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Two Procedures with Randomized Controls for the Parameters' Confidence Region of Linear Plant under External Arbitrary Noise

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Abstract— **In this paper we join the previous results about asymptotic properties of a randomized control strategy with new one based on a non-asymptotic approach of LSCR (Leave-out Sign-dominant Correlation Regions) methods. Our consideration is focused on problems of the identification or adaptive optimal control for the linear plant with unknown parameters and dynamics disturbed by an external noise. The assumptions about the noise are reduced to a minimum: it can virtually be arbitrary, but, independently of it, the user must be able to add test perturbations through the input channel. The theoretical results are illustrated by simulations.**

I. Introduction

The severity of the identification problem is commonly caused by the insufficient variability of an input signal. In control systems test control signals can be fed to the input of a control plant, which alleviates the problem of the plant unknown parameters reconstruction. For example, under the assumption that noise lacks a harmonic signal arriving at the input of a linear stable stationary plant is transformed into a harmonic output signal once the transient is completed. The amplitude of this signal is proportional to the value of the plant's transfer function at a frequency of the harmonic signal. On this basis when varying a frequency one can construct the plant transfer function, that is, in essence, the signal can be identified. In a similar manner, the plant's unit impulse response can be reconstructed for input impulses (step functions).

With test signals as control signals the identification of a plant is possible for additive noise acting on it, too. Noise does not necessarily possess any useful stochastic properties and does not need to be stochastic at all. The reconstruction of unknown values of parameters is provided with properties of a test signal, which is mixed with a control signal. The introduction of a test signal in a control channel can deteriorate the control performance. However, in an appropriate decision about the intensity of a test signal the output process will be indistinguishable from an optimal process through time (if the intensity of a test signal is diminished rapidly with time it is not necessary that the identification process is complete).

The investigation of identification techniques with test signals was first used in [1] and subsequently extended in [2] to closed control systems. In these works an assumption was made of a plant a priori stability, a disturbance was assumed to be a white noise process, and in addition, a relatively limiting constraint was placed on the noisy control.

In [3], [5] for the non a priori stable case there were suggested the algorithm when special randomized test signals in the input channel allow to identify asymptotically the control plant unknown parameters under almost arbitrary additive noise in a plant model. The procedure is valid for any noise v_t and does not require a priori knowledge of its characteristics; noise may be not random or may be white or correlated, with zero-mean or bias; a signal-noise ratio may be high or low. Recently similar randomized control strategies were put forward in [6], [7]. The recovery of unknown parameter values is provided by the properties of randomized test signals which are added together with an intrinsic adaptive control signal from a closed loop. This approach follows from Feldbaum's concept of *dual control* [8]: *control must be not only directing, but also learning*.

In [5], [9], for the case of an arbitrary noise (e. g., *unknown but bounded noise*), the randomization was used to develop an identification algorithm which allowed for obtaining the asymptotically confidence region of an indefinitely small size. These results were extended to the case of time-varying parameters in [10], [11]. The information about the maximum possible amplitude of the noise has only been used in the formulas for estimating the rate of convergence, i. e., this knowledge is not required for operability of an identification algorithm.

The identification method discussed below is based on the reparametrization of the mathematical model of a plant (instead of plant coefficients as its initial parameters, some alternative parameters are convenient to use which are in an one-to-one correspondence to the initial parameters). This enables the plant to be written in the form which is not too different from a "linear observation scheme". Then justified recurrent algorithms such as stochastic approximation algorithms can be applied for estimating unknown values of the parameters.

The main purpose of this paper is to join the former asymptotic results and new procedure from [12], [13] which gives rigorously guaranteed nonasymptotic confidence regions for unknown parameters of a linear dynamical control plant which is disturbed by an arbitrary noise. The new procedure consists of simple input design steps followed by an algorithm named LSCR (Leave-out Sign-dominant Correlation Regions), which is mostly promoted by M. Campi and E. Weyer [6]. In [6] authors consider the finite time problem setting and they prove the result about the

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proposed algorithm efficiency when an amount of data points increase infinitely. It is possible to achieve an arbitrary small level of an accuracy but the algorithm's complexity increase significantly. The former algorithm from [3] has a much more simple recurrent form and it uses the same control strategy as in [12], [13] which is different from the control strategy of [6]. It has randomized items in the input channel only one time per some interval when the control strategy of [6] has randomized part of inputs on each iteration. Both this facts allow to prove in [13] more weak assumptions about external noise and to modify the original procedures for the infinite time case.

The paper is organized as follows: At the beginning we give a preliminary example for illustrative purposes. Then, in Section III, we formulate a formal problem setting. Section IV provides the rules to form control inputs (control synthesis). Section V introduces the main assumptions and describes a special method of transfer function reparameterization. Stochastic approximation algorithm is considered in Section VI. Next Section VII summarizes the result of [13] about properties of confidence regions in the case of finite number of observations. At the end, we make conclusions.

II. Preliminary Example

Is it possible to get smart estimates under arbitrary external noise?

For example, let's consider the simple problem of estimating of an unknown parameter $\theta^\star$ from the observations:

$$y_t = \theta_\star \cdot u_t + v_t, \tag{1}$$

where we can

- to chose the inputs (control actions) u_t, $t = 1, 2, ..., N$,
- to measure the outputs y_t (see Fig. 1).

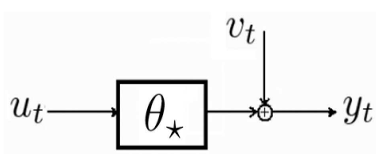

Fig. 1. The model of observations.

The problem is *to find or estimate* the unknown parameter $\theta_\star \in \mathbb{R}$ by the sequence of inputs and outputs $\{u_t, y_t\}$ without any restrictions for the sequence $\{v_t\}$ of external noises.

Does not it seem absurd such a statement of the problem?

From the deterministic point of view, yes, of course! There can be no deterministic algorithm which gives at least some common sense answer (other than a meaningless solution — the entire real axis). For the fixed number of observations and for any proposed answer as a number or a finite interval one can always choose such v_t that the following observation will be wrong for the proposed answer.

The algorithm of sequential estimation of an unknown parameter $\theta_\star$ of (1) consists of two steps:

1) Input (control actions) u_t selection.
2) Estimation of the parameter $\theta_\star$ based on the data obtained u_t, y_t (for example, calculation of an estimate $\widehat{\theta}_t$ or set $\widehat{\Theta}_t$ containing $\theta_\star$).

If in addition to the problem setting we would be to assume a random (probabilistic) nature of the noise v_t then under the conditions of the strong law of large numbers we could be talking about estimating an unknown parameter $\theta_\star$ by simply averaging the data of observation. The simulation results with the real parameter $\theta_\star = 3$ and the observations which were made with an uniformly distributed on the interval $[-0.5, 0.5]$ noise v_t are given in Table 1, row 5 indicate the proximity estimates $\widehat{\theta}_7 = \frac{1}{7}\sum_{i=1}^{7} y_i = 2.99$ to the real parameter $\theta_\star = 3$.

Table 1.

t	1	2	3	4	5	6	7
u_t	1	1	1	1	1	1	1
$v_t = rand() - 0.5$							
y_t	2.9	2.8	3.2	3.3	2.6	3.4	2.7
$\widehat{\theta}_t$	2.9	2.85	2.97	3.05	2.96	3.03	2.99
$v_t = rand() - 0.5 + m$, $m = 1$							
y_t	3.9	3.8	4.2	4.3	3.6	3.9	4.2
$\widehat{\theta}_t$	3.9	3.85	3.97	4.05	3.96	4.03	3.99

If the observations were carried out also with the random noise but with the unknown expectation $m = E\{v_t\}$ (for example, $m = 1$, Table 1, row 6) then the simulation results (Table 1, row 8) shows that the algorithm failed: $\widehat{\theta}_7 = 3.99$, and this value is substantially exceeds the $\theta_\star = 3$.

Despite the seeming absurdity of the statement of the problem of an estimation under arbitrary external noise it is often still have to solve from the practical needs.

Consider the following rule of a random input selection for the first step

$$u_t = \begin{cases} +1, & \text{with probability } \frac{1}{2}, \\ -1, & \text{with probability } \frac{1}{2}. \end{cases} \tag{2}$$

At the second step from the known values (u_t, y_t) we form a value

$$\tilde{y}_t = u_t \cdot y_t.$$

For the "new" sequence of observations we have a similar to (1) model

$$\tilde{y}_t = \theta_\star \cdot \tilde{u}_t + \tilde{v}_t,$$

where $\tilde{u}_t = u_t^2$ and $\tilde{v}_t = u_t \cdot v_t$.

Let's suppose, as before in the simulation, that v_t is a random noise but with an unknown expectation. If v_t is an external noise it is natural to assume that they are independent of our randomized inputs (controls) at step 1. Hence we have

$$E\{\tilde{v}_t\} = E\{u_t \cdot v_t\} = E\{u_t\} \cdot E\{v_t\} = 0 \cdot m = 0,$$

i. e. in the "new model" the "hard" observation problem of estimating an unknown parameter $\theta^\star$ of (1) is converted by using a random selection rule for inputs (controls) in step 1 to the "standard" problem of estimating an unknown parameter $\theta_\star$ observed on the background of independent centered noise.

In Table 2 we summarize the corresponding results of simulation.

Table 2.

t	1	2	3	4	5	6	7
u_t	-1	1	-1	1	1	1	-1
$v_t = rand() - 0.5 + m,\ m = 1$							
y_t	-2.1	3.8	-1.8	4.3	3.6	4.4	-2.3
$\bar{u}_t$	1	1	1	1	1	1	1
$\bar{y}_t$	2.1	3.8	1.8	4.3	3.6	4.4	2.3
$\widehat{\theta}_t$	2.1	2.95	2.57	3.00	3.12	3.33	3.19

The comparison of the result Table 2, row 7 with previous one from Table 1, row 8 shows that new estimates are substantially better but the quality of evaluations turned out lower than in the more relevant results from Table 2, row 5 because the "new errors" $\tilde{v}_t$ have a bigger variance compared to v_t.

The probability of a wrong decision making can be estimated asymptotically using the assessment of the correspondence mean rate of the convergence in [9] and Chebyshev's inequality. For every t and for any $\varepsilon > 0$ we have

$$Prob\{|\widehat{\theta}_t - \theta^\star| \geq \varepsilon\} \leq \frac{1}{t}\frac{\mathrm{E}\{v_t^2\}}{\varepsilon^2} + o\left(\frac{1}{t}\right).$$

For the finite number of observations ($N = 7$) a new rigorous mathematical result of a guaranteed set of possible values of the unknown parameter $\theta_\star$ can be obtained for an arbitrary external noise v_t following by the method described by M. Campi in [14]:

1) Let be $M = 8$ and select randomly seven $(= M - 1)$ different groups of four indexes $T_1, \ldots, T_7$.
2) Compute the partial sums $\bar{s}_i = \sum_{j \in T_i} \bar{y}_j$, $i = 1, \ldots, 7$.
3) Build the confidence interval

$$\widehat{\Theta} = [\min_{i \in 1:7} \bar{s}_i; \max_{i \in 1:7} \bar{s}_i],$$

which contains $\theta^\star$ with the probability $p = 75\%$ $(= 1 - 2 \cdot 1/M)$.

By the method described for the data $\{(u_t, y_t)\}$ in Table 2 we obtain:

Table 3.

i	T_i	$\bar{s}_i$
1	{2, 3, 4, 5}	3.375
2	{1, 3, 4, 6}	3.15
3	{2, 3, 5, 6}	3.4
4	{1, 2, 6, 7}	3.15
5	{1, 4, 5, 7}	3.075
6	{2, 3, 5, 7}	2.875
7	{1, 4, 6, 7}	3.275

Hence

- the unknown parameter $\theta_\star$ belongs to the interval $\widehat{\Theta} = [2.875; 3.4]$ with probability $p = 75\%$.

For the problem of an unknown parameter estimation under arbitrary external noise which seems absurd and can not be handle by any deterministic algorithm in principle the randomization in the process of the input data selection can get quite reasonable results.

it Remark. An alternative probabilistic approach is a Bayesian estimation when the noise v_t probability is attributed a priori to a nature Q. But Bayesian and randomized approaches are quite different from the practical point of view. In a Bayesian approach the probability Q describes a probability of a value of v_t in a comparison with other, i. e. the choice of Q is a part of the problem model. In contrast, the probability P in a randomized approach is selected artificially. P exists only in our algorithm, and therefore, there is no a traditional problem of "a bad model" as can happen with the Q in a Bayesian approach.

III. Problem Statement

Consider a dynamical system

$$y_t = G_\star(z^{-1})u_t + v_t \tag{3}$$

with input u_t and output y_t shown in Fig. 2.

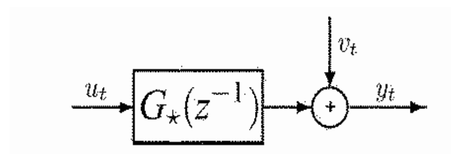

Fig. 2. Dynamical system.

Noise v_t describes all other sources, apart from u_t, which cause variation in y_t. z^{-1} is a delay operator: $z^{-1}u_t = u_{t-1}$. The transfer function $G_\star(z^{-1})$ belongs to a set of transfer functions $G(\theta, z^{-1})$ parameterized by θ, i. e. $G_\star(z^{-1}) = G(\theta_\star, z^{-1})$ for some $\theta_\star$. The structure of the model class $G(\theta, z^{-1})$ is known but $\theta_\star$ itself is unknown. The problem under consideration is to determine, based on a finite set of data of inputs and outputs collected at time $t = 1, 2, \ldots, N$, a confidence region $\widehat{\Theta}$ for $\theta_\star$ with a specified probability chosen by a user. Moreover, $\widehat{\Theta}$ must be constructed without any a priori knowledge of the noise level, distribution, or correlation.

The procedure discussed further is intended to identify the unknown parameters of a dynamic scalar linear control plant which is described by an autoregressive moving average model. It is based on the reparameterization of a plant mathematical model. Instead of the natural parameters of the plant — dynamic coefficients — it is convenient to use some other parameters which are in one-to-one correspondence with them. Such reparameterization is a result of rewriting the plant's equation in moving average model form which makes it possible to use the LSCR procedure for building the confidence region, even in the cases if an adaptive algorithm is used in the feedback channel.

We assume that a control plant has scalar inputs and outputs and it is described by Equation (3) in discrete time with $G_\star(z^{-1}) = B_\star(z^{-1})/A_\star(z^{-1})$, where

$$A_\star(\lambda) = 1 + a_\star^1\lambda + \cdots + a_\star^{n_a}\lambda^{n_a},$$
$$B_\star(\lambda) = b_\star^l\lambda^l + b_\star^{l+1}\lambda^{l+1} + \cdots + b_\star^{n_b}\lambda^{n_b},$$

the natural numbers n_a, n_b are the output and input (control) model orders; l is a delay in control, $1 \leq l \leq n_b$; $a_\star^1, \ldots, a_\star^{n_a}, b_\star^l, \ldots, b_\star^{n_b}$ are the plant parameters, a part of which is unknown.

It is required to define, with a given probability, an area of reliability for unknown coefficients of the plant (3) by the observations of outputs $\{y_t\}$ on a finite interval of time $t=1,2,\ldots,N$, and known inputs (controls) $\{u_t\}$, which can be chosen.

IV. Control Actions with Randomized Test Signals

Let $s \le n_a+n_b-l+1$ be a positive integer number. (It is usually equal to the quantity of unknown parameters of plant (3)). And let $N = s\cdot N_\Delta$ be with some N_Δ.

Let us choose a sequence of independent random variables, which are symmetrically distributed around zero, (a randomized test perturbation) $\Delta_0,\Delta_1,\ldots,\Delta_{N_\Delta-1}$ and add them to the input channel once per every s time moments (at the beginning of each time interval) in order to "enrich" the variety of observations.

To be more precise, we will build controls $\{u_t\}_{t=0}^{N-l}$ by the rule

$$u_{sn+i-l} = \begin{cases} \Delta_n + \bar{u}_{sn-l}, & i=0, \\ \bar{u}_{sn+i-l}, & i=1,2,\ldots,s-1, \end{cases} \quad n=0,\ldots,N_\Delta-1,$$

where "intrinsic" controls $\{\bar{u}_t\}$ are determined by an adjustable feedback law

$$\bar{u}_t = \mathscr{U}_t(y_t, y_{t-1},\ldots,\bar{u}_{t-1},\ldots),\, t\ge 0,\ \bar{u}_{-k}=0,\ k>0.$$

The type and characteristics of a feedback depend on specific practical problems. In particular, it is possible to use a trivial law of "intrinsic" feedback: $\bar{u}_t=0,\ t=0,1,\ldots,N-l$, or to use a stabilized regulator

$$C(z^{-1},\tilde{\tau}_t)\bar{u}_t = D(z^{-1},\tilde{\tau}_t)y_t \tag{4}$$

with parameters $\tilde{\tau}_t=\widehat{\tau}_{t-s}$ which are tuning by the "Strip" algorithm [3] with choosing sufficiently large noise level C_v

$$\widehat{\tau}_t = \widehat{\tau}_{t-1} - \frac{(\varphi_t^{\mathrm{T}}\widehat{\tau}_{t-1}-y_t)\mathbf{1}_{\{|\varphi_t^{\mathrm{T}}\widehat{\tau}^{t-1}-y_t|-2C_v-\delta\|\varphi_t\|>0\}}}{\|\varphi_t\|^2}\varphi_t, \tag{5}$$

where $\mathbf{1}_{\{\cdot\}}$ is a set characteristic function, $\delta\ge 0$ is a constant, $\varphi_t=(-y_{t-1},\ldots,-y_{t-n_a},u_{t-l},\ldots,u_{t-n_b})^{\mathrm{T}}$.

The feedback regulator (4) is determined by such polinomials $C(\lambda,\tau)$ and $D(\lambda,\tau)$ that a charecterized polinomial $A(\lambda,\tau)C(\lambda,\tau)-B(\lambda,\tau)D(\lambda,\tau)$ is a stable polinomial.

V. Main Assumptions and Reparameterization of the Transfer Function

Main assumption

A1. The user can choose Δ_n and this choice does not affect to the external noise $v_{sn},\ldots,v_{s(n+1)-1}$. (In the mathematical sense, Δ_n does not depend on $\{v_t\}_{t=1}^{s(n+1)-1}$.)

Note, that no assumptions are made about the noise v_t and about the upper limits of the noise amplitudes. If the noise is random, there are no assumptions about the zero-mean or any autocorrelation properties.

For time sn, $n=0,\ldots,N_\Delta-1$, we can denote $\bar{v}_{sn}=v_{sn}+(1-A_\star(z^{-1}))y_{sn}+(B_\star(z^{-1})-b_\star^l z^{-l})u_{sn}$ and rewrite Equation (3) in the following form:

$$y_{sn} = \Delta_n\theta_\star^1 + \theta_\star^1\bar{u}_{sn-l} + \bar{v}_{sn},$$

where $\theta_\star^1=b_\star^l$. This equation shows a direct relation between observation y_{sn} and test signal Δ_n which does not depend on the "new" noise $\bar{v}_{sn}$.

Similarly, we rewrite Equation (3) for the rest of time instances $sn+k-1$, $k=2,\ldots,s$, sequentially excluding the variables $y_{sn+k-1},\ldots,y_{sn}$ from the left-hand side of the Equation (3) using the same equation (3) for early time instants

$$y_{sn+k-1} = \Delta_n\theta_\star^k + \sum_{i=0}^{k-1}\theta_\star^{k-i}\bar{u}_{sn-l+i} + \bar{v}_{sn+k-1}, \tag{6}$$

where $\theta_\star^{k-i}$, $i=0,\ldots,k-1$ are the corresponding coefficients of the remaining right-hand side terms with $\bar{u}_{sn-l+i}$.

In [3] and [5], the authors suggest forming new parameters as s-vector $\theta_\star$ of coefficients $\theta_\star^k$ obtained in (6). They also give conditions for the invertibility of such reparameterization procedure.

The next formula follows immediately from the above definition $\theta_\star=\mathbb{A}^{-1}\mathbb{B}$, where $s\times s$ matrix $\mathbb{A}$ and s-vector $\mathbb{B}$ are

$$\mathbb{A} = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ a_\star^1 & 1 & \ldots & 0 & 0 \\ a_\star^2 & a_\star^1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & a_\star^{n_a}\ldots & a_\star^1 & 1 \end{pmatrix},\ \mathbb{B} = \begin{pmatrix} b_\star^l \\ \vdots \\ b_\star^{n_b} \\ \vdots \\ 0 \end{pmatrix}.$$

Consider the conditions of the existence of a corresponding inverse function.

Assumption

A2. Let s be a positive integer such that a set of the plant's unknown parameters is uniquely determined by some function $\tau(\theta)$ from the above-defined vector $\theta_\star$.

By Lemma 2.2 on p. 117 from [5] Assumption **A2** holds for $s=n_a+n_b-l+1$ when the plant's orders n_a,n_b are known and the following assumption is satisfied.

A3. The polynomials $z^{n_a}A_\star(z^{-1})$ and $z^{n_b}B_\star(z^{-1})$ are mutually prime.

In [5] there is also an algorithm for the inverse function $\tau(\theta)$.

In practice, usually only part of plant parameters are unknown. Sometimes, unknown parameters correspond to the low degrees of z^{-1} which are smaller than some $\bar{n}_a$ and $\bar{n}_b$ respectively. In this case, we can choose $s=\bar{n}_a+\bar{n}_b-l+1$ which is significantly less than n_a+n_b-l+1. Moreover, the "new" noise $\bar{v}_{sn+k-1}$ in (6) can be divided into two parts: nonmeasurable $\tilde{v}_{sn+k-1}$ and measurable ψ_{sn+k-1}. The measurable part is determined by observable inputs and outputs with known coefficients (see the example below).

Example. Consider the second-order plant

$$y_t + a_\star^1 y_{t-1} + y_{t-2} = b_\star^1 u_{t-1} + 1.6u_{t-2} + v_t, \tag{7}$$
$$t=1,2,\ldots,N,$$

with unknown coefficients $a_\star^1$ and $b_\star^1\ne 0$.

Denote

$$\tau_\star = \begin{pmatrix} a_\star^1 \\ b_\star^1 \end{pmatrix}.$$

Let be $s=2$ and vector $\theta_\star$ of the "new" parameters be

$$\theta_\star = \begin{pmatrix} b^1_\star \\ 1.6 - a^1_\star b^1_\star \end{pmatrix} \in \mathbb{R}^2.$$

In this case, the inverse function $\tau(\theta)$ is

$$\tau(\theta) = \begin{pmatrix} \frac{1.6-\theta^2}{\theta^1} \\ \theta^1 \end{pmatrix}.$$

Equations (6) have the following forms:

$$y_{2n} = \Delta_n \theta^1_\star + \theta^1_\star \bar{u}_{2n-1} + \psi_{2n} + \tilde{v}_{2n},$$

$$y_{2n+1} = \Delta_n \theta^2_\star + \theta^2_\star \bar{u}_{2n-1} + \theta^1_\star \bar{u}_{2n} + \psi_{2n+1} + \tilde{v}_{2n+1},$$

where $\psi_{2n+k} = 1.6\bar{u}_{2n-2+k} - y_{2n-2+k}$, $k=0,1$, $\tilde{v}_{2n} = v_{2n} - a^1_\star y_{2n-1}$, $\tilde{v}_{2n+1} = v_{2n+1} + a^1_\star(a^1_\star y_{2n-1} + y_{2n-2} - 1.6\bar{u}_{2n-2} - v_{2n})$.

VI. Stochastic Approximation Algorithm

Let be for $\{\Delta_n\}$ following conditions holds

$$\begin{aligned} &\mathrm{E}\{\Delta_n\} = \mathrm{E}\{\Delta_n^3\} = 0, \qquad |\Delta_n| \le \frac{C_\Delta R_n}{\sqrt{1+\ln\{n\}}}, \\ &\mathrm{E}\{\Delta_n^2\} = \frac{\sigma_\Delta^2 R_n^2}{1+\ln\{n\}}, \quad \mathrm{E}\{\Delta_n^4\} \le \frac{M_4^4 R_n^4}{(1+\ln\{n\})^2}, \end{aligned} \quad (8)$$

where

$$R_n = C_R\left(1 + \sum_{j=1}^{p} |y_{sn+k-j}| + \sum_{j=1}^{p-k} |\bar{u}_{sn-j}|\right)$$

and σ_Δ^2, M_4^4, C_Δ, $C_R > 0$ are positive constants.

Consider the estimation algorithm

$$\begin{cases} \tau_t = \bar{\tau}(\widehat{\theta}_{n-1}),\ s(n-1) < t \le sn,\ s = 1,2,\ldots,n = 1,2,\ldots, \\ \widehat{\theta}_n = \widehat{\theta}_{n-1} - \Gamma \frac{1+\ln\{n\}}{nR_n} \Delta_n \left(\frac{\Delta_n}{R_n} \widehat{\theta}_{n-1} - (Y_n - \psi_n(\widehat{\theta}_{n-1})) \right), \end{cases} \quad (9)$$

where Γ is a positive define matrix:$2\lambda_{\min}(\Gamma)\sigma_\Delta^2 \ge 1$, Y_n is a vector of current observations.

For sufficiently large number of $\bar{R} > 0$ let be "intrinsic" control $\{\bar{u}_t\}$ is built by the feedback controller (4) with adjustable parameters

$$\tilde{\tau}_t = \begin{cases} \tau_t, & \text{if } |y_t| + |u_{t-1}| < \bar{R}, \\ \widehat{\tau}_{t-s} & \text{— othewise}, \end{cases} \quad (10)$$

where estimates $\widehat{\tau}_t$ are formed by "Strip"-algorithm (5).

Theorem 1. If the conditions **A1–A3** and $2\lambda_{\min}(\Gamma)\sigma_\Delta^2 \ge 1$ are satisfied **then** for an arbitrary initial condition $\widehat{\tau}^0 \in \mathbb{R}^s$ the algorithm (9) ensures the estimates $\{\tau_t\}$ such that for an arbitrary $\rho > 0$ the following limit relations are valid with probability 1 and in the mean square sense:

$$\lim_{n\to\infty} n^{1-\rho} ||\widehat{\theta}_n - \theta^\star|| = 0,\ \lim_{t\to\infty} t^{1-\rho} ||\widehat{\tau}_t - \tau^\star|| = 0.$$

Remark. From (8) it follows that $\Delta_n \to 0$ as $t \to \infty$ with probability 1, implying that a test signal vanishes with time. That is why adaptive systems can be synthesized with the identification algorithm described in such a way that with time their output becomes indistinguishable from the output of an optimal system synthesized for a known parameter of a control plant.

The proof of Theorem 1. This proof is similar to the corresponding proof from [4]. By virtue the algorithm, the vectors $q_n^{(\rho)} = n^{\frac{1-\rho}{1}}(\theta^\star - \widehat{\theta}_n)$ are related by the following formula:

$$q_{n+1}^{(\rho)} \approx (1 - \frac{2\gamma v_n^2 - 1 + \rho}{2n}) q_n^{(\rho)} + \frac{\gamma\sqrt{\ln\{n\}}}{n^{\frac{1+\rho}{2}}} \psi_n.$$

From this point on the sign $\approx$ is taken to mean that the equality is satisfied up to values of the highest order of smallness when $n \to \infty$ and can be disregarded. Under the conditions of Theorem 1 the following inequality is valid:

$$\mathrm{E}\{q_{n+1}^{(\rho)} | u_0^{ns-1}, y_0^{ns}\} \approx (1 - \frac{2\gamma\sigma_v^2 - 1 + \rho}{2n}) |q_n^{(\rho)}|^2 +$$

$$+ \gamma^2 \frac{2\ln\{n\}}{n^{1+\rho}} \mathrm{E}\{|\psi_n|^2 | u_0^{ns-1}, y_0^{ns}\}.$$

Since

$$\sum_{n=1}^{\infty} \frac{\ln\{n\}}{n^{1+\rho}},\ \sum_{n=1}^{\infty} \frac{2\gamma\sigma_v^2 - 1 + \rho}{n} = \infty$$

then by the familiar Doob's Theorem on the convergence of semi-martingales the limit equalities of Theorem 1 are valid.

VII. Procedure for Constructing Confidence Regions

The previous result has an asymptotic nature. For the finite number of observations we can use the following procedure.

1) Using observational data, we can write predictors as a function of θ

$$\hat{y}_{sn+k-1}(\theta) = \Delta_n \theta^k + \sum_{i=0}^{k-1} \theta^{k-i} \bar{u}_{sn+k-l-i}, \quad (11)$$

$$n = 0, \ldots, N_\Delta - 1,\ k = 1, \ldots, s.$$

2) We can calculate the prediction error

$$\varepsilon_t(\theta) = y_t - \hat{y}_t(\theta),\ t = 1, \ldots, N.$$

3) According to the observed data, we form a set of functions of θ

$$f_{sn+k-1}(\theta) = \Delta_n \varepsilon_{sn+k-1}(\theta),\ n = 0, \ldots, N_\Delta - 1,$$

$$k = 1, \ldots, s.$$

4) Choose a positive integer $M > 2s$ and construct M different binary stochastic strings (of zeros and ones) $(h_{i,1}, \ldots, h_{i,N})$, $i = 0, 1, \ldots, M-1$, as follows: $h_{0,j} = 0$, $j = 1, \ldots, N$, all the other elements $h_{i,j}$ take the values of zero or one with the equal probability $\frac{1}{2}$. We calculate

$$g_i^k(\theta) = \sum_{n=0}^{N_\Delta - 1} h_{i,ns+k} \cdot f_{ns+k-1}(\theta),\ i = 0, \ldots, M-1,$$

$$k = 1, \ldots, s.$$

5) Choose q from the interval $[1; M/2s]$. For $k = 1, \ldots, s$, construct a region $\widehat{\Theta}^k$ such that at least q of the $g_i^k(\theta)$

functions are strictly higher than 0 and at least q are strictly lower than 0.

We define the confidence set by the formula

$$\widehat{\Theta} = \bigcap_{k=1}^{s} \widehat{\Theta}^k. \quad (12)$$

Remarks. 1. The procedure described above is similar to the one suggested in [6] but it has two significant differences from it. First, we consider a confidence set $\widehat{\Theta}$ in the state space $\mathbb{R}^s$ instead of $\mathbb{R}^{n_a+n_b}$. The confidence regions $\widehat{\Theta}^k$, $k = 1,\ldots,s$, are the subsets of $\mathbb{R}^k$ instead of $\widehat{\Theta}^k \subset \mathbb{R}^{n_a+n_b}$. Second, randomized trial perturbations are included through the input channel only once per every s time instants instead of permanent perturbations in [6].

2. If we can divide the "new" noise $\bar{v}_{sn+k-1}$ in (6) into two parts — $\tilde{v}_{sn+k-1}$ and ψ_{sn+k-1} — where the first part is nonmeasurable whereas the second is determined by observable inputs and outputs with known coefficients then in the above-described procedure we can use stronger predictors instead of (11)

$$\hat{y}_{sn+k-1}(\theta) = \Delta_n \theta^k + \sum_{i=0}^{k-1} \theta^{k-i} \bar{u}_{sn+k-l-i} + \psi_{sn+k-1}.$$

The probability that $\theta_\star$ belongs to each of regions $\widehat{\Theta}^k$, $k = 1,2,\ldots,s$, is given in the following theorem.

Theorem 2: Let condition **A1** be satisfied. Consider $k \in \{1,2,\ldots,s\}$ and assume that $Prob(g_i^k(\theta_\star) = 0) = 0$. **Then**

$$Prob\{\theta_\star \in \widehat{\Theta}^k\} = 1 - 2q/M \quad (13)$$

where M, q and $\widehat{\Theta}^k$ are from steps 4 and 5 of the above-described procedure.

Proof: See [13].

The next corollary follows directly from Theorem 2.

Corollary 3: Under the conditions of Theorem 2

$$Prob\{\theta_\star \in \widehat{\Theta}\} \geq 1 - 2sq/M \quad (14)$$

where $\widehat{\Theta}$ is taken from (12).

Note, that, as it was pointed out in [6], the value of the probability in (13) is accurate but not the lower limit. Inequality in (14) is obtained because the events $\{\theta_\star \notin \widehat{\Theta}^k\}$, $k = 1,\ldots,s$ may overlap.

From the above, it is easy to derive.

Theorem 4: Let conditions **A1–A2** be satisfied and assume that $Prob(g_i^k(\theta_\star) = 0) = 0$. **Then** the set $\tau(\widehat{\Theta})$ is the confidence set for unknown parameters of plant (3) with a confidence level of no less than $1 - 2sq/M$.

VIII. Conclusion

From the theoretical point of view, an important feature of the suggested procedure is that it operates without any significant assumptions about the external noise. It is also a vital importance from the practical point of view since in practical applications it is difficult to obtain a priori knowledge about the noise characteristics. The resulting confidence set is not conservative because it gives a rather good description of the uncertainties in the model.

In the future work we plan to use above theoretical results in our practical project: multiagents group of UAV [15], [16]. Algorithms of a fly optimization are one of the most important topics for the development of UAVs control programs. One of possibilities is to use above described randomized algorithms. Other way is to accumulate energy and increase the flight range by using the thermal updrafts which are formed in the lower atmosphere due to disruption of warm air from the surface when it is heated by sunlight [17].

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