\frac{\epsilon}{2}$ in Theorem 1 conditions and for R_1 when $\alpha=\epsilon$ we have

$$R_1\geq\frac{P_1^2(\ln 4-\ln\epsilon)}{\epsilon^2\sqrt{2N(\ln 2-\ln\epsilon)}}.$$

Proof:

$$\mathrm{P}\{|\hat{P}-P|\geq\alpha\}=\mathrm{P}\{|\frac{1}{N}\sum_{i=1}^{N}\hat{P}_i-P|\geq\alpha\}=\mathrm{P}\{|\frac{1}{N}\sum_{i=1}^{N}\hat{P}_i-\sum_{k\in I_1}\mathrm{P}(s_k)p_k|\geq\alpha\}=$$

$$=\mathrm{P}\{|\sum_{k\in I_1}\mathrm{P}(s_k)\left(\frac{1}{N\mathrm{P}(s_k)}\sum_{i\in T_k}\hat{P}_i-p_k\right)|\geq\alpha\}\leq\mathrm{P}\{P_1\max_{k\in I_1}|\frac{1}{N\mathrm{P}(s_k)}\sum_{i\in T_k}\hat{P}_i-p_k|\geq\alpha\}.$$

Denote

$$k_m=\operatorname{argmax}\left\{\max_{k\in I_1}|\frac{1}{N\mathrm{P}(s_k)}\sum_{i\in T_k}\hat{P}_i-p_k|\right\},$$

(k_m is the random variable). We have

$$\mathrm{P}\{|\hat{P}-P|\geq\alpha\}\leq\mathrm{P}\{N_{k_m}<p_{1,\min}N/2\}\mathrm{P}\{\max_{k\in I_1}|\frac{1}{N\mathrm{P}(s_k)}\sum_{i\in T_k}\hat{P}_i-p_k|\geq\frac{\alpha}{P_1}|k_m<p_{1,\min}N/2\}+$$

$$+\mathrm{P}\{N_{k_m}\geq p_{1,\min}N/2\}\mathrm{P}\{\max_{k\in I_1}|\frac{1}{N\mathrm{P}(s_k)}\sum_{i\in T_k}\hat{P}_i-p_k|\geq\frac{\alpha}{P_1}|N_{k_m}\geq p_{1,\min}N/2\})\leq$$

$$\leq\mathrm{P}\{N_{k_m}<p_{1,\min}N/2\}+\mathrm{P}\{\max_{k\in I_1}|\frac{1}{N\mathrm{P}(s_k)}\sum_{i\in T_k}\hat{P}_i-p_k|\geq\frac{\alpha}{P_1}|N_{k_m}\geq p_{1,\min}N/2\}\leq$$

By virtue Lemma 1 and the condition on N we get

$$\mathrm{P}\{|\hat{P}-P|\geq\alpha\}\leq\frac{\epsilon}{2}+\mathrm{P}\{\max_{k\in I_1}|\frac{1}{N_k}\sum_{i\in T_k}\frac{N_k}{N\mathrm{P}(s_k)}\hat{P}_i-p_k|\geq\frac{\alpha}{P_1}|N_{k_m}\geq p_{1,\min}N/2\}=$$

$$(\star)\qquad=\frac{\epsilon}{2}+\mathrm{P}\{\max_{k\in I_1}|\frac{1}{N_kR_1}\sum_{i\in T_k}\sum_{l=1}^{R_1}\frac{N_k}{N\mathrm{P}(s_k)}\hat{P}_{il}-p_k|\geq\frac{\alpha}{P_1}|N_{k_m}\geq p_{1,\min}N/2\}.$$

Since $E(\frac{N_k}{N\mathrm{P}(s_k)}\hat{P}_{il})=p_k$ then by Lemma 2 and the condition on R_1 we have

$$\mathrm{P}\{|\hat{P}-P|\geq\epsilon\}\leq\frac{\epsilon}{2}+2e^{-p_{1,\min}NR_1\alpha^2P_1^{-2}}\leq\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

Theorem 4: If Assumption **(A')** holds and $p_{j,\min} > 0$, $j = 1, \dots, M$

$$N \geq 2\frac{\ln 2 - \ln \epsilon}{p_{1,\min}^2},$$

$$NR_1R_2\cdots R_j \geq 2^{j+1}\frac{(j+1)\ln 2 - \ln \epsilon}{p_{1,\min}p_{2,\min}\cdots p_{j,\min}p_{j+1,\min}^2},\ 1 \leq j < M$$

and

$$NR_1R_2\cdots R_M \geq 2^{M+1}\frac{((M+1)\ln 2 - \ln \epsilon)P_{M/M-1}^2P_{M-1/M-2}^2\cdots P_{1/0}^2}{p_{1,\min}p_{2,\min}\cdots p_{M,\min}\alpha^2}$$

then

$$\mathrm{P}\{|\hat{P} - P| \geq \alpha\} \leq \epsilon.$$

8 CONCLUDING REMARKS

9 APPENDIX

References

[1] A. Lagnoux, "Rare event simulation," 2003.