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Optimal Step-Size of a Local Voting Protocol for Differentiated Consensuses Achievement in a Stochastic Network with Cost Constraints

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Abstract— **This paper deals with a "differentiated consensuses" problem in a distributed stochastic network system with priorities and cost constraints on system topology. The network is considered as a set of heterogeneous agents that process incoming tasks with different importance (priority) levels. The observations about neighbors' states are supposed to be obtained with random noise and delays and the topology could switch over time. Several consensus objectives are to be achieved. To maintain almost balanced load under cost constraints on network topology, i.e. approximate consensus *for every priority class* across the network, a new family of control protocols that use different step-size parameters for each task class is introduced. An instrument for choosing optimal step-sizes for the proposed control strategy is given. In addition, a numerical example that illustrates the proposed control strategy and the results of simulations are provided.**

I. Introduction

Recently, the consensus approach was widely used to solve numerous practical problems such as cooperative control of multivehicle networks [1–3], distributed control of robotic networks [4], flocking problem [5, 6], optimal control of sensor networks [7, 8] and others. A lot of attention was paid to obtain the corresponding consensus conditions for such systems (see e.g. [9–16]).

One of important practical problems is a load balancing problem. This is the problem of tasks redistribution between agents. It arises in various types of network systems, such as computer, production, transport, logistics, and other service networks. This could be networks consisting of heterogeneous agents that work together to achieve some practical goal, e.g. to process all incoming tasks as fast as possible. However, there could be other goals as well. In our previous work [17] it was shown that the problem of almost optimal task distribution among agents could be reformulated as a problem of the consensus achievement in the network. A centralized algorithm was considered in [18]. The multi-agent approach was developed in [17]. Some control strategies for the load balancing that redistribute tasks in accordance with a current loads of agents and their productivities were also introduced. It is also important to look at the specifics of tasks since in real practical applications some of the incoming tasks could be more important then others (there could be different priorities of tasks). This condition should also be taken into account when we build a control strategy in [19, 20]. For such networks with tasks of different importance (priority) levels there could be several consensus objectives. We call this problem a *differentiated consensuses problem*, which defines a consensus problem for systems with multiple classes, where a consensus is targeted for each class and may be different among classes. Ultimately, the control goal of the network is to achieve a consensus within each class separately. In [19] for a distributed stochastic network with priorities we introduced a control strategy that allocates the resources of the network in a randomized way with corresponding probabilities for each priority class. Also corresponding conditions for the achievement of differentiated consensuses throughout the whole network were obtained. In [21] we considered a choice of an optimal step-size for task redistribution among agents in a stochastic network with randomized priorities. It was shown that while choosing the step-size one has to make a trade-off between noise sensitivity of control protocol and the rate of convergence it provides. We proposed a way to choose step-size that maximizes precision of convergence.

In this paper we extend the results of our previous work [21] and introduce a control strategy that uses different step-size parameters for each task class under different cost constraints on network topology for each task class. We also provide a way to choose optimal step-sizes for the proposed family of control protocols.

The paper is organized as follows. In Section II the notation and the problem formulation are given. The new family of control protocols for achieving the differentiated consensus is introduced in Section III. In Section IV the main assumptions and main results are presented. Simulation results are included in Section V. Section VI contains conclusion remarks.

II. Problem Statement

Consider a dynamic network system of n agents, which collaborate among themselves, and a set of tasks of different classes, which have to be executed by the system. Tasks are fed to different agents of the system in different discrete

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time instants $t = 0, 1, \ldots$. Agents process incoming tasks in parallel. Tasks can be redistributed among agents based on a feedback. Note that a task cannot be interrupted if it is being processed by an agent, i.e. the system is non-preemptive.

Without loss of generality, agents in the system are numbered. Assume, that $N = \{1, \ldots, n\}$ denotes the set of agents in the network system. Let $i \in N$ be the number of an agent. The network topology may switch over time. Let the dynamic network topology be modeled by a sequence of digraphs $\{(N, E_t)\}_{t \geq 0}$, where $E_t \subset E$ denotes the set of edges at time t of topology graph (N, E_t). The corresponding adjacency matrices are denoted as $A_t = [a_t^{i,j}]$, where $a_t^{i,j} > 0$ if agent j is connected with agent i and $a_t^{i,j} = 0$ otherwise. Here and below, an upper index of agent i is used as the corresponding number of an agent (while not as an exponent). Denote $\mathscr{G}_{A_t}$ as the corresponding graph.

To introduce some properties of the network topology, the following definitions from the graph theory will be used. Define *the weighted in-degree* of node i as the sum of i-th row of matrix A: $indeg^i(A) = \sum_{j=1}^n a^{i,j}$; $\mathscr{D}(A) = \mathrm{diag}\{indeg^i(A)\}$ is the corresponding diagonal matrix; $indeg_{\max}(A)$ is the maximum in-degree of graph $\mathscr{G}_A$. Let $\mathscr{L}(A) = \mathscr{D}(A) - A$ denote the *Laplacian* of graph $\mathscr{G}_A$; $\cdot^{\mathrm{T}}$ is a vector or matrix transpose operation; $||A||$ is the Euclidian norm: $||A|| = \sqrt{\sum_i \sum_j (a^{i,j})^2}$; $Re(\lambda_2(A))$ is the real part of the second eigenvalue of matrix A ordered by the absolute magnitude; $\lambda_{\max}(A)$ is the maximum eigenvalue of matrix A.

It is said that digraph $\mathscr{G}_B$ is a subgraph of a digraph $\mathscr{G}_A$ if $b^{i,j} \leq a^{i,j}$ for all $i, j \in N$.

Digraph $\mathscr{G}_A$ is said to contain a *spanning tree* if there exists a directed tree $\mathscr{G}_{tr} = (N, E_{tr})$ as a subgraph of $\mathscr{G}_A$ which includes all vertices of $\mathscr{G}_A$.

Let $(\Omega, \mathscr{F}, P)$ be the underlying probability space corresponding to the sample space, the collection of all events, and the probability measure respectively, and E be a mathematical expectation symbol.

We suppose that tasks (jobs) belong to different classes $k = 1, \ldots, m$ and every agent has m queues — one for each task class.

The behavior of agent $i \in N$ is described by characteristics of two types:

- lengths of m queues of tasks of each class k at time instant t: $q_t^{i,k}$, $k = 1, \ldots, m$,
- productivity: p^i.

Each agent should distribute its own productivity among all task classes in such a way that, on the one hand the priorities for task classes are provided and on the other hand the "starvation problem" is taken into account i.e. tasks of the lower priority classes do not wait for execution for too long. This is achieved by making use of the probabilistic priority discipline [22]. Each task class is given a productivity fraction P_k, $k = 1, \ldots, m$ which is the same for a certain class k on every agent in the system. On each agent the tasks from its queues are chosen for execution randomly according to the following formula: $\tilde{p}_t^{i,k} = \begin{cases} \frac{P_k}{\sum_{q_t^{i,l} > 0} P_l}, & \text{if } q_t^{i,k} > 0; \\ 0, & \text{otherwise,} \end{cases}$ where $\tilde{p}_t^{i,k}$ is the probability of choosing a task of class k for execution on agent i at a time instant t. Therefore the bigger fraction P_k corresponds to the higher chance of that a task of class k to be executed. Thus agent's productivity is distributed among all classes of tasks in the following way: $p_t^{i,k} = \tilde{p}_t^{i,k} p^i$. Here $p_t^{i,k}$ is a number of operations allocated for tasks of class k on agent i at a time instant t if the productivity p^i means the whole number of operations which agent i is able to proceed during the time from t till $t+1$. Note that according to the definition of $\tilde{p}_t^{i,k}$ if at certain time instant t' the queue of tasks of class k' on the agent i' is empty, no operations would be allocated for tasks of class k'. Instead $p_{t'}^{i',k'}$ operations would be distributed among other task classes in proportions of their productivity fractions P_k, $k \neq k'$.

For all $i \in N$, $t = 0, 1, \ldots$, the dynamics of the network system in a vector form is as follows

$$\mathbf{q}_{t+1}^i = \mathbf{q}_t^i - \mathbf{p}_t^i + \mathbf{z}_t^i + \mathbf{u}_t^i, \qquad (1)$$

where $\mathbf{q}_t^i = [q_t^{i,k}]$ is a vector whose k-th element is defined by the amount of tasks of k-th class; $\mathbf{p}_t^i = [p_t^{i,k}]$, and $\mathbf{z}_t^i = [z_t^{i,k}]$ is an m-vector whose k-th element $z_t^{i,k}$ is the amount of new tasks of class k, which came to the system and were received by agent i at time instant t; $\mathbf{u}_t^i \in \mathbb{R}^m$ is a vector of control actions (redistributed tasks of class k to agent i at time instant t), which could (and should) be chosen based on some information about queue lengths of neighbors $\mathbf{q}_t^j$, $j \in N_t^i$, where N_t^i is the set $\{j \in N : a_t^{i,j} > 0\}$.

Denote $\bar{p}_t^{i,k} = \mathrm{E} p_t^{i,k}$ and

$$x_t^{i,k} = \begin{cases} q_t^{i,k} / \bar{p}_t^{i,k}, & \text{if } \bar{p}_t^{i,k} > 0; \\ 0, & \text{otherwise} \end{cases} \qquad (2)$$

the *load* of agent $i \in N$ for priority class $k = 1, \ldots, m$. Assume, that $p^i \neq 0$, $\forall i \in N$ and $P_k \neq 0$, $k = 1, \ldots, m$. In [17] it was proven that from all possible options for the redistribution of all tasks the minimum operation time of the system is achieved when loads $x_t^{i,k}$ are equalized throughout the network. Hence, it is important to consider the achievement of the following goal.

Now we define the cost of a chosen topology $\{N_t^i, i \in N\}$

$$C(\{N_t^i, i \in N\}) = \max_{i \in N} \sum_{j \in N_t^i} a_t^{i,j}. \qquad (3)$$

Tasks have different priorities and, for each priority, the maximum cost of the network graphs that could be used is defined.

For each time instant t, consider m ways (which may be different and each corresponds to one class) to select the topology subgraphs $\mathscr{G}_t^k : \mathscr{G}_t^m \subseteq \mathscr{G}_t^{m-1} \subseteq \ldots \subseteq \mathscr{G}_t^1$ of the graph $\mathscr{G}_{A_t}$, which allows to use redistribution protocols for tasks with priority k. Let B_t^k be the corresponding adjacency matrices. Note that one of the possible ways of choosing $\mathscr{G}_t^k$ is to use $\mathscr{G}_{A_t}$ for all k.

Definition *1:* We will say that network topology decomposition $\{\mathscr{G}_t^k\}$ satisfies average cost constraint $\{c_k\}$ if for every priority class k

$$d_{\max}(B^k_{av}) = \mathrm{E}d_{\max}(B^k_t) = \mathrm{E}\max_{i\in N}\sum_{j\in N^{i,k}_t} b^{i,j,k}_t \le c_k, \quad (4)$$

where $N^{i,k}_t$ is the neighbors set of agent i at time t formed in accordance with the topology $\mathscr{G}^k_t$. (Consider the example in section V.)

We will consider control protocols that satisfy some specific constraint on the cost of the topology for each task priority class.

It is required to maintain balanced (equal) loads across the network for every priority class and, at the same time, to meet the cost constraint requirement.

At this setting we can consider the consensus problem for states $\mathbf{x}^i_t = [x^{i,k}_t]$ of agents, where $\mathbf{x}^i_t$ is a state vector of agent $i \in N$, consisting of loads $x^{i,k}_t$ for m classes. We use the following definitions.

Definition *2:* n agents of a network are said to reach a *consensus* at time t if $\mathbf{x}^i_t = \mathbf{x}^j_t \quad \forall i,j \in N, i \neq j$.

Definition *3:* n agents are said to achieve *asymptotic mean square ε-consensus* for $\varepsilon > 0$ when

$$\overline{\lim}_{t\to\infty}\mathrm{E}\|\mathbf{x}^i_t - \mathbf{x}^j_t\|^2 \le \varepsilon.$$

To ensure balanced loads across the network (e.g., in order to increase the overall throughput of the system and to reduce the execution time), it is naturally to use a redistribution protocol over time. We assume that to form the control (redistribution) strategy each agent $i \in N$ has noisy and possibly delayed observations about its neighbors' states

$$\mathbf{y}^{i,j}_t = \mathbf{x}^j_{t-d^{i,j}_t} + \mathbf{w}^{i,j}_t,\ j \in N^i_t, \quad (5)$$

where $\mathbf{w}^{i,j}_t$ is a noise vector, $0 \le d^{i,j}_t \le \bar{d}$ are integer-valued delays, and $\bar{d}$ is a maximum of possible delays.

III. Control Protocol

In [17], properties of a control algorithm, called local voting protocol, were studied for stochastic networks in the context of load balancing problem. For each agent the control (amount of redistributed tasks) was determined by the weighted sum of differences between the information about the state of the agent and the information about its neighbors' states. Let's consider a similar family of protocols as follows. For each $k = 1,\ldots,m$ we define

$$u^{i,k}_t = \gamma_k \bar{p}^{i,k}_t \sum_{j\in\bar{N}^i_t} b^{i,j}_t (y^{i,j,k}_t - x^{i,k}_t), \quad (6)$$

where $\gamma_k > 0$, $k = 1,\ldots,m$ are step-sizes of the control protocol and $\bar{N}^i_t \subset N^i_t$ is the neighbor set of agent i (note, that we could use not all the available connections, but some subset of them), $b^{i,j}_t$ are protocol coefficients, satisfying(4). In [19] it was shown that differentiated consensuses could be achieved throughout the whole network via control protocol similar to the one mentioned above but with the same step-size for balancing queues lengths of tasks of each class. However, since tasks of different classes may be fed to the system with different intensities it is reasonable to treat them differently and choose step-sizes separately for each task class.

Let $B_t = [b^{i,j}_t]$ be the matrices of task redistribution protocols for every time instant t. (We set $b^{i,j}_t = 0$ when $a^{i,j}_t = 0$ or $j \notin \bar{N}^i_t$.) The corresponding graph $\mathscr{G}_{B_t}$ may have the same topology as graph $\mathscr{G}_{A_t}$ of matrix A_t or more poor.

Let's assume $\bar{d} = 0$ and $\bar{p}^{i,k}_{t+1} = \bar{p}^{i,k}_t$. Then the dynamics of the closed loop system with protocol (6) will be as follows:

$$x^{i,k}_{t+1} = x^{i,k}_t - \tilde{r}^{i,k}_t + \tilde{z}^{i,k}_t + \gamma_k \sum_{j\in\bar{N}^i_t} b^{i,j}_t (y^{i,j,k}_t - x^{i,k}_t) =$$

$$x^{i,k}_t - \tilde{r}^{i,k}_t + \tilde{z}^{i,k}_t + \gamma_k \left(\sum_{j\in\bar{N}^i_t} b^{i,j}_t x^{j,k}_t\right) - \gamma_k d^i(B_t) x^{i,k}_t + \gamma_k \tilde{w}^{i,k}_t, \quad (7)$$

where $i \in N$, $k = 1,\ldots m$ and $\tilde{w}^{i,j,k}_t = b^{i,j}_t w^{i,j,k}_t$,

$$\tilde{r}^{i,k}_t = \begin{cases} p^{i,k}_t/\bar{p}^{i,k}_t, & \text{if } \bar{p}^{i,k}_t > 0; \\ 0, & \text{otherwise,} \end{cases} \quad \tilde{z}^{i,k}_t \begin{cases} z^{i,k}_t/\bar{p}^{i,k}_t, & \text{if } \bar{p}^{i,k}_t > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Let us rewrite Eq. (7) in a more compact form. Define the $\mathbb{R}^n$-valued vectors $\mathbf{X}^k_t = [x^{i,k}]$, $\mathbf{R}^k_t = [\tilde{r}^{i,k}_t]$, $\mathbf{Z}^k_t = [\tilde{z}^{i,k}_t]$ and $\mathbf{W}^k_t = \sum_{j\in\bar{N}^i_t} b^{i,j}_t \mathbf{w}^{i,j,k}_t$. The dynamics of the closed loop system with protocol (6) may be represented as

$$\mathbf{X}^k_{t+1} = \mathbf{X}^k_t + \gamma_k(B_t - \mathscr{D}(B_t))\mathbf{X}^k_t - \mathbf{R}^k_t + \mathbf{Z}^k_t + \gamma_1 \mathbf{W}^k_t. \quad (8)$$

If $\bar{d} > 0$ we "artificially" add $n\bar{d}$ new agents to the current network topology. At each time instant t the new "fictitious" agents have states which are equal to the corresponding states of "real" agents at previous time instants $t-1$, $t-2,\ldots,t-\bar{d}$. The same is done for every class $k = 1\ldots m$. Let $x^{i,k}_t \equiv 0$, $i \in N$ for $-\bar{d} \le t < 0$. Denote $\bar{\mathbf{X}}^k_t \in \mathbb{R}^{\bar{n}}$, $\bar{n} = n(\bar{d}+1)$, as an extended state vector for $t = 0,1,\ldots$ which consist of $\bar{d}+1$ (n)-vectors $\mathbf{X}^k_t, \mathbf{X}^k_{t-1},\ldots,\mathbf{X}^k_{t-\bar{d}}$, i.e. it includes all the components with all kinds of delays not exceeding $\bar{d}$. Introduce the extended $\bar{n}\times\bar{n}$ matrices $\bar{B}^k_t$ of control protocol (6) which consist of zeros at all places except $|\bar{N}^i_t|$ entries $\bar{b}^{i,j+ns^{i,j}_t,k}_t$ in each $i \in N$, $j \in \bar{N}^i_t$ of n first lines, which are equal to $b^{i,j}_t$ and $\bar{b}^{i,i-n,k}_t = 1/\gamma_k$ in next $n\bar{d}$ lines, $i = n+1,\ldots,\bar{n}$.

Due to the view of Laplacian matrices $\mathscr{L}(\bar{B}^k_t)$ we can rewrite the dynamics of the system in the following vector-matrix form:

$$\bar{\mathbf{X}}^k_{t+1} = \bar{\mathbf{X}}^k_t - \gamma_k \mathscr{L}(\bar{B}^k_t)\bar{\mathbf{X}}^k_t + \begin{pmatrix} -\mathbf{R}^k_t + \mathbf{Z}^k_t + \gamma_k \mathbf{W}^k_t \\ 0 \end{pmatrix}. \quad (9)$$

IV. Main Results

A. Assumptions

Assume that the following conditions are satisfied:

- **A1**. **a)** For all $i \in N$, $j \in N^i_t$, observation noise vectors $\mathbf{w}^{i,j}_t$ are zero-mean, independent identically distributed (i.i.d.) random vectors with bounded variances: $\mathrm{E}(\mathbf{w}^{i,j}_t)^2 \le \sigma^2_w$.

  b) Graphs $\mathscr{G}_{B_t}$, $t = 1,\ldots$ are i.i.d. (independent identically distributed), i.e. the random events of appearance of of "time-varying" edge (j,i) in graph $\mathscr{G}_{B_t}$ are independent and identically distributed for the fixed pair (j,i), $i \in N, j \in N^i_{\max} = \cup_t \bar{N}^i_t$. For all $i \in N$, $j \in N^i_t$

weights $b_t^{i,j}$ in the control protocol are independent random variables with mean values (mathematical expectations): $\mathrm{E}b_t^{i,j} = b_{av}^{i,j}$, and bounded variances: $\mathrm{E}(b_t^{i,j} - b_{av}^{i,j})^2 \leq \sigma_b^2$. Let B_{av}^k be the corresponding adjacency matrix.

c) For all $i \in N, j \in N^i$ there exists a finite value $\bar{d} \in \mathbb{N}$: $d_t^{i,j} \leq \bar{d}$ with probability 1, and integer-valued delays $d_t^{i,j}$ are i.i.d. random variables taking values $l = 0, \dots, \bar{d}$ with probability $D_l^{i,j}$.

d) For all $k = 1, \dots, m$, $i \in N$, $t = 0, 1, \dots$ random values $z_t^{i,k}$ are independent with expectations: $\mathrm{E}z_t^{i,k} = \bar{z}^k$ which do not depend on i, and variances: $\mathrm{E}(z_t^{i,k} - \bar{z}^k)^2 \leq \sigma_{z,k}^2$.

e) For all $i \in N$, $t = 0, 1, \dots$ random vectors $\mathbf{p}_t^i$ are i.i.d. and consist of independent components. Random values $\tilde{r}_t^{i,k}$, $k = 1, \dots, m$, have expectations: $\mathrm{E}\tilde{r}_t^{i,k} = \bar{r}^k$ and bounded variances: $\mathrm{E}(\tilde{r}_t^{i,k} - \bar{r}^k)^2 \leq \sigma_{r,k}^2$ which do not depend on i.

Additionally, all mentioned in Assumption **A1** independent random variables and vectors are mutually independent.

- **A2**. Graph $\mathscr{G}_{B_{av}}$ has a spanning tree (for the consensuses to be achievable throughout the system [10]).
- **A3.** For step-sizes γ_k, $k = 1 \dots m$ of control protocols (6) the following conditions are satisfied:

$$0 < \gamma_k < \frac{1}{indeg_{\max}(B_{av}^k)},\ 0 < \delta_k(\gamma_k) < 1, \tag{10}$$

where $\delta_k(\gamma_k) = R - \gamma_k Re(\lambda_{\max}(Q))$, $R = 1 - D_{\max} + D_{\max}|Re(\lambda_2(\mathscr{L}(B_{av}^k)))|$, $D_{\max} = \max_{i,j,l} D_l^{i,j}$, $Q = \mathrm{E}(\mathscr{L}(\mathrm{E}\bar{B}_t^k) - \mathscr{L}(\bar{B}_t^k))^{\mathrm{T}}(\mathscr{L}(\mathrm{E}\bar{B}_t^k) - \mathscr{L}(\bar{B}_t^k))$.

Note that $|Re(\lambda_2(\mathscr{L}(B_{av}^k)))| > 0$ when Assumption **A2** holds (see [16]).

B. Averaged Models

Let $x_0^{\star,k}$, $k = 1, \dots, m$ be the weighted average of the initial states

$$x_0^{\star,k} = \frac{\sum_i g_i x_0^{i,k}}{\sum_i g_i}$$

where g^T is the left eigenvector of matrix B_{av}^k [16] ($x_0^{\star,k} = \frac{1}{n}\sum_{i=1}^n x_0^{i,k}$ in the case of balanced topology graph $\mathscr{G}_{B_{av}^k}$) and $\{x_t^{\star,k}\}$ is the trajectory of averaged systems

$$x_{t+1}^{\star,k} = x_t^{\star,k} + \bar{z}^k - \bar{r}^k,\ \ k = 1, \dots, m. \tag{11}$$

where $\bar{z}^k$ and $\bar{r}$ are expectations which are defined by Assumptions **A1.d,e**.

C. Differentiated Consensuses

Theorem *1:* **If** Assumption **A2** holds **then** for any average costs constraint $\{c_k\}$, $c_k > 0$, there exists network topology decomposition $\{\mathscr{G}_t^k\}$ that satisfies the averaged costs constrains $\{c_k\}$ and for which all averaged graphs $\mathscr{G}_{av}^k$ have spanning trees.

Proof: The proof is given in [20]. ■

Consider vectors $\bar{\mathbf{X}}_t^{\star,k} \in \mathbb{R}^{\bar{n}}$, $t = 0, 1, \dots$, $k = 1, \dots, m$ which consist of $x_t^{\star,k}, \dots, x_t^{\star,k}, x_{t-1}^{\star,k}, \dots, x_{t-1}^{\star,k}, \dots, x_{t-\bar{d}}^{\star,k}, \dots, x_{t-\bar{d}}^{\star,k}$.

Theorem *2:* **If** Assumptions **A1–A3** hold **then** for averaged squared difference $v_t^k = \mathrm{E}||\bar{\mathbf{X}}_t^k - \bar{\mathbf{X}}_t^{\star,k}||^2$ of trajectories of closed-loop systems (7) and (11) following inequalities are satisfied:

$$v_t^k \leq \frac{\gamma_k^2 H^k + S_k}{\gamma_k \delta_k(\gamma_k)} + (1 - \gamma_k \delta_k(\gamma_k))^t \left(v_0^k - \frac{\gamma_k^2 H^k + S_k}{\gamma_k \delta_k(\gamma_k)} \right),\ \tag{12}$$

where $k = 1, \dots, m$, $H^k = 2\sigma_w^2(n^2\sigma_b^2 + ||B_{av}^k||^2)$, $S_k = n(\sigma_{z,k}^2 + \sigma_{r,k}^2)$, i.e. **if** additionally $v_0^k < \infty$, **then** the asymptotic mean square ε_k-consensus in (7) is achieved with $\varepsilon_k = \frac{\gamma_k^2 H^k + S_k}{\gamma_k \delta_k(\gamma_k)}$.

Proof: The proof is similar to the proof in [21] or we can use results from [19] with minor revisions. ■

Remarks 1: At this point, we highlight that, the result of Theorem 2 shows that queues with different priorities achieve m different consensus levels separately. This behavior is termed as *differentiated consensuses*.

2: If Assumptions **A1.b** and **A1.c** hold, the averaged matrices $\bar{B}_{av}^k$ consist of elements

$$\bar{b}_{av}^{i,j,k} = \begin{cases} D_{j \div n}^{i,j \bmod n} b^{i,j \bmod n}, \text{ if } i \in N,\ j \bmod n \neq 0 \\ D_{j \div n}^{i,n} b^{i,n}, \text{ if } i \in N,\ j \bmod n = 0 \\ 1/\gamma_k, \text{ if } i = n+1, \dots, \bar{n},\ j = i - n, \\ 0, \text{ otherwise.} \end{cases} \tag{13}$$

Here, operation mod is a remainder of division, and $\div$ is a division without remainder. Note, that if $\bar{d} = 0$, then $\bar{B}_{av}^k = B_{av}$.

3: Agent productivities' for different task classes are defined according to productivity fractions which are the same for each agent in the system. Concerning this, in the case of continuous task flow when agents' queues for all priorities are nonempty, productivity vectors $\mathbf{p}_t^i$ are independent of each other, i.e. assumption **A1.e** holds.

4: In control protocols (6) we can choose γ_k depending on our intentions. We may either want to reduce noise sensitivity asymptotically in (12), in which case we should take smaller γ_k, or it may be more important for us to exchange the incoming tasks faster so we should take larger γ_k. In that case agents would exchange their tasks faster but the noise will have larger impact on the system. So, here we have a trade-off between the noise sensitivity and tasks exchanging in our system. The next theorem gives an asymptotically optimal solution.

Theorem *3:* **If** Assumptions **A1–A3** hold **then** optimal step-sizes $\gamma_k^\star$, $k = 1, \dots, m$, of each control protocol from (6) can be calculated by formulas:

$$\gamma_k^\star = -\frac{S_k}{H^k}\Delta^k + \sqrt{\frac{S_k^2}{H^{k2}}\Delta^{k2} + \frac{S_k}{H^k}} \tag{14}$$

where $\Delta^k = \frac{Re(\lambda_{\max}(Q))}{R}$.

Proof: Let's find the minimum of possible upper bounds for ε_k. For derivative of $\frac{\gamma_k^2 H^k + S_k}{\gamma_k \delta_k(\gamma_k)}$ by γ_k we have

$$\left(\frac{\gamma_k^2 H^k + S_k}{\gamma_k \delta_k(\gamma_k)} \right)' = \left(\frac{\gamma_k H^k + \frac{1}{\gamma_k} S_k}{R - \gamma_k Re(\lambda_{\max}(Q))} \right)' =$$

$$\frac{H^k - \frac{1}{\gamma_k^2} S_k}{R - \gamma_k Re(\lambda_{\max}(Q))} + \frac{(\gamma_k H^k + \frac{1}{\gamma_k} S_k) Re(\lambda_{\max}(Q))}{(R - \gamma_k Re(\lambda_{\max}(Q)))^2}.$$

Let's now set the obtained expression to zero and solve the quadratic equation:

$$\gamma_k^2 H^k + 2\gamma_k S_k \Delta^k - S_k = 0.$$

As a result we get optimal values for $\gamma_k^\star$ which are coincide with (14).

$\blacksquare$

Remarks 5: The optimality of step-sizes of control protocol (6) is understood in a sense that it provides the speed of convergence and noise tolerance of the protocol needed for achieving the maximal possible convergence (or minimal deviation from the consensus value) in the system under the given conditions.

V. Simulation Results

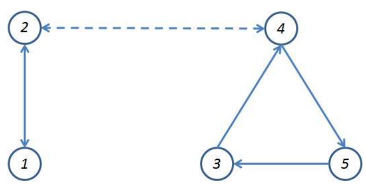

Fig. 1. The multi-agent system topology.

Let's consider an example of a network of five agents, having a topology shown on Fig. 1. Agents form two groups $\{1,2\}$ and $\{3,4,5\}$. Assume that links between agents in groups are stable and do not switch and link between groups may disappear with probability $\frac{1}{5}$. Let's also assume that cost of using links in groups is rather low and equals 1 while cost of link between groups is comparatively high and equals 5. Maximum delay for the information exchange $\bar{d}$ equals 1 and the probability of delay is $\frac{1}{3}$ and is the same for all edges. Noise in communication channels is normally distributed zero-mean random variable with parameter $\sigma_w = 3$. Agents' productivities are 8, 2, 1, 4 and 10 computational instructions per time unit. Number of incoming tasks is Poisson random variable with parameter $\lambda = 1$. Assume tasks of two different classes arrive at the agents. Let the productivity fractions be given as $2:1$, i.e. with all nonempty queues the agent's productivity will be divided among classes as $\frac{2}{3}p$ and $\frac{1}{3}p$ correspondingly.

In this case matrix A of network system topology will have the following form:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Note, that given topology graph is balanced. Let's assume that the following topology cost constraints are given: $\{c_k\}_{k=1,2} = \{6, 1.5\}$. While cost constraint for priority 1 is met if matrix of control protocol B_{av}^1 is taken equal to A, we have to reduce usage of costly $2-4$ and $4-2$ links for priority 2 to meet cost constraint. This is done by using these links randomly with probability $\frac{1}{10}$. Taking delays in account, extended matrix of control protocol for priority 2 will have the following form:

$$\bar{B}_{av}^2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \frac{5}{10}\frac{4}{5} & 0 & 0 & 0 & 0 & \frac{5}{10}\frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{10}\frac{4}{5} & 1 & 0 & 0 & 0 & \frac{5}{10}\frac{1}{5} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\gamma_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\gamma_2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

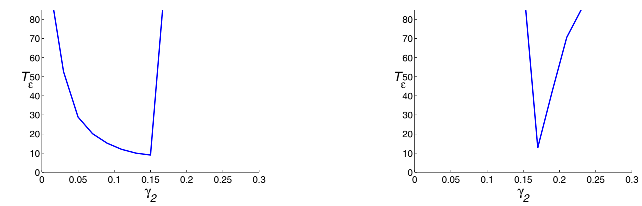

Fig. 2. Dependence between $\gamma^{\star,k}$ and the time to consensus achievement for task of priority class $k = 1, 2$.

Let's compute optimal step-size $\gamma_1^\star$ for balancing agents' loads for task of class 1. Among all agents maximal $\sigma_{r,k}$ is presented at the agent with largest productivity. In our example $\sigma_{r,1}^2 = 0.9497$ for agent with productivity 10. So S_k defined in (12) equals 9.7484 and $H^k = 2 \cdot 3^2 (5^2 0.16 + 1.4^2) = 107.28$ for H^k defined in (12) since $\sigma_b^2 = 0.16$. $\Delta^k = \frac{1.7761}{1.2764} = 1.3915$ since $D_{\max} = \frac{2}{3}$, $\|Re(\lambda_2(\mathscr{L}(B_{av}^k)))\| = 1.4146$, $R = 1.2764$ and $Re(\lambda_{max}(Q)) = 1.7761$. for Q defined in (10). According to formula (14) we have

$$\gamma_1^\star = -\frac{9.7484}{107.28} 1.3915 + \sqrt{\frac{9.7484^2}{107.28^2} 1.3915^2 + \frac{9.7484}{107.28}} \approx 0.2004$$

and according to Theorem 1 corresponding minimum value $\varepsilon_1^\star$ equals to 76.2045.

Let's make sure that $\gamma_1^\star$ satisfies inequality (10). Since $d_{max}(B_{av}^k) = \frac{7}{5}$, $\delta_1(\gamma_1^\star) = 1.2764 - 0.2004 \cdot 1.7761 \approx 0.9205 < \frac{7}{5}$ therefore γ_1 should be less than $\frac{5}{7}$ that is true in our case.

Making analogous calculations we get optimal step-sizes for other task classes $\gamma_2^\star = 0.2045$.

Simulation results in Fig. 2 show that optimal step-size $\gamma_1^\star$ also provides close to optimal rate of convergence of agent's states in the system. Horizontal axis corresponds to γ_1 and vertical axis gives T_ε time to ε-consensus ($\varepsilon = 76$ was taken in experiments).

Fig. 3 shows the behavior of agents' loads during task distribution via the described protocol with optimal step-size parameters. It could be seen from Fig. 3 that agent's loads achieve consensus for every task class. Higher lines correspond to loads for priority 2. Due to comparatively rare use of link between groups agents achieve consensus first inside their clusters. During transmission between clusters information exchange is rather extensive and loads of agents that have links to another cluster converge with high rate. After that new consensus value is reached inside clusters. The process continues until consensus in the whole system is reached. Behavior of agents' loads for priority 1 is the opposite. In absence of restrictions on link usage and due to higher rate of information exchange on "expensive" links the first to converge are states of agents connected with these links. After that agents' states in clusters converge to consensus value reached by agents linked to another cluster.

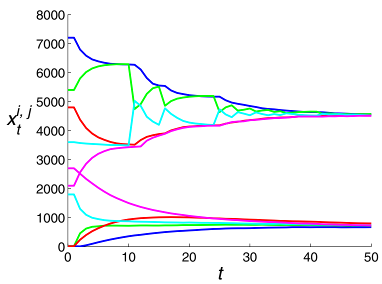

Fig. 3. Evaluation of agents' loads in the example for 3 classes of tasks.

VI. Conclusion

In this paper we examined a *differentiated consensuses* problem for a distributed stochastic network with cost constraints on system topology for different priorities of incoming tasks. The network model was assumed to have switched topology, noise and delays in measurements. We introduced the new control strategy (a modification of a local voting protocol) that uses different step-size parameters for each task class according to corresponding cost constraints and proposed an instrument to choose optimal step-size parameters. As in our previous work [19] for this model we obtained the conditions of achieving *differentiated consensuses* (m different consensus levels separately).

To illustrate the new theoretical results we presented simulation results that show the performance of the control protocol. It was shown that the larger step-size allows to achieve consensus among agents' loads faster. However, due to the larger noise sensitivity the deviation from the consensus value is relatively high compared to that for the smaller step-size. On the contrary, if the step-size is too low, though being tolerant to noise, the system achieves consensus rather slowly. It was shown how to choose the optimal step-size in the trade-off between the speed of convergence and noise tolerance in the system under given conditions.

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