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Optimal Step-Size of a Local Voting Protocol for Differentiated Consensuses Achievement in a Stochastic Network with Priorities

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***Abstract*— This paper deals with a "differentiated consensuses" problem in a distributed stochastic network system with priorities. The network is considered as a set of heterogeneous agents that process incoming tasks with different importance (priority) levels. The observations about neighbors' states are supposed to be obtained with random noise and delays and the topology could switch over time. Several consensus objectives are to be achieved. To maintain almost balanced load, i.e. approximate consensus *for every priority class* across the network, a new family of control protocols that use different step-size parameters for each task class is introduced. An instrument for choosing optimal step-sizes for the proposed control strategy is given. In addition, a numerical example that illustrates the proposed control strategy and the results of simulations are provided.**

I. Introduction

Recently, the consensus approach was widely used to solve numerous practical problems such as cooperative control of multivehicle networks [1–3], distributed control of robotic networks [4], flocking problem [5, 6], optimal control of sensor networks [7, 8], distributed control of learning or educational processes [9], and others. A lot of attention was paid to obtain the corresponding consensus conditions for such systems (see e.g. [10–19]).

One of important practical problems is a load balancing problem. This is the problem of tasks redistribution between agents. It arises in various types of network systems, such as computer, production, transport, logistics, and other service networks. This could be networks consisting of heterogeneous agents that work together to achieve some practical goal, e.g. to process all incoming tasks as fast as possible. However, there could be other goals as well. Basically, all the control strategies which are commonly used are intended to increase the efficiency of the network system. In our previous work [20] it was shown that the problem of almost optimal task distribution among agents could be reformulated as a problem of the consensus achievement in the network. A centralized algorithm was considered in [21]. The multi-agent approach was developed in [20, 22]. Some control strategies for the load balancing that redistribute tasks in accordance with a current loads of agents and their productivities were also introduced. It is also important to look at the specifics of tasks since in real practical applications some of the incoming tasks could be more important then others (there could be different priorities of tasks). This condition should also be taken into account when we build a control strategy in [23, 24]. For such networks with tasks of different importance (priority) levels there could be several consensus objectives. We call this problem a *differentiated consensuses problem*, which defines a consensus problem for systems with multiple classes, where a consensus is targeted for each class and may be different among classes. Ultimately, the control goal of the network is to achieve a consensus within each class separately. In [23] for a distributed stochastic network with priorities we introduced a control strategy that allocates the resources of the network in a randomized way with corresponding probabilities for each priority class. Also corresponding conditions for the achievement of differentiated consensuses throughout the whole network were obtained. In [25] these results were extended for the case of unknown agents' productivities. In that control strategy we used the same step-size parameter for balancing queues lengths of tasks of each class. However, since tasks of different classes may be fed to the system with different intensities and complexities it is reasonable to treat them differently and choose the step-size separately for each task class.

In this paper we extend the results of our previous work [23] and introduce a new control strategy that uses different step-size parameters for each task class. In addition, in this work we provide a way to choose optimal step-sizes for the proposed family of control protocols.

The paper is organized as follows. In Section II the notation and the problem formulation are given. The new family of control protocols for achieving the differentiated consensus is introduced in Section III. In Section IV the main assumptions and main results are presented. Simulation results are included in Section V. Section VI contains conclusion remarks.

II. Problem Statement

Consider a dynamic network system of n agents, which collaborate among themselves, and a set of tasks of different classes, which have to be executed by the system. Tasks are fed to different agents of the system in different discrete

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time instants $t = 0, 1, \ldots$. Agents process incoming tasks in parallel. Tasks can be redistributed among agents based on a feedback. Note that a task cannot be interrupted if it is being processed by an agent, i.e. the system is non-preemptive.

Without loss of generality, agents in the system are numbered. Assume, that $N = \{1, \ldots, n\}$ denotes the set of agents in the network system. Let $i \in N$ be the number of an agent. The network topology may switch over time. Let the dynamic network topology be modeled by a sequence of digraphs $\{(N, E_t)\}_{t \geq 0}$, where $E_t \subset E$ denotes the set of edges at time t of topology graph (N, E_t). The corresponding adjacency matrices are denoted as $A_t = [a_t^{i,j}]$, where $a_t^{i,j} > 0$ if agent j is connected with agent i and $a_t^{i,j} = 0$ otherwise. Here and below, an upper index of agent i is used as the corresponding number of an agent (while not as an exponent). Denote $\mathscr{G}_{A_t}$ as the corresponding graph.

To introduce some properties of the network topology, the following definitions from the graph theory will be used. Define *the weighted in-degree* of node i as the sum of i-th row of matrix A: $indeg^i(A) = \sum_{j=1}^n a^{i,j}$; $\mathscr{D}(A) = \mathrm{diag}\{indeg^i(A)\}$ is the corresponding diagonal matrix; $indeg_{\max}(A)$ is the maximum in-degree of graph $\mathscr{G}_A$. Let $\mathscr{L}(A) = \mathscr{D}(A) - A$ denote the *Laplacian* of graph $\mathscr{G}_A$; $\cdot^{\mathrm{T}}$ is a vector or matrix transpose operation; $||A||$ is the Euclidian norm: $||A|| = \sqrt{\sum_i \sum_j (a^{i,j})^2}$; $Re(\lambda_2(A))$ is the real part of the second eigenvalue of matrix A ordered by the absolute magnitude; $\lambda_{\max}(A)$ is the maximum eigenvalue of matrix A.

It is said that digraph $\mathscr{G}_B$ is a subgraph of a digraph $\mathscr{G}_A$ if $b^{i,j} \leq a^{i,j}$ for all $i, j \in N$.

Digraph $\mathscr{G}_A$ is said to contain a *spanning tree* if there exists a directed tree $\mathscr{G}_{tr} = (N, E_{tr})$ as a subgraph of $\mathscr{G}_A$.

We suppose that tasks (jobs) belong to different classes $k = 1, \ldots, m$ and every agent has m queues — one for each task class.

The behavior of agent $i \in N$ is described by characteristics of two types:

- lengths of m queues of tasks of each class k at time instant t: $q_t^{i,k}$, $k = 1, \ldots, m$,
- productivity: p^i.

Each agent should distribute its own productivity among all task classes in such a way that, on the one hand the priorities for task classes are provided and on the other hand the "starvation problem" is taken into account i.e. tasks of the lower priority classes do not wait for execution for too long. This is achieved by making use of the probabilistic priority discipline [26]. Each task class is given a productivity fraction P_k, $k = 1, \ldots, m$ which is the same for a certain class k on every agent in the system. On each agent the tasks from its queues are chosen for execution randomly according to the following formula:

$$\tilde{p}_t^{i,k} = \begin{cases} \frac{P_k}{\sum_{q_t^{i,l} > 0} P_l}, & \text{if } q_t^{i,k} > 0; \\ 0, & \text{otherwise,} \end{cases}$$

where $\tilde{p}_t^{i,k}$ is the probability of choosing a task of class k for execution on agent i at a time instant t. Therefore the bigger fraction P_k corresponds to the higher chance of that a task of class k to be executed. Thus agent's productivity is distributed among all classes of tasks in the following way:

$$p_t^{i,k} = \tilde{p}_t^{i,k} p^i. \tag{1}$$

Here $p_t^{i,k}$ is a number of operations allocated for tasks of class k on agent i at a time instant t if the productivity p^i means the whole number of operations which agent i is able to proceed during the time from t till $t+1$. Note that according to the definition of $\tilde{p}_t^{i,k}$ if at certain time instant t' the queue of tasks of class k' on the agent i' is empty, no operations would be allocated for tasks of class k'. Instead $p_{t'}^{i',k'}$ operations would be distributed among other task classes in proportions of their productivity fractions P_k, $k \neq k'$.

For all $i \in N$, $t = 0, 1, \ldots$, the dynamics of the network system is as follows

$$\begin{cases} q_{t+1}^{i,1} = q_t^{i,1} - p_t^{i,1} + z_t^{i,1} + u_t^{i,1} \\ q_{t+1}^{i,2} = q_t^{i,2} - p_t^{i,2} + z_t^{i,2} + u_t^{i,2} \\ \vdots \\ q_{t+1}^{i,m} = q_t^{i,m} - p_t^{i,m} + z_t^{i,m} + u_t^{i,m} \end{cases} \tag{2}$$

or in a more compact vector form,

$$\mathbf{q}_{t+1}^i = \mathbf{q}_t^i - \mathbf{p}_t^i + \mathbf{z}_t^i + \mathbf{u}_t^i,$$

where $\mathbf{q}_t^i = [q_t^{i,k}]$ is a vector whose k-th element is defined by the amount of tasks of k-th class; $\mathbf{p}_t^i = [p_t^{i,k}]$, and $\mathbf{z}_t^i = [z_t^{i,k}]$ is an m-vector whose k-th element $z_t^{i,k}$ is the amount of new tasks of class k, which came to the system and were received by agent i at time instant t; $\mathbf{u}_t^i \in \mathbb{R}^m$ is a vector of control actions (redistributed tasks of class k to agent i at time instant t), which could (and should) be chosen based on some information about queue lengths of neighbors $\mathbf{q}_t^j$, $j \in N_t^i$, where N_t^i is the set $\{j \in N : a_t^{i,j} > 0\}$.

Denote

$$x_t^{i,k} = \begin{cases} q_t^{i,k} / \tilde{p}_t^{i,k}, & \text{if } \tilde{p}_t^{i,k} > 0; \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

the *load* of agent $i \in N$ for priority class $k = 1, \ldots, m$. Assume, that $p^i \neq 0$, $\forall i \in N$ and $P_k \neq 0$, $k = 1, \ldots, m$. In [20] it was proven that from all possible options for the redistribution of all tasks the minimum operation time of the system is achieved when loads $x_t^{i,k}$ are equalized throughout the network. Hence, it is important to consider the achievement of the following goal.

It is required to maintain balanced (equal) loads across the network for every priority class.

At this setting we can consider the consensus problem for states $\mathbf{x}_t^i = [x_t^{i,k}]$ of agents, where $\mathbf{x}_t^i$ is a state vector of agent $i \in N$, consisting of loads $x_t^{i,k}$ for m classes. We use the following definitions.

Definition *1:* n agents of a network are said to reach a *consensus* at time t if $\mathbf{x}_t^i = \mathbf{x}_t^j \quad \forall i, j \in N, i \neq j$.

Definition *2:* n agents are said to achieve *asymptotic mean square ε-consensus* for $\varepsilon > 0$ when

$$\overline{\lim}_{t \to \infty} \mathrm{E} \|\mathbf{x}_t^i - \mathbf{x}_t^j\|^2 \leq \varepsilon.$$

To ensure balanced loads across the network (e.g., in order to increase the overall throughput of the system and to reduce the execution time), it is naturally to use a redistribution protocol over time. We assume that to form the control (redistribution) strategy each agent $i \in N$ has noisy and possibly delayed observations about its neighbors' states

$$\mathbf{y}_t^{i,j} = \mathbf{x}^j_{t-d_t^{i,j}} + \mathbf{w}_t^{i,j},\ j \in N_t^i, \tag{4}$$

where $\mathbf{w}_t^{i,j}$ is a noise vector, $0 \le d_t^{i,j} \le \bar{d}$ are integer-valued delays, and $\bar{d}$ is a maximum of possible delays.

III. Control Protocol

In [20, 22], properties of a control algorithm, called local voting protocol, were studied for stochastic networks in the context of load balancing problem. For each agent the control (amount of redistributed tasks) was determined by the weighted sum of differences between the information about the state of the agent and the information about its neighbors' states. Let's consider a similar family of protocols as follows. For each $k = 1,\dots,m$ we define

$$u_t^{i,k} = \gamma_k \tilde{p}_t^{i,k} \sum_{j \in \bar{N}_t^i} b_t^{i,j}(y_t^{i,j,k} - x_t^{i,k}), \tag{5}$$

where $\gamma_k > 0$, $k = 1,\dots,m$ are step-sizes of the control protocol and $\bar{N}_t^i \subset N_t^i$ is the neighbor set of agent i (note, that we could use not all the available connections, but some subset of them), $b_t^{i,j}$ are protocol coefficients. In [23] it was shown that differentiated consensuses could be achieved throughout the whole network via control protocol similar to the one mentioned above but with the same step-size for balancing queues lengths of tasks of each class. However, since tasks of different classes may be fed to the system with different intensities it is reasonable to treat them differently and choose step-sizes separately for each task class.

Let $B_t = [b_t^{i,j}]$ be the matrices of task redistribution protocols for every time instant t. (We set $b_t^{i,j} = 0$ when $a_t^{i,j} = 0$ or $j \notin \bar{N}_t^i$.) The corresponding graph $\mathscr{G}_{B_t}$ may have the same topology as graph $\mathscr{G}_{A_t}$ of matrix A_t or more poor.

Let's assume $\bar{d} = 0$. Then the dynamics of the closed loop system with protocol (5) will be as follows: for $k = 1,\dots m$

$$x_{t+1}^{i,k} = x_t^{i,k} - \tilde{r}_t^{i,k} + \tilde{z}_t^{i,k} + \gamma_k \sum_{j \in \bar{N}_t^i} b_t^{i,j}(y_t^{i,j,k} - x_t^{i,k}) =$$

$$x_t^{i,k} - \tilde{r}_t^{i,k} + \tilde{z}_t^{i,k} + \gamma_k \left(\sum_{j \in \bar{N}_t^i} b_t^{i,j} x_t^{j,k} \right) - \gamma_k d^i(B_t) x_t^{i,k} + \gamma_k \tilde{w}_t^{i,k},\ i \in N, \tag{6}$$

where $\tilde{w}_t^{i,j,k} = b_t^{i,j} w_t^{i,j,k}$ and

$$\tilde{r}_t^{i,k} = \begin{cases} p_t^{i,k}/\tilde{p}_t^{i,k}, & \text{if } \tilde{p}_t^{i,k} > 0; \\ 0, & \text{othewise}, \end{cases} \quad \tilde{z}_t^{i,k} \begin{cases} z_t^{i,k}/\tilde{p}_t^{i,k}, & \text{if } \tilde{p}_t^{i,k} > 0; \\ 0, & \text{othewise}. \end{cases}$$

Let us rewrite Eq. (6) in a more compact form. Define the $\mathbb{R}^n$-valued vectors $\mathbf{X}_t^k = [x^{i,k}]$, $\mathbf{R}_t^k = [\tilde{r}_t^{i,k}]$, $\mathbf{Z}_t^k = [\tilde{z}_t^{i,k}]$ and $\mathbf{W}_t^k = \sum_{j \in \bar{N}_t^i} b_t^{i,j} \mathbf{w}_t^{i,j,k}$. The dynamics of the closed loop system with protocol (5) may be represented as

$$\mathbf{X}_{t+1}^k = \mathbf{X}_t^k + \gamma_k (B_t - \mathscr{D}(B_t)) \mathbf{X}_t^k - \mathbf{R}_t^k + \mathbf{Z}_t^k + \gamma_1 \mathbf{W}_t^k. \tag{7}$$

If $\bar{d} > 0$ we "artificially" add $n\bar{d}$ new agents to the current network topology. At each time instant t the new "fictitious" agents have states which are equal to the corresponding states of "real" agents at previous time instants $t-1$, $t-2,\dots,t-\bar{d}$. The same is done for every class $k = 1\dots m$. Let $x_t^{i,k} \equiv 0$, $i \in N$ for $-\bar{d} \le t < 0$. Denote $\bar{\mathbf{X}}_t^k \in \mathbb{R}^{\bar{n}}$, $\bar{n} = n(\bar{d}+1)$, as an extended state vector for $t = 0,1,\dots$ which consist of $\bar{d}+1$ (n)-vectors $\mathbf{X}_t^k, \mathbf{X}_{t-1}^k, \dots, \mathbf{X}_{t-\bar{d}}^k$, i.e. it includes all the components with all kinds of delays not exceeding $\bar{d}$. Introduce the extended $\bar{n} \times \bar{n}$ matrices $\bar{B}_t^k$ of control protocol (5) which consist of zeros at all places except $|\bar{N}_t^i|$ entries $\bar{b}_t^{i,j+ns_t^{i,j},k}$ in each $i \in N$, $j \in \bar{N}_t^i$ of n first lines, which are equal to $b_t^{i,j}$ and $\bar{b}_t^{i,i-n,k} = 1/\gamma_k$ in next $n\bar{d}$ lines, $i = n+1,\dots,\bar{n}$.

Due to the view of Laplacian matrices $\mathscr{L}(\bar{B}_t^k)$ we can rewrite the dynamics of the system in the following vector-matrix form:

$$\bar{\mathbf{X}}_{t+1}^k = \bar{\mathbf{X}}_t^k - \gamma_k \mathscr{L}(\bar{B}_t^k) \bar{\mathbf{X}}_t^k + \begin{pmatrix} -\mathbf{R}_t^k + \mathbf{Z}_t^k + \gamma_k \mathbf{W}_t^k \\ 0 \end{pmatrix}. \tag{8}$$

IV. Main Results

A. Assumptions

Let $(\Omega, \mathscr{F}, P)$ be the underlying probability space corresponding to the sample space, the collection of all events, and the probability measure respectively, and E be a mathematical expectation symbol.

Assume that the following conditions are satisfied:

- **A1**. **a)** For all $i \in N,\ j \in N_t^i$, observation noise vectors $\mathbf{w}_t^{i,j}$ are zero-mean, independent identically distributed (i.i.d.) random vectors with bounded variances: $\mathrm{E}(\mathbf{w}_t^{i,j})^2 \le \sigma_w^2$.
  b) Graphs $\mathscr{G}_{B_t}$, $t = 1,\dots$ are i.i.d. (independent identically distributed), i.e. the random events of appearance of of "time-varying" edge (j,i) in graph $\mathscr{G}_{B_t}$ are independent and identically distributed for the fixed pair (j,i), $i \in N, j \in N_{\max}^i = \cup_t \bar{N}_t^i$. For all $i \in N,\ j \in N_t^i$ weights $b_t^{i,j}$ in the control protocol are independent random variables with mean values (mathematical expectations): $\mathrm{E}b_t^{i,j} = b_{av}^{i,j}$, and bounded variances: $\mathrm{E}(b_t^{i,j} - b_{av}^{i,j})^2 \le \sigma_b^2$. Let B_{av} be the corresponding adjacency matrix.
  c) For all $i \in N, j \in N^i$ there exists a finite value $\bar{d} \in \mathbb{N}$: $d_t^{i,j} \le \bar{d}$ with probability 1, and integer-valued delays $d_t^{i,j}$ are i.i.d. random variables taking values $l = 0,\dots,\bar{d}$ with probability $D_l^{i,j}$.
  d) For all $k = 1,\dots,m$, $i \in N$, $t = 0,1,\dots$ random values $z_t^{i,k}$ are independent with expectations: $\mathrm{E}z_t^{i,k} = \bar{z}^k$ which do not depend on i, and variances: $\mathrm{E}(z_t^{i,k} - \bar{z}^k)^2 \le \sigma_{z,k}^2$.
  e) For all $i \in N$, $t = 0,1,\dots$ random vectors $\mathbf{p}_t^i$ are i.i.d. and consist of independent components. Random values $\tilde{r}_t^{i,k}$, $k = 1,\dots,m$, have expectations: $\mathrm{E}\tilde{r}_t^{i,k} = \bar{r}^k$ and bounded variances: $\mathrm{E}(\tilde{r}_t^{i,k} - \bar{r}^k)^2 \le \sigma_{r,k}^2$ which do not depend on i.
  Additionally, all mentioned in Assumption **A1** independent random variables and vectors are mutually independent.

- **A2**. Graph $\mathscr{G}_{B_{av}}$ has a spanning tree (for the consensuses to be achievable throughout the system [11]).
- **A3.** For step-sizes γ_k, $k = 1 \dots m$ of control protocols (5) the following conditions are satisfied:

$$0 < \gamma_k < \frac{1}{indeg_{\max}(B_{av})},\ 0 < \delta_k(\gamma_k) < 1, \qquad (9)$$

where $\delta_k(\gamma_k) = R - \gamma_k Re(\lambda_{\max}(Q))$, $R = 1 - D_{\max} + D_{\max}|Re(\lambda_2(\mathscr{L}(B_{av})))|$, $D_{\max} = \max_{i,j,l} D_l^{i,j}$, $Q = \mathrm{E}(\mathscr{L}(\mathrm{E}\bar{B}_t^k) - \mathscr{L}(\bar{B}_t^k))^{\mathrm{T}}(\mathscr{L}(\mathrm{E}\bar{B}_t^k) - \mathscr{L}(\bar{B}_t^k))$.
Note that $|Re(\lambda_2(\mathscr{L}(B_{av})))| > 0$ when Assumption **A2** holds (see [19]).

B. Averaged Models

Let $x_0^{\star,k}$, $k = 1,\dots,m$ be the weighted average of the initial states

$$x_0^{\star,k} = \frac{\sum_i g_i x_0^{i,k}}{\sum_i g_i}$$

where g^T is the left eigenvector of matrix B_{av} [19] ($x_0^{\star,k} = \frac{1}{n}\sum_{i=1}^n x_0^{i,k}$ in the case of balanced topology graph $\mathscr{G}_{B_{av}}$) and $\{x_t^{\star,k}\}$ is the trajectory of averaged systems

$$x_{t+1}^{\star,k} = x_t^{\star,k} + \bar{z}^k - \bar{r}^k,\ k = 1,\dots,m. \qquad (10)$$

where $\bar{z}^k$ and $\bar{r}$ are expectations which are defined by Assumptions **A1.d,e**.

C. Differentiated Consensuses

Consider vectors $\bar{\mathbf{X}}_t^{\star,k} \in \mathbb{R}^{\bar{n}}$, $t = 0,1,\dots,$ $k = 1,\dots,m$ which consist of $x_t^{\star,k},\dots,x_t^{\star,k},x_{t-1}^{\star,k},\dots,x_{t-1}^{\star,k},\dots,x_{t-\bar{d}}^{\star,k},\dots,x_{t-\bar{d}}^{\star,k}$.

Theorem *1:* **If** Assumptions **A1–A3** hold **then** for averaged squared difference $v_t^k = \mathrm{E}||\bar{\mathbf{X}}_t^k - \bar{\mathbf{X}}_t^{\star,k}||^2$ of trajectories of closed-loop systems (6) and (10) following inequalities are satisfied:

$$v_t^k \le \frac{\gamma_k^2 C + S_k}{\gamma_k \delta_k(\gamma_k)} + (1 - \gamma_k\delta_k(\gamma_k))^t \left(v_0^k - \frac{\gamma_k^2 C + S_k}{\gamma_k\delta_k(\gamma_k)}\right), \quad (11)$$

where $k = 1,\dots,m$, $C = 2\sigma_w^2(n^2\sigma_b^2 + ||B_{av}||^2)$, $S_k = n(\sigma_{z,k}^2 + \sigma_{r,k}^2)$, i.e. **if** additionally $v_0^k < \infty$, **then** the asymptotic mean square ε_k-consensus in (6) is achieved with $\varepsilon_k = \frac{\gamma_k^2 C + S_k}{\gamma_k\delta_k(\gamma_k)}$.

Proof: For each $k = 1,\dots,m$, the proof is similar to the corresponding proof in [25] or we can use results from [23] with minor revisions. ∎

Remarks 1: At this point, we highlight that, the result of Theorem 1 shows that queues with different priorities achieve m different consensus levels separately. This behavior is termed as *differentiated consensuses*.

2: If Assumptions **A1.b** and **A1.c** hold, the averaged matrices $\bar{B}_{av}^k$ consist of elements

$$\bar{b}_{av}^{i,j,k} = \begin{cases} D_{j \div n}^{i,j \bmod n} b^{i,j \bmod n}, \text{ if } i \in N,\ j \bmod n \neq 0 \\ D_{j\div n}^{i,n} b^{i,n}, \text{ if } i \in N,\ j \bmod n = 0 \\ 1/\gamma_k, \text{ if } i = n+1,\dots,\bar{n},\ j = i - n, \\ 0, \text{ otherwise.} \end{cases} \qquad (12)$$

Here, operation mod is a remainder of division, and $\div$ is a division without remainder. Note, that if $\bar{d} = 0$, then $\bar{B}_{av}^k = B_{av}$.

3: Assumption **A1.e** is the most controversial. It is very hard to prove. Moreover the independence of $p_t^{i,k}$ is doubtful. However we make this assumption for technical reasons to simplify the proof and we would like to omit it in future works.

4: In control protocols (5) we can choose γ_k depending on our intentions. We may either want to reduce noise sensitivity asymptotically in (11), in which case we should take smaller γ_k, or it may be more important for us to exchange the incoming tasks faster so we should take larger γ_k. In that case agents would exchange their tasks faster but the noise will have larger impact on the system. So, here we have a trade-off between the noise sensitivity and tasks exchanging in our system. The next theorem gives an asymptotically optimal solution.

Theorem *2:* **If** Assumptions **A1–A3** hold **then** optimal step-sizes $\gamma_k^\star$, $k = 1,\dots,m$, of each control protocol from (5) can be calculated by formulas:

$$\gamma_k^\star = -\frac{S_k}{C}\Delta + \sqrt{\frac{S_k^2}{C^2}\Delta^2 + \frac{S_k}{C}} \qquad (13)$$

where $\Delta = \frac{Re(\lambda_{\max}(Q))}{R}$.

Proof: Let's find the minimum of possible upper bounds for ε_k. For derivative of $\frac{\gamma_k^2 C + S_k}{\gamma_k\delta_k(\gamma_k)}$ by γ_k we have

$$\left(\frac{\gamma_k^2 C + S_k}{\gamma_k\delta_k(\gamma_k)}\right)' = \left(\frac{\gamma_k C + \frac{1}{\gamma_k}S_k}{R - \gamma_k Re(\lambda_{\max}(Q))}\right)' =$$

$$\frac{C - \frac{1}{\gamma_k^2}S_k}{R - \gamma_k Re(\lambda_{\max}(Q))} + \frac{(\gamma_k C + \frac{1}{\gamma_k}S_k)Re(\lambda_{\max}(Q))}{(R - \gamma_k Re(\lambda_{\max}(Q)))^2}.$$

Let's now set the obtained expression to zero and solve the quadratic equation:

$$\gamma_k^2 C + 2\gamma_k S_k \Delta - S_k = 0.$$

As a result we get optimal values for $\gamma_k^\star$ which are coincide with (13). ∎

Remarks 5: The optimality of step-sizes of control protocol (5) is understood in a sense that it provides the speed of convergence and noise tolerance of the protocol needed for achieving the maximal possible convergence (or minimal deviation from the consensus value) in the system under the given conditions.

V. Simulation Results

Let's consider an example of a network of five agents, connected as a circle (see Fig. 1). Assume, that the links between agents may disappear with probability of $\frac{1}{5}$, and "diagonal" links may also appear with the same probability. Maximum delay for the information exchange $\bar{d}$ equals to 1 and the probability of delay appearance equals $\frac{1}{3}$ and is the same for all edges. Noise in communication channels is normally distributed zero-mean random variable with parameter $\sigma_w = 5$. Agents' productivities are 8, 2, 1, 4 and 10 computational instructions per time unit. Number of incoming tasks is Poisson random variable with parameter $\lambda = 2$. It is taken that tasks of three different classes arrive at the agents. Let the productivity fractions be given as $3:2:1$, i.e. with all nonempty queues the agent's productivity will be divided among classes as $\frac{3}{3+2+1}p = \frac{1}{2}p$, $\frac{1}{3}p$ and $\frac{1}{6}p$ correspondingly.

In this case matrix B_{av} will have the following form:

$$B_{av} = \begin{pmatrix} 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{4}{5} \\ \frac{4}{5} & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{4}{5} & 0 & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{4}{5} & 0 & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{4}{5} & 0 \end{pmatrix}$$

Note, that given average topology graph $\mathscr{G}_{B_{av}}$ is balanced. Matrix of control protocol for priority class 1 is:

$$\bar{B}^1_{av} = \begin{pmatrix} 0 & \frac{2}{15} & \frac{2}{15} & \frac{2}{15} & \frac{8}{15} & 0 & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{4}{15} \\ \frac{8}{15} & 0 & \frac{2}{15} & \frac{2}{15} & \frac{2}{15} & \frac{4}{15} & 0 & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{2}{15} & \frac{8}{15} & 0 & \frac{2}{15} & \frac{2}{15} & \frac{1}{15} & \frac{4}{15} & 0 & \frac{1}{15} & \frac{1}{15} \\ \frac{2}{15} & \frac{2}{15} & \frac{8}{15} & 0 & \frac{2}{15} & \frac{1}{15} & \frac{1}{15} & \frac{4}{15} & 0 & \frac{1}{15} \\ \frac{2}{15} & \frac{2}{15} & \frac{2}{15} & \frac{8}{15} & 0 & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{4}{15} & 0 \\ \frac{1}{\gamma_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\gamma_1} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let's compute optimal step-size $\gamma_1^\star$ for balancing agents' loads for task of class 1. Among all agents maximal $\sigma_{r,k}$ is presented at the agent with largest productivity. In our example $\sigma^2_{r,1} = 0.9497$ for agent with productivity 10. So S_k defined in (11) equals 9.7484 and $C = 2 \cdot 3^2(5^2 0.16 + 1.4^2) = 107.28$ for C defined in (11) since $\sigma_b^2 = 0.16$. $\Delta = \frac{1.7761}{1.2764} = 1.3915$ since $D_{\max} = \frac{2}{3}$, $\|Re(\lambda_2(\mathscr{L}(B_{av})))\| = 1.4146$, $R = 1.2764$ and $Re(\lambda_{max}(Q)) = 1.7761$. for Q defined in (9). According to formula (13) we have

$$\gamma_1^\star = -\frac{9.7484}{107.28}1.3915 + \sqrt{\frac{9.7484^2}{107.28^2}1.3915^2 + \frac{9.7484}{107.28}} \approx 0.2004$$

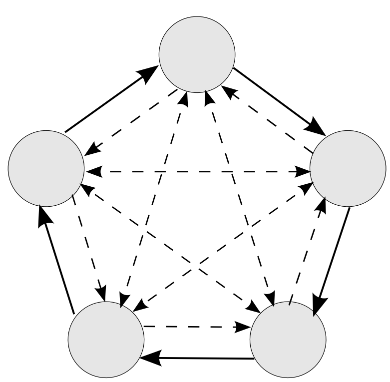

Fig. 1. The multi-agent system topology.

and according to Theorem 1 corresponding minimum value $\varepsilon_1^\star$ equals to 76.2045.

Let's make sure that $\gamma_1^\star$ satisfies inequality (9). Since $indeg_{\max}(B_{av}) = \frac{7}{5}$, $\delta_1(\gamma_1^\star) = 1.2764 - 0.2004 \cdot 1.7761 \approx 0.9205 < \frac{7}{5}$ therefore γ_1 should be less than $\frac{5}{7}$ that is true in our case.

Simulation results in Fig. 2 show that optimal step-size $\gamma_1^\star$ also provides close to optimal rate of convergence to consensus of agent's states in the system. Horizontal axis corresponds to γ_1 and vertical axis gives T_ε time to ε-consensus ($\varepsilon = 76$ was taken in experiments).

Making analogous calculations we get optimal step-sizes for other task classes $\gamma_2^\star = 0.2045$, $\gamma_3^\star = 0.2212$. Fig. 3 shows the behavior of agents' loads during task distribution via the described protocol with optimal step-size parameters $\gamma_1^\star$, $\gamma_2^\star$ and $\gamma_3^\star$. ε-consensus time T_ε equals 4 for all priorities. It could be seen from Fig. 3 that agent's loads achieve consensus for every task class. Initial agents' loads are chosen randomly from interval $[0, 400]$ for every priority so queue lengths for every task class have nearly the same average value. Loads however achieve three different consensus values being defined as queue length of tasks of certain class divided by probability of choosing a task of this class for processing (3). Probability fractions are taken as $3:2:1$, i.e. probabilities of choosing tasks for processing are $\frac{3}{6}$, $\frac{2}{6}$ and $\frac{1}{6}$. So consensus values for loads settled in proportions nearly $2:3:6$, that is expected in case queue lengths for all priorities have roughly the same average value.

VI. Conclusion

In this paper we examined a *differentiated consensuses* problem for a distributed stochastic network with different priorities of incoming tasks. The network model was assumed to have switched topology, noise and delays in measurements. We introduced the new control strategy (a modification of a local voting protocol) that uses different step-size parameters for each task class and proposed an instrument to choose optimal step-size parameters. As in our

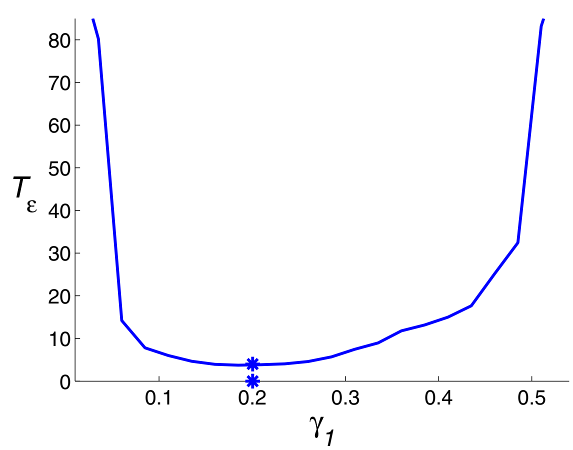

Fig. 2. Dependence between γ_1 and the time to consensus achievement for task of priority class 1.

previous work [23] for this model we obtained the conditions of achieving *differentiated consensuses* (m different consensus levels separately).

To illustrate the new theoretical results we presented simulation results that show the performance of the control protocol. It was shown that the larger step-size allows to achieve consensus among agents' loads faster. However, due to the larger noise sensitivity the deviation from the consensus value is relatively high compared to that for the smaller step-size. On the contrary, if the step-size is too low, though being tolerant to noise, the system achieves consensus rather slowly. It was shown how to choose the optimal step-size in the trade-off between the speed of convergence and noise tolerance in the system under given conditions.

In this paper standard assumptions on statistical properties for the topology, noise and delays in measurements are considered. In [27] these conditions are weakened and some approaches how to deal with the systems under the influence of almost arbitrary external perturbations are suggested.

REFERENCES

[1] W. Ren, R. Beard, and E. Atkins, "Information consensus in multivehicle cooperative control," *Control Systems, IEEE*, vol. 27, no. 2, pp. 71–82, 2007.

[2] O. Granichin, P. Skobelev, A. Lada, I. Mayorov, and A. Tsarev, "Comparing adaptive and non-adaptive models of cargo transportation in multi-agent system for real time truck scheduling," *Proceedings of the 4th International Joint Conference on Computational Intelligence*, pp. 282–285, 2012.

[3] K. Amelin, N. Amelina, O. Granichin, O. Granichina, and B. Andrievsky, "Randomized algorithm for uavs group flight optimization," *11th IFAC International Workshop on Adaptation and Learning in Control and Signal Processing*, pp. 205–208, 2013.

[4] F. Bullo, J. Cortés, and S. Martinez, *Distributed control of robotic networks: a mathematical approach to motion coordination algorithms*. Princeton University Press, 2009.

[5] W. Yu, G. Chen, and M. Cao, "Distributed leader–follower flocking control for multi-agent dynamical systems with time-varying velocities," *Systems & Control Letters*, vol. 59, no. 9, pp. 543–552, 2010.

[6] C. Virágh, G. Vásárhelyi, N. Tarcai, T. Szörényi, G. Somorjai, T. Nepusz, and T. Vicsek, "Flocking algorithm for autonomous flying robots," *Bioinspiration & biomimetics*, vol. 9, no. 2, p. 025012, 2014.

[7] S. Kar and J. M. Moura, "Distributed consensus algorithms in sensor networks: Quantized data and random link failures," *Signal Processing, IEEE Transactions on*, vol. 58, no. 3, pp. 1383–1400, 2010.

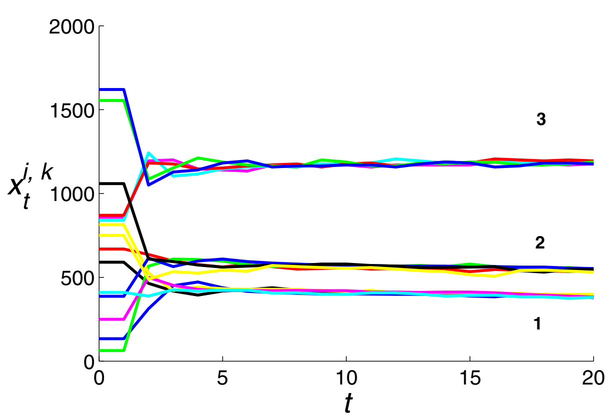

Fig. 3. Evaluation of agents' loads in the example for 3 classes of tasks.

[8] A. Chilwan, N. Amelina, Z. Mao, Y. Jiang, and D. Vergados, "Consensus based report-back protocol for improving the network lifetime in underwater sensor networks," *Lecture Notes in Computer Science (LNCS)*, vol. 8846, pp. 26–37, 2014.

[9] S. Komarov, A. Terekhov, and O. Granichina, "Integrated-distributed automated information system for a large scientific and educational institutions," *Vestnik St. Petersburg University. Ser. 10: Applied Mathematics, Informatics and Control Processes*, no. 1, pp. 87–94, 2008.

[10] W. Ren and R. Beard, *Distributed consensus in multi-vehicle cooperative control: theory and applications*. Springer, 2007.

[11] P. Y. Chebotarev and R. P. Agaev, "Coordination in multiagent systems and laplacian spectra of digraphs," *Automation and Remote Control*, vol. 70, pp. 469–483, 2009.

[12] T. Li and J. Zhang, "Mean square average-consensus under measurement noises and fixed topologies: Necessary and sufficient conditions," *Automatica*, vol. 45, no. 8, pp. 1929–1936, 2009.

[13] W. Yu, G. Chen, and M. Cao, "Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems," *Automatica*, vol. 46, no. 6, pp. 1089–1095, 2010.

[14] K. Amelin, N. Amelina, O. Granichin, and O. Granichina, "Multi-agent stochastic systems with switched topology and noise," in *Proc. 13th ACIS International Conference on Software Engineering, Artificial Intelligence, Networking and Parallel & Distributed Computing (SNPD)*, Kyoto, Japan, 2012, pp. 438–443.

[15] N. Amelina, A. Fradkov, and K. Amelin, "Approximate consensus in multi-agent stochastic systems with switched topology and noise," in *Proc. IEEE 2012 Multiconference on Systems and Control (MSC-2012)*, Dubrovnik, Croatia, 2012, pp. 445–450.

[16] M. Huang, "Stochastic approximation for consensus: a new approach via ergodic backward products," *IEEE Transactions on Automatic Control*, vol. 57, no. 12, pp. 2994–3008, 2012.

[17] A. V. Proskurnikov, "Average consensus in networks with nonlinearly delayed couplings and switching topology," *Automatica*, vol. 49, no. 9, pp. 2928–2932, 2013.

[18] N. Amelina and A. Fradkov, "Approximate consensus in multi-agent nonlinear stochastic systems," in *2014 European Control Conference (ECC-2014)*, 2014, pp. 2833–2838.

[19] F. Lewis, H. Zhang, K. Hengster-Movric, and A. Das, *Cooperative Control of Multi-Agent Systems: Optimal and Adaptive Design Approaches (Communications and Control Engineering)*. Springer, 2014.

[20] N. Amelina, A. Fradkov, Y. Jiang, and D. Vergados, "Approximate consensus in stochastic networks with application to load balancing," *IEEE Transactions on Information Theory*, vol. 61, no. 4, pp. 1739–1752, 2015.

[21] O. Granichin and N. Amelina, "Simultaneous perturbation stochastic approximation for tracking under unknown but bounded disturbances," *IEEE Transactions on Automatic Control*, vol. 60, no. 5, 2015.

[22] N. Amelina, O. Granichin, and A. Kornivetc, "Local voting protocol in decentralized load balancing problem with switched topology, noise, and delays," *Proc. of 52nd IEEE Conference on Decision and Control (CDC 2013)*, pp. 4613–4618, 2013.

[23] N. Amelina, O. Granichin, O. Granichina, Y. Ivanskiy, and Y. Jiang, "Differentiated consensuses in a stochastic network with priorities," in *IEEE Multi-Conference on Systems and Control (MSC 2014)*, 2014, pp. 4613–4618.

[24] N. Amelina, O. Granichin, O. Granichina, and Y. Jiang, "Differentiated consensuses in decentralized load balancing problem with randomized topology, noise, and delays," *IEEE Conference on Decision and Control (CDC 2014)*, pp. 6969–6974, 2014.

[25] N. Amelina, O. Granichin, O. Granichina, Y. Ivanskiy, and Y. Jiang, "Local voting protocol for differentiated consensuses in a stochastic network with priorities," in *1st IFAC Conference on Modelling, Identification and Control of Nonlinear Systems*, Saint Petersburg, Russia, 2015.

[26] Y. Jiang, C.-K. Tham, and C.-C. Ko, "A probabilistic priority scheduling discipline for multi-service networks," *Computer Communications*, vol. 25, no. 13, pp. 1243–1254, 2002.

[27] O. Granichin, Z. V. Volkovich, and D. Toledano-Kitai, *Randomized Algorithms in Automatic Control and Data Mining*. Springer, 2015.