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Differentiated Consensuses in a Stochastic Network with Priorities

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Abstract— **In this paper a distributed stochastic network system with incoming tasks that are classified with priorities is studied. The network system is assumed to have variable topology, and agents are not necessarily always connected to each other. In addition, the observations about neighbors' states are supposed to be obtained with random noise and delays. To ensure efficient operation of this network system, a novel control strategy is proposed. With this strategy, network resources are allocated in a randomized way with probabilities corresponding to each priority class. To maintain the balanced load across the network for different priorities, a so-called "differentiated consensuses" problem is examined. This consensus problem is that, in a system with multiple classes, consensus is targeted for each class, which may be different among classes. In this paper, the ability of the proposed control protocol to maintain almost balanced load, i.e. approximate consensus *for every priority class* across the network, is proved. In addition, a numerical example that illustrates the proposed control strategy and the results of simulations are provided.**

I. Introduction

To date, the consensus problem has been widely used to solve practical challenges such as information exchange in multiprocessor networks, distributed control of robotic networks, optimal task allocation in groups of unmanned areal vehicles, distribution of tasks in transportation networks, distributed computing, distributed control of learning or educational processes etc. [1], [2], [3], [4], [5], [6], [7]. These works usually consider networks consisting of agents which work together to accomplish certain tasks. Agents not necessarily have to be identical, e.g., they could have different capacities or productivities. Approaches and algorithms for distributed control in such networks have attracted a lot of research interest. Essentially the question is how to distribute tasks among the agents to ensure the system working more efficiently.

In general, for a network serving incoming tasks that have to be distributed among agents, various approaches for the allocation of these tasks can be used. In previous works [4], [5], it was shown that the problem of optimal (or near optimal) task distribution in a network among agents can be reformulated as a problem of reaching a consensus in the network. Based on this finding, one approach is to distribute the tasks among agents in accordance with the current loads of agents and in a view of their productivities [4], [5]. A limitation of this approach is that it treats all jobs as the same type, and does not take into account the specifics of tasks. In a real systems, however, some tasks could be more important or urgent then others and the control strategy should consider different priorities for them.

Recently in [8], it was discussed that the tasks in a network could be of different types, e.g. they have different importance (priority) levels. For such networks, the same task allocation question applies. To address the question, it is important to take into account the types of incoming jobs and the specifics of the entire system in the control. Specifically, it was highlighted [8] that there could be several consensus objectives for such a network, since, in practice, many network systems support more than one class where service differentiation exists. For example, in a system, the deadline requirement for a class of urgent tasks may be different from that for a class of normal tasks. This calls for *differentiated consensuses*, which we define as a consensus problem for systems with multiple classes, where a consensus is targeted for each class and may be different among classes. Ultimately, for the control goal of the network we want to achieve a consensus within each class, separately.

In [9], [10] the group consensus in multi-agent network is considered. Unlike group consensus, differentiated consensus has to be achieved throughout the whole network, not just among certain group of agents and thus the consensus value for every priority class has to be the same for every agent.

In [8], we considered a network with cost constrains where tasks are of different (strict) priorities and at an agent, when multiple tasks are present, the tasks at the highest priority level are served first. For this network, we proposed to distribute tasks among agents in accordance with the current load of each priority and the productivity of each agent. With the proposed strategy, it was proved [8] that the network can achieve *differentiated consensuses* and satisfy different cost constrains for all groups of priorities.

In this paper, we consider another general setting for the network with priorities. In particular, the scheduling discipline among different types of tasks at each agent is Strict Priority in [8], which has the known "starvation problem" for low priority tasks and may reduce the efficiency of the network [11]. In the present work, we extend our previous results to consider that the scheduling discipline at each agent is Probabilistic Priority [12], [13]. Essentially, we add to each priority class a probability. Specifically, the high priority tasks are executed with a high probability, but

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the execution of jobs with lower priority also have nonzero probability (but, of course, lower than the high priority tasks). Hence, depending on the priorities of tasks, they will be allocated with different amounts of network resources.

More specifically, we investigate *differentiated consensuses* in a distributed stochastic network with incoming tasks that are classified with priorities. In the considered network the communication links between agents may change over time, i.e. the topology is variable, and the information about states of neighbors (e.g., queue lengths) is to be obtained with noise and delays. We introduce a new control strategy for task redistribution among agents. Following this control strategy we use the randomized way to distribute resources, with corresponding probabilities. Depending on the priority level of tasks in a queue we allocate different amount of resources (more resources for high priority tasks, less resources for low priority tasks). We propose a control protocol and prove its ability to achieve almost balanced (equal) load, i.e. approximate consensus, *for every priority class* across the network.

The paper is organized as follows. Section II contains the problem formulation and notations used in the paper. In Section III the control protocol for achieving the differentiated consensus is suggested. The main result along with the proof and necessary assumptions are in the Section IV. Simulation results are presented in Section V. Section VI provides conclusion and plans for future work.

II. Problem Statement

Consider a dynamic network system of n agents, which collaborate with each other, and a set of tasks of different classes, which have to be executed in the system. Tasks come to possibly different agents of the system in different discrete time instants $t = 0,1,\dots,T$. Agents process incoming tasks in parallel. Tasks can be redistributed among agents based on a feedback. Note that a task cannot be interrupted after it is being processed by an agent, i.e. the system is non-preemptive.

Without loss of generality, agents in the system are numbered. Let i, $i = 1,\dots,n$, be the number of an agent. Assume, that $N = \{1,\dots,n\}$ denotes the set of agents in the network system. The network topology may switch over time. Let the dynamic network topology be modeled by a sequence of digraphs $\{(N,E_t)\}_{t\geq 0}$, where $E_t \subset E$ denotes the set of edges at time t of topology graph (N,E_t) and may change over time. The corresponding adjacency matrices are denoted as $A_t = [a_t^{i,j}]$, where $a_t^{i,j} > 0$ if agent j is connected with agent i and $a_t^{i,j} = 0$ otherwise. Here and below, an upper index of agent i is used as the corresponding number of an agent (while not as an exponent). Matrix $A_t = [a_t^{i,j}]$ is the adjacency matrix of the graph at time t. Denote $\mathscr{G}_{A_t}$ as the corresponding graph.

To introduce some properties of the network topology, the following definitions from the graph theory will be used. Define *the weighted in-degree* of node i as the sum of i-th row of matrix A: $d^i(A) = \sum_{j=1}^n a^{i,j}$; $D(A) = \mathrm{diag}\{d^i(A)\}$ is the corresponding diagonal matrix; $d_{\max}(A)$ is the maximum in-degree of the graph $\mathscr{G}_A$. Let $\mathscr{L}(A) = D(A) - A$ denote the *Laplacian* of the graph $\mathscr{G}_A$; $\cdot^{\mathrm{T}}$ is a vector or matrix transpose operation; $||A||$ is the Euclidian norm: $||A|| = \sqrt{\sum_i \sum_j (a^{i,j})^2}$; $Re(\lambda_2(A))$ is the real part of the second eigenvalue of matrix A ordered by absolute magnitude; $\lambda_{\max}(A)$ is the maximum eigenvalue of matrix A.

The digraph $\mathscr{G}_B$ is to be said a subgraph of a digraph $\mathscr{G}_A$ if $b^{i,j} \leq a^{i,j}$ for all $i,j \in N$.

The digraph $\mathscr{G}_A$ is said to contain a *spanning tree* if there exists a directed tree $\mathscr{G}_{tr} = (N,E_{tr})$ as a subgraph of $\mathscr{G}_A$.

We suppose that tasks (jobs) belong to different classes $k = 1,\dots,m$ and every agent has m queues — one for each task class.

Particularly, the behavior of an agent $i \in N$ is described by two characteristics:

- the m-vector of queue lengths of tasks $\mathbf{q}_t^i = [q_t^{i,k}]$ at time t whose k-th element is defined by the amount of tasks of k-th class $k = 1,\dots,m$;
- the productivity p^i.

Agents, having each its own probability or the number of operations the agent can execute during the time instant, should distribute it among all tasks classes in such a way that, on the one hand the priorities for task classes are provided and on the other hand the "starvation problem" is taken into account i.e. tasks of the lower priority classes don't wait for execution for too long. This is achieved by making use of the probabilistic priorities. Each task class is given a productivity fraction P_k, $k = 1,\dots,m$ which is the same for certain class k on every agent in the system. On each agent the tasks from its queues are chosen for execution randomly according to the following formula:

$$\tilde{p}_t^{i,k} = \begin{cases} \frac{P_k}{\sum_{q_t^{i,l}>0} P_l}, & \text{if } q_t^{i,k} > 0; \\ 0, & \text{otherwise,} \end{cases}$$

where $\tilde{p}_t^{i,k}$ is the probability of choosing a task of class k for execution on agent i at a time instant t. Therefore the bigger the fraction P_k is, the higher is the chance for the task of class k to be executed. Thus agent's productivity is distributed among all classes of tasks in the following way:

$$p_t^{i,k} = \tilde{p}_t^{i,k} p^i. \tag{1}$$

Here $p_t^{i,k}$ is a number of operations allocated for tasks of class k on agent i at a time instant t. Note that according to the way $\tilde{p}_t^{i,k}$ is defined if at certain time instant t' the queue of tasks of class k' on the agent i' is empty, no operations would be allocated for tasks of class k'. Instead $p_{t'}^{i',k'}$ operations would be distributed among other task classes in proportions of their productivity fractions P_k, $k \neq k'$.

For all $i \in N$, $t = 0,1,\dots,T$, the dynamics of the network system is as follows

$$\mathbf{q}_{t+1}^i = \mathbf{q}_t^i - \mathbf{p}_t^i + \mathbf{z}_t^i + \mathbf{u}_t^i, \tag{2}$$

where $\mathbf{p}_t^i = [p_t^{i,k}]$, and $\mathbf{z}_t^i = [z_t^{i,k}]$ is an m-vector whose k-th element $z_t^{i,k}$ is the amount of new tasks of class k, which came to the system and were received by agent i at time instant

t; $\mathbf{u}_t^i \in \mathbb{R}^m$ is an m-vector of control actions (redistributed tasks of class k to agent i at time instant t), which could (and should) be chosen based on some information about queue lengths of neighbors $\mathbf{q}_t^j,\ j \in N_t^i$, where N_t^i is the set $\{j \in N : a_t^{i,j} > 0\}$.

Assume, that $p^i \neq 0,\ \forall i \in N$ and $P_k \neq 0,\ k = 1,\dots,m$. In [4] it was proven that from all possible options for the redistribution of all tasks the minimum operation time of the system is achieved when the load (defined as the ratio of the queue length over the productivity) is equalized throughout the network. Hence, it is important to consider the achievement of the following goal.

It is required to maintain balanced (equal) load across the network for every priority class.

At this setting we can consider the consensus problem for states $\mathbf{x}_t^i = [x_t^{i,k}]$ of agents, where

$$x_t^{i,k} = \begin{cases} q_t^{i,k}/\tilde{p}_t^{i,k}, & \text{if } \tilde{p}_t^{i,k} > 0; \\ 0, & \text{otherwise.} \end{cases}$$

We emphasize, that $\mathbf{x}_t^i$ *is a state vector, consisting of states for m classes.*

To ensure balanced load across the network (e.g., in order to increase the overall throughput of the system and to reduce the execution time), it is naturally to use a redistribution protocol over time.

We assume that to form the control strategy $\mathbf{u}_t^i$ each agent $i \in N$, has noisy and possibly delayed observations about its neighbors' states

$$\mathbf{y}_t^{i,j} = \mathbf{x}^j_{t-s_t^{i,j}} + \mathbf{w}_t^{i,j},\ j \in N_t^i, \tag{3}$$

where $\mathbf{w}_t^{i,j}$ is a noise vector, $0 \le s_t^{i,j} \le \bar{s}$ are integer-valued delays, and $\bar{s}$ is a maximum of possible delays.

III. Control Protocol

In [4], [5], properties of a control algorithm, called local voting protocol, for load balancing problem of a stochastic network were studied. The control value of the local voting protocol for each agent was determined by the weighted sum of differences between the information about the state of the agent and the information about its neighbors' states.

Let's consider a similar family of protocols as follows. For each $k = 1,\dots,m$ we define

$$u_t^{i,k} = \gamma \tilde{p}_t^{i,k} \sum_{j \in \bar{N}_t^i} b_t^{i,j}(y_t^{i,j,k} - x_t^{i,k}), \tag{4}$$

where $\gamma > 0$ is a step-size of the control protocol and $\bar{N}_t^i \subset N_t^i$ is the neighbor set of agent i (note, that we could use not all the available connections, but some subset of them), $b_t^{i,j}$ are protocol coefficients. With the use of this protocol, the system works in such a way that within each priority tasks are distributed evenly. Let $B_t = [b_t^{i,j}]$ be the matrices of task redistribution protocols for every time instant t. (We set $b_t^{i,j} = 0$ when $a_t^{i,j} = 0$ or $j \notin \bar{N}_t^i$.)

Due to a construction method of matrices B_t, the corresponding graph $\mathscr{G}_{B_t}$ most of the time may has the same topology as graph $\mathscr{G}_{A_t}$ of matrix A_t or more poor.

Let's assume $\bar{s} = 0$. Then the dynamics of the closed loop system with protocol (4) will be as follows

$$\mathbf{x}_{t+1}^i = \mathbf{x}_t^i - \tilde{\mathbf{r}}_t^i + \tilde{\mathbf{z}}_t^i + \gamma \sum_{j \in N_t^i} b_t^{i,j}(\mathbf{y}_t^{i,j} - \mathbf{x}_t^i) =$$

$$\mathbf{x}_t^i - \tilde{\mathbf{r}}_t^i + \tilde{\mathbf{z}}_t^i + \gamma \left(\sum_{j \in N_t^i} b_t^{i,j} \mathbf{x}_t^j \right) - \gamma d^i(B_t)\mathbf{x}_t^i + \gamma \tilde{\mathbf{w}}_t^i,\ i \in N, \tag{5}$$

where vectors $\tilde{\mathbf{r}}_t^i = [\tilde{r}_t^{i,k}]$ and $\tilde{\mathbf{z}}_t^i = [\tilde{z}_t^{i,k}]$ consist of components

$$\tilde{r}_t^{i,k} = \begin{cases} p_t^{i,k}/\tilde{p}_t^{i,k}, & \text{if } \tilde{p}_t^{i,k} > 0; \\ 0, & \text{othewise,} \end{cases} \quad \tilde{z}_t^{i,k} \begin{cases} z_t^{i,k}/\tilde{p}_t^{i,k}, & \text{if } \tilde{p}_t^{i,k} > 0; \\ 0, & \text{othewise,} \end{cases}$$

and $\tilde{\mathbf{w}}_t^i = \sum_{j \in \bar{N}_t^i} b_t^{i,j} \mathbf{w}_t^{i,j}$.

Let us rewrite Eq. (5) in a more compact form. Define the $\mathbb{R}^{nk}$-valued vectors $\mathbf{X}_t$, $\mathbf{R}_t$, $\mathbf{Z}_t$, $\mathbf{Y}_t$ and $\mathbf{W}_t$ by concatenation of corresponding vectors $\tilde{\mathbf{x}}_t^i$, $\tilde{\mathbf{r}}_t^i$, $\tilde{\mathbf{z}}_t^i$, and $\tilde{\mathbf{w}}_t^i$. The dynamics of the closed loop system with protocol (4) reduces to

$$\mathbf{X}_{t+1} = \mathbf{X}_t + \gamma(B_t \otimes I_m)\mathbf{X}_t - \gamma(D(B_t) \otimes I_m)\mathbf{X}_t - \mathbf{R}_t + \mathbf{Z}_t + \gamma \mathbf{W}_t, \tag{6}$$

where $B_t \otimes I_m$ is the Kronecker product the $nm \times nm$ which is the block matrix:

$$B_t \otimes I_m = \begin{bmatrix} b_t^{1,1} I_m & \cdots & b_t^{1,n} I_m \\ \vdots & \ddots & \vdots \\ b_t^{n,1} I_m & \cdots & b_t^{n,n} I_m \end{bmatrix}.$$

If $\bar{s} > 0$ we "artificially" add $n\bar{s}$ new agents to the current network topology. At each time instant t the new "fictitious" agents have states which are equal to the corresponding states of "real" agents at previous time instants $t-1, t-2, \dots, t-\bar{s}$. The same is done for every class $k = 1\dots m$. Let $x_t^{i,k} \equiv 0, i \in N$ for $-\bar{s} \le t < 0$. Denote $\bar{\mathbf{X}}_t \in \mathbb{R}^{\bar{n}}$, $\bar{n} = nm(\bar{s}+1)$, as an extended state vector for $t = 0,1,\dots$ which consists of $\bar{s}+1$ (nm)-vectors $\mathbf{X}_t, \mathbf{X}_{t-1}, \dots, \mathbf{X}_{t-\bar{s}}$, i.e. it includes all the components with all kinds of delays not exceeding $\bar{s}$. Denote $n(\bar{s}+1)$ as $\tilde{n}$. Introduce the extended $\tilde{n} \times \tilde{n}$ matrix $\bar{B}_t$ of the control protocol (4) which consists of zeros at all places except $|\bar{N}_t^i|$ entries $\bar{b}_t^{i,j+ns_t^{i,j}}$ in each $i \in N,\ j \in \bar{N}_t^i$ of n first lines, which are equal to $b_t^{i,j}$ and $\bar{b}_t^{i,i-n} = 1/\gamma$ in next $n\bar{s}$ lines, $i = n+1, \dots, \tilde{n}$.

Due to the view of the Laplacian matrix $\mathscr{L}(\bar{B}_t \otimes I_m)$ we can rewrite the dynamics of the system in the following vector-matrix form:

$$\bar{\mathbf{X}}_{t+1} = \bar{\mathbf{X}}_t - \gamma \mathscr{L}(\bar{B}_t \otimes I_m)\bar{\mathbf{X}}_t + \begin{pmatrix} -\mathbf{R}_t + \mathbf{Z}_t + \gamma \mathbf{W}_t \\ 0 \end{pmatrix}. \tag{7}$$

IV. Main Results

A. Assumptions

Let $(\Omega, \mathscr{F}, P)$ be the underlying probability space corresponding to the sample space, the collection of all events, and the probability measure respectively, and E be a mathematical expectation symbol.

We assume that graphs $\mathscr{G}_{B_t}$, $t = 1, \dots$ are i.i.d. (independent identically distributed), i.e. the random events of appearance of edge (j,i) are independent and identically distributed for the fixed (j,i). Let $b_{av}^{i,j}$ define mean values (mathematical expectations) of $b_t^{i,j}$, and B_{av} is the corresponding adjacency matrix.

Assume that the following conditions are satisfied:

- **A1**. Graph $\mathscr{G}_{B_{av}}$ has a spanning tree, for the consensuses to be achievable throughout the system [14].
- **A2**. **a)** For all $i \in N,\ j \in N_t^i$, observation noise vectors $\mathbf{w}_t^{i,j}$ are zero-mean, independent identically distributed (i.i.d.) random vectors with bounded variances: $\mathrm{E}(\mathbf{w}_t^{i,j})^2 \leq \sigma_w^2$.
  b) For all $i \in N, j \in N_{\max}^i = \cup_t \bar{N}_t^i$ the appearance of "time-varying" edges (j,i) in graph $\mathscr{G}_{B_t}$ is independent random event. For all $i \in N,\ j \in N_t^i$ weights $b_t^{i,j}$ in the control protocol are independent random variables with expectations: $\mathrm{E}\tilde{b}_t^{i,j} = b^{i,j}$, and bounded variances: $\mathrm{E}(\tilde{b}_t^{i,j} - b^{i,j})^2 \leq \sigma_b^2$.
  c) For all $i \in N, j \in N^i$ there exists a finite value $\bar{s} \in \mathbb{N}$: $s_t^{i,j} \leq \bar{s}$ with probability 1, and integer-valued delays $s_t^{i,j}$ are i.i.d. random variables taking value $l = 0,\dots,\bar{s}$ with probability $p_l^{i,j}$.
  d) For all $k = 1,\dots,m,\ i \in N,\ t = 0,1,\dots$ random values $z_t^{i,k}$ are independent with expectations: $\mathrm{E}z_t^{i,k} = \bar{z}^k$ which do not depend on i, and variances: $\mathrm{E}(z_t^{i,k} - \bar{z}^k)^2 \leq \sigma_{z,k}^2$.
  e) For all $i \in N,\ t = 0,1,\dots$ random vectors $\mathbf{p}_t^i$ are independent and consist of independent components. Random values $\tilde{r}_t^{i,k},\ k = 1,\dots,m$, have expectations: $\mathrm{E}\tilde{r}_t^{i,k} = \bar{r}^k$ which do not depend on i.
  Additionally, all mentioned in Assumptions **A2.a–A2.d** independent random variables and vectors are mutually independent.
  In general, if Assumptions **A2.b** and **A2.c** hold, the averaged matrices $\bar{B}_{av} = \mathrm{E}\bar{B}_t$, consist of elements
  $$\bar{b}_{av}^{i,j} = \begin{cases} p_{j \div n}^{i,(j \bmod n)+1} b^{i,(j \bmod n)+1}, \text{ if } i \in N,\ j = 1,\dots,\tilde{n}, \\ 1/\gamma, \text{ if } i = n+1,\dots,\tilde{n},\ j = i-n, \\ 0, \text{ otherwise.} \end{cases} \tag{8}$$
  Here, the operation mod is a remainder of division, and $\div$ is a division without remainder.
  Note, that if $\bar{s} = 0$, then $\bar{B}_{av} = B_{av}$.
- **A3.** For the step-size of the control protocol $\gamma > 0$ the following conditions are satisfied:
  $$\delta = |Re(\lambda_2(\bar{B}_{av} \otimes I_m))| - \gamma Re(\lambda_{\max}(Q)) > 0, \tag{9}$$
  where
  $$Q = \mathrm{E}C_t^{\mathrm{T}} C_t,\ \ C_t = \mathscr{L}(\bar{B}_{av} \otimes I_m) - \mathscr{L}(\bar{B}_t \otimes I_m).$$
  and
  $$\gamma \leq \frac{1}{\max\{d_{\max}(B_{av}), \delta\}}. \tag{10}$$

B. Averaged Models

Let $\mathbf{x}_0^\star$ be the weighted average m-vector of the initial states
$$\mathbf{x}_0^\star = \frac{\sum_i g_i \mathbf{x}_0^i}{\sum_i g_i}$$
where g^T is the left eigenvector of matrix B_{av} [15] and $\{\mathbf{x}_t^\star\}$ is the trajectory of averaged systems
$$\mathbf{x}_{t+1}^\star = \mathbf{x}_t^\star + \bar{\mathbf{z}} - \bar{\mathbf{r}}, \tag{11}$$
where m-vectors $\bar{\mathbf{z}} = [\bar{z}^k]$ and $\bar{\mathbf{r}} = [\bar{r}^k]$ consist of expectations which are defined by Assumptions **A2.d, A2.e**.

Note that in the case of balanced topology graph $\mathscr{G}_{B_{av}}$, $\mathbf{x}_0^\star = \frac{1}{n}\sum_{i=1}^n \mathbf{x}_0^i$.

C. Differentiated Consensuses

Consider vectors $\bar{\mathbf{X}}_t^\star \in \mathbb{R}^{\tilde{n}},\ t = 0,1,\dots,$ which consist of $\mathbf{1}_n \otimes \mathbf{x}_t^\star, \mathbf{1}_n \otimes \mathbf{x}_{t-1}^\star, \dots, \mathbf{1}_n \otimes \mathbf{x}_{t-\bar{s}}^\star$.

Theorem *1:* If Assumptions **A1–A3** hold for trajectories of closed-loop systems (5) and (11) the following inequality holds:
$$\mathrm{E}||\bar{\mathbf{X}}_t - \bar{\mathbf{X}}_t^\star||^2 \leq \frac{\Delta}{\gamma\delta} + (1-\gamma\delta)^t \left(||\bar{\mathbf{X}}_0 - \bar{\mathbf{X}}_0^\star||^2 - \frac{\Delta}{\gamma\delta} \right),$$
where
$$\Delta = 2m\sigma_w^2\gamma^2(n^2\sigma_b^2 + ||B_{av}||^2) + n\sum_{k=1}^m (\sigma_{z,k}^2 + (1-P_k)^2),$$
i.e. if additionally $\mathrm{E}||\bar{\mathbf{X}}_0 - \bar{\mathbf{X}}_0^\star||^2 < \infty$, **then** the asymptotic mean square ε-consensus in (5) is achieved with
$$\varepsilon \leq \frac{\Delta}{\gamma\delta}.$$

Proof: Consider vectors $\bar{\mathbf{X}}_t^\star \in \mathbb{R}^{\tilde{n}},\ t = 0,1,\dots,$ which consist of $\mathbf{1}_n \otimes \mathbf{x}_t^\star, \mathbf{1}_n \otimes \mathbf{x}_{t-1}^\star, \dots, \mathbf{1}_n \otimes \mathbf{x}_{t-\bar{s}}^\star$ and satisfy the equation:
$$\bar{\mathbf{X}}_{t+1}^\star = U\bar{\mathbf{X}}_t^\star + \begin{pmatrix} \bar{\mathbf{Z}} - \bar{\mathbf{R}} \\ 0 \end{pmatrix}, \tag{12}$$
where $\mathbf{1}_n$ is the n-vector of units, $\bar{\mathbf{Z}} = \mathbf{1}_n \otimes \bar{z}$, $\bar{\mathbf{R}} = \mathbf{1}_n \otimes \bar{r}$, and U is a $\tilde{n} \times \tilde{n}$ matrix:
$$U = \begin{pmatrix} I_{nk} & 0 & \dots & 0 & 0 \\ I_{nk} & 0 & \dots & 0 & 0 \\ 0 & I_{nk} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & I_{nk} & 0 \end{pmatrix}.$$

The vector $\mathbf{1}_{\tilde{n}}$ is the right eigenvector of Laplacian-type matrices $\bar{\mathscr{L}}_t = \gamma\mathscr{L}(\bar{B}_t \otimes I_m)$ and $\tilde{\mathscr{L}} = \gamma\mathscr{L}(\bar{B}_{av} \otimes I_m)$ corresponding to the zero eigenvalue: $\bar{\mathscr{L}}_t \mathbf{1}_{\tilde{n}} = \tilde{\mathscr{L}}\mathbf{1}_{\tilde{n}} = 0$. Sums of all elements in rows of matrices $\bar{\mathscr{L}}_t$ or $\tilde{\mathscr{L}}$ are equal to zero and, moreover, all the diagonal elements are positive and equal to the absolute value of the sum of all other elements in the row.

Due to the definition of matrices $\bar{\mathscr{L}}_t$ for differences $v_{t+1} = \bar{\mathbf{X}}_{t+1} - \bar{\mathbf{X}}_{t+1}^\star$ of trajectories of systems (7) and (12) we have
$$v_{t+1} = \bar{\mathbf{X}}_t - \bar{\mathscr{L}}_t \bar{\mathbf{X}}_t + \begin{pmatrix} \mathbf{F}_t + \gamma\mathbf{W}_t \\ 0 \end{pmatrix} - U\bar{\mathbf{X}}_t^\star - \begin{pmatrix} \bar{\mathbf{F}} \\ 0 \end{pmatrix}$$
$$= v_t - \bar{\mathscr{L}}_t v_t + \begin{pmatrix} \gamma\mathbf{W}_t + \mathbf{F}_t - \bar{\mathbf{F}} \\ 0 \end{pmatrix},$$

where $\mathbf{F}_t = \mathbf{Z}_t - \mathbf{R}_t$ and $\bar{\mathbf{F}} = \bar{\mathbf{Z}} - \bar{\mathbf{R}}$. Further, by adding and subtracting $\tilde{\bar{\mathscr{L}}}\nu_t$ we get

$$\nu_{t+1} = (I_{\tilde{n}} - \tilde{\mathscr{L}})\nu_t + (\tilde{\mathscr{L}} - \bar{\mathscr{L}}_t)\nu_t + \begin{pmatrix} \gamma \mathbf{W}_t + \mathbf{F}_t - \bar{\mathbf{F}} \\ 0 \end{pmatrix}.$$

Let $\tilde{\mathscr{F}}_t$ denote σ-algebra of all probabilistic events, generated by the random elements $\mathbf{x}_0^i, \mathbf{w}_0^{i,j}, \dots, \mathbf{w}_{t-1}^{i,j}, \mathbf{z}_0^i, \dots, \mathbf{z}_{t-1}^i, s_0^{i,j}, \dots, \quad s_{t-1}^{i,j}, \mathbf{p}_0^i, \dots, \mathbf{p}_{t-1}^i, b_0^{i,j}, \dots, b_{t-1}^{i,j}, b_t^{i,j}$, $i,j \in N$, Consider the conditional mathematical expectation of the squared norm ν_{t+1}:

$$\mathrm{E}_{\tilde{\mathscr{F}}_t}||\nu_{t+1}||^2 = ||(I_{\tilde{n}} - \tilde{\mathscr{L}})\nu_t||^2 + 2\nu_t^{\mathrm{T}}(I_{\tilde{n}} - \tilde{\mathscr{L}})^{\mathrm{T}}(\tilde{\mathscr{L}} - \bar{\mathscr{L}}_t)\nu_t +$$
$$+2\nu_t^{\mathrm{T}}(I_{\tilde{n}} - \bar{\mathscr{L}}_t)^{\mathrm{T}} \begin{pmatrix} \gamma \mathrm{E}_{\tilde{\mathscr{F}}_t}\mathbf{W}_t + \mathrm{E}_{\tilde{\mathscr{F}}_t}(\mathbf{F}_t - \bar{\mathbf{F}}) \\ 0 \end{pmatrix} + \quad (13)$$
$$+2\gamma \mathrm{E}_{\tilde{\mathscr{F}}_t}\mathbf{W}_t^{\mathrm{T}}(\mathbf{F}_t - \bar{\mathbf{F}}) + \nu_t^{\mathrm{T}}(\tilde{\mathscr{L}} - \bar{\mathscr{L}}_t)^{\mathrm{T}}(\tilde{\mathscr{L}} - \bar{\mathscr{L}}_t) +$$
$$+\gamma^2 \mathrm{E}_{\tilde{\mathscr{F}}_t}||\mathbf{W}_t||^2 + \mathrm{E}_{\mathscr{F}_{t-1}}||\mathbf{F}_t - \bar{\mathbf{F}}||^2.$$

By Assumption **A2.d,A2.e** and the independence of $\mathbf{F}_t$ from σ-algebra $\mathscr{F}_{t-1}$ we have

$$\mathrm{E}_{\tilde{\mathscr{F}}_t}(\mathbf{F}_t - \bar{\mathbf{F}}) = \mathrm{E}(\mathbf{F}_t - \bar{\mathbf{F}}) = 0, \quad (14)$$
$$\mathrm{E}_{\tilde{\mathscr{F}}_t}||\mathbf{F}_t - \bar{\mathbf{F}}||^2 = \mathrm{E}||\mathbf{F}_t - \bar{\mathbf{F}}||^2 = n\sum_{k=1}^{m} \sigma_{z,k}^2 + (1-P_k)^2.$$

Due to Assumption **A2.a**, mutual independence of $w_t^{i,j}$, $i,j \in N$, and their independence from $\mathbf{F}_t$ and σ-algebra $\tilde{\mathscr{F}}_t$, we obtain

$$\mathrm{E}_{\tilde{\mathscr{F}}_t}\sum_{j\in\bar{N}_t^1} b_t^{i,j}(\mathbf{w}_t^{i,i} - \mathbf{w}_t^{i,j}) = \sum_{j\in\bar{N}_t^1} b_t^{i,j}\mathrm{E}(\mathbf{w}_t^{i,i} - \mathbf{w}_t^{i,j}) = 0,$$
$$\mathrm{E}_{\tilde{\mathscr{F}}_t}\sum_{j\in\bar{N}_t^1} b_t^{i,j}(\mathbf{w}_t^{i,i} - \mathbf{w}_t^{i,j})(\mathbf{F}_t - \bar{\mathbf{F}}) = \quad (15)$$
$$\sum_{j\in\bar{N}_t^1} b_t^{i,j}\mathrm{E}(\mathbf{w}_t^{i,i} - \mathbf{w}_t^{i,j})\mathrm{E}(\mathbf{F}_t - \bar{\mathbf{F}}) = 0,$$

and

$$\mathrm{E}_{\tilde{\mathscr{F}}_t}\left(\sum_{j\in\bar{N}_t^i} b_t^{i,j}(\mathbf{w}_t^{i,i} - \mathbf{w}_t^{i,j})\right)^2 = \quad (16)$$
$$= \sum_{j\in\bar{N}_t^i}(b_t^{i,j})^2(\mathrm{E}(\mathbf{w}_t^{i,i})^2 + \mathrm{E}(\mathbf{w}_t^{i,j})^2) = 2m\sigma_w^2\sum_{j\in\bar{N}_t^i}(b_t^{i,j})^2.$$

Taking into account the above relations (14)–(16) and denoting by $\bar{\mathbf{b}}_t$ the vector, consisting of the components $\sum_{j\in\bar{N}_t^1}(b_t^{1,j})^2, \dots, \sum_{j\in\bar{N}_t^n}(b_t^{n,j})^2$, we derive from (13)

$$\mathrm{E}_{\tilde{\mathscr{F}}_t}||\nu_{t+1}||^2 = ||(I_{\tilde{n}} - \tilde{\mathscr{L}})\nu_t||^2 + 2\nu_t^{\mathrm{T}}(I_{\tilde{n}} - \tilde{\mathscr{L}})^{\mathrm{T}}(\tilde{\mathscr{L}} - \bar{\mathscr{L}}_t)\nu_t +$$
$$+\nu_t^{\mathrm{T}}(\tilde{\mathscr{L}} - \bar{\mathscr{L}}_t)^{\mathrm{T}}(\tilde{\mathscr{L}} - \bar{\mathscr{L}}_t) + 2m\sigma_w^2\gamma^2\bar{\mathbf{b}}_t + n\sum_{k=1}^{m}(\sigma_{z,k}^2 + (1-P_k)^2). \quad (17)$$

Let $\mathscr{F}_t$ denote the σ-algebra of probabilistic events, generated by all random elements $\mathbf{x}_0^i, \mathbf{w}_0^{i,j}, \dots, \mathbf{w}_t^{i,j}, \mathbf{z}_0^i, \dots, \mathbf{z}_t^i, s_0^{i,j}, \dots, \quad s_t^{i,j}, \mathbf{p}_0^i, \dots, \mathbf{p}_t^i, b_0^{i,j}, \dots, b_t^{i,j}$, $i,j \in N$, that were implemented before time t. Consider conditional expectations of both sides of (17). Due to stochastic properties of the uncertainties **A2b,c** and the independence of $\bar{B}_t^k$ and $\bar{\mathbf{b}}_t$ from σ-algebra $\tilde{\mathscr{F}}_t$ we obtain

$$\mathrm{E}_{\mathscr{F}_{t-1}}||\nu_{t+1}||^2 = ||(I_{\tilde{n}} - \tilde{\mathscr{L}})\nu_t||^2 + \gamma^2\nu_t^{\mathrm{T}}Q\nu_t +$$
$$+2m\sigma_w^2\gamma^2\mathrm{E}_{\mathscr{F}_{t-1}}\bar{\mathbf{b}}_t + n\sum_{k=1}^{m}(\sigma_{z,k}^2 + (1-P_k)^2) \le$$
$$(1 - \gamma|Re(\lambda_2(\bar{B}_{av}\otimes I_m))| + \gamma^2\lambda_{\max}(Q))||\nu_t||^2 +$$
$$+2m\sigma_w^2\gamma^2(n^2\sigma_b^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=0}^{\bar{s}} p_k^{i,j}(b^{i,j})^2) +$$
$$+n\sum_{k=1}^{m}(\sigma_{z,k}^2 + (1-P_k)^2) = (1-\gamma\delta)||\nu_t||^2 +$$
$$+2m\sigma_w^2\gamma^2(n^2\sigma_b^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}(b^{i,j})^2) +$$
$$+n\sum_{k=1}^{m}(\sigma_{z,k}^2 + (1-P_k)^2) = (1-\gamma\delta)||\nu_t||^2 + \Delta.$$

We take the unconditional expectation and get:

$$\mathrm{E}||\nu_{t+1}||^2 \le (1-\gamma\delta)\mathrm{E}||\nu_t||^2 + \Delta,$$

By Lemma 1 of Chapter 2 of [16] it follows that inequality (12), which is the first part of Theorem 1, holds.

The second conclusion about the asymptotic mean square ε-consensus follows from inequality (12) if $t \to \infty$. Since Assumption **A4** is satisfied, we obtain that $|1-\gamma\delta| < 1$, and, therefore, the second term of (12) exponentially tends to zero. ■

At this point, we highlight that, the result of Theorem 1 shows that queues with different priorities achieve m different consensus levels separately. This behavior is termed as *differentiated consensuses*.

V. Simulation Results

Let's consider the example of a network of five agents, connected as a circle. Assume the links between agents may disappear with probability of $1/5$, and "diagonal" links may also appear with the same probability. Maximum delay for information exchange $\bar{s}$ equals 1 and the probability of delay appearance is equal to $1/3$ and is the same for all edges. Let there be tasks of three different classes arriving at the agents. The productivity fractions are 4 : 2 : 1, i. e. with all queues nonempty the agent's productivity will be divided among classes as $\frac{4}{4+2+1}p = \frac{4}{7}p$, $\frac{2}{7}p$ and $\frac{1}{7}p$ correspondingly.

So, in this case matrix B_{av} will have the form

$$B_{av} = \begin{pmatrix} 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{4}{5} \\ \frac{4}{5} & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{4}{5} & 0 & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{4}{5} & 0 & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{4}{5} & 0 \end{pmatrix}$$

Note that given average topology graph $\mathscr{G}_{B_{av}}$ is balanced.

Let's choose $\gamma = 1/2$ For B_{av} we have $d_{\max}(B_{av}) = 7/5$.

Matrix $\bar{B}_{av}$ will look like this:

$$\bar{B}_{av} = \begin{pmatrix} 0 & \frac{2}{15} & \frac{2}{15} & \frac{2}{15} & \frac{8}{15} & 0 & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{4}{15} \\ \frac{8}{15} & 0 & \frac{2}{15} & \frac{2}{15} & \frac{2}{15} & \frac{4}{15} & 0 & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{2}{15} & \frac{8}{15} & 0 & \frac{2}{15} & \frac{2}{15} & \frac{1}{15} & \frac{4}{15} & 0 & \frac{1}{15} & \frac{1}{15} \\ \frac{2}{15} & \frac{2}{15} & \frac{8}{15} & 0 & \frac{2}{15} & \frac{1}{15} & \frac{1}{15} & \frac{4}{15} & 0 & \frac{1}{15} \\ \frac{2}{15} & \frac{2}{15} & \frac{2}{15} & \frac{8}{15} & 0 & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{4}{15} & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this case we can compute $|Re(\lambda_2(\bar{B}_{av} \otimes I_m)| = 1.5396$, $Re(\lambda_{\max}(Q)) = 0.4963$. We see, that both $\delta = Re(\lambda_2(\bar{B}_{av} \otimes I_m) - \gamma Re(\lambda_{\max}(Q)) = 1.5396 - 0.5 * 0.4963 = 1.2915 > 0$ and $1/2 = \gamma \leq \frac{1}{\max\{d_{\max}(B_{av}),\delta\}} = \frac{1}{\max\{7/5,\, 1.2915\}} = \frac{1}{7.5} = 5/7$ are satisfied, so (9) and (10) are true here.

Fig. 1 shows behavior of agents' queue lengths during task distribution via described protocol. In the experiment agents are given tasks with exponentially distributed "complexities" with parameter 1 while agents' productivities also equal 1.

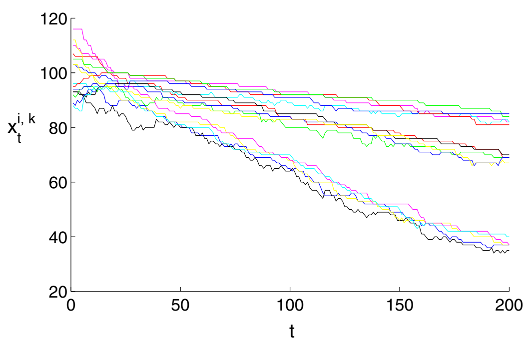

Fig. 1. Evaluation of queue lengths in the example for 3 classes of tasks

The lines on the figure correspond to agent's queue lengths for every class of tasks, that is $x_t^{i,k}$. In the experiment shown above there are five agents in the system with three queues each.

In Fig. 1 typical behaviors of queue lengths of tasks of different classes are shown. The states of different agents for the same class k congregate and reduce due to tasks execution. The more the productivity fraction for the class is, the faster corresponding states of agents reduce.

VI. Conclusion

In this paper we examined a new consensus problem, called *differentiated consensuses*. Specifically, we considered a distributed stochastic network, where incoming tasks have different priorities. The network model was supposed to have switched topology, noise and delays in measurements. For this network, we obtained the conditions of achieving m (possibly different) consensus levels separately. A new control strategy that allocates the resources of the network in a randomized way with corresponding probabilities for each priority class was introduced. To illustrate the theoretical results we considered a numerical example which shows the performance of the control protocol.

Standard assumptions on statistical properties for the topology, noise and delays in measurements are considered. In [17] these conditions are weakened and some approaches how to deal with the systems under the influence of almost arbitrary external perturbations are suggested. We plan to extend these approaches to consider the case with deterministic and *unknown but bounded noise*.

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