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Online Parameter Estimation for MPC Model Uncertainties Based on LSCR Approach

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Abstract— **The paper discusses a novel probabilistic approach for online parameter estimation of the predictor model used in an MPC (Model Predictive Control) setting in the presence of model uncertainties and external disturbances. Model uncertainty makes it hard to compute an optimal control in general case, because it is needed to take into account all possible values of model parameters. Therefore, it is a good way for optimisation to shrink a set of possible model parameters. The proposed method iteratively estimates model parameters using randomized control strategy and algorithm based on LSCR (Leave-out Sign-dominant Correlation Regions) and computes a new control for the estimated parameters using robust MPC. The theoretical results are demonstrated via a model simulation example with two unknown parameters.**

I. Introduction

Model Predictive Control is a control technique that deals with multi variable systems, constraints, and uncertainty [1], [2]. At each sampling time, a finite horizon optimal control problem is solved based on a given model of the system.

Often, in the real world we do not exactly know actual values of model parameters, for instance in case of presence of model uncertainties. Moreover, it is often required to use MPC in the presence of external disturbances. This two cases of uncertainties will be considered as assumptions below. One of the possible way to compute a suitable control in the presence of uncertainties is to use robust MPC [3]–[6]. Robust MPC solves the same problem as standard MPC, but guarantees stability and constraints satisfaction also in the presence of model uncertainties and external disturbances. But even in this case model uncertainty limits our possibilities to construct optimal control, because a set of possible values of model parameters can be relatively large. It leads us to situation when the set of parameters includes many redundant values that do not ever (or rarely) will take place on practice. So one of the possible solutions is to shrink this set somehow, for instance, using online parameter estimation. At the same time, it is clear that such shrinking potentially leads to more "suitable" control.

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But what does "suitable" actually mean in that case? At each time instance we could analyse system output and decide what the next set of possible parameters should be. The proposed method uses a stabilized regulator $\hat{u}$ additionally disturbed by a random value Δ (see [7]). And because of this random disturbance Δ, the control cannot be exactly robust, but it can be near to the robust one. That is, the value of a cost function on the disturbed control can be very close to the value of a cost function on the robust control. This fact is demonstrated in $Theorem\ 1$ below.

The main contribution of the paper is a combination of the two following approaches – robust MPC (RMPC) [3] (based on Scenario Approach [8]) and modified LSCR approach [7] based on [9], [10]. Theoretical results in the present paper show how these methods work together. Illustrative example at the end of the paper demonstrates applicability of the proposed method. In [11] the authors use asymptotic methods for uncertainty evaluation in model which deals with iterative robust control. But the proposed approach uses non-asymptotic methods for uncertainty evaluation and deals with RMPC.

Briefly the proposed method can be described as follows. At first, we select a size of predicted horizon for RMPC. Within that window robust control is constructed using Scenario Approach (see [8]) - the method which is designed to minimize a linear objective to a number of convex constraints, one for each instance of the uncertainty. Then we use constructed robust control to estimate parameters of a given system. To do that we add some random perturbation with specific known statistical properties to constructed control and apply this new control to the system. This allows us to construct a confidential set for parameters of the model using modified LSCR approach [7].

Applicability of the proposed method is demonstrated via a simulation example with two unknown parameters. Also we have simulated the following problem related to fault tolerant control. Initially, all parameters of a model vary only within some unknown bounded set. Then, let us suppose that a fault occurs at some time instant, that is one of parameters was changed and moved outside of the initial set. And we recompute the set on the next step with added fault parameter. Therefore, on the next time instant we construct a new control on the newly re-constructed set.

The paper is organized as follows. Section II presents the problem statement. Main result given in Sections III and IV. Results of numerical experiments are shown in Section V, followed by Conclusions and Future work discussion.

II. Problem statement

Consider the following discrete time model:

$$\begin{cases} x_{t+1} = A^*(z^{-1})x_t + B^*(z^{-1})u_t + w_t, \\ w_t \in W, \end{cases} \tag{1}$$

where x_t are scalar outputs, u_t are scalar inputs, w_t is an external disturbance, and $t = 1, 2, \dots, T$ is a time index. In (1) z^{-1} is a delay operator: $z^{-1}x_t = x_{t-1}$ and $z^{-1}u_t = u_{t-1}$. Operators A^* and B^* are defined as follows:

$$\begin{aligned} A^*(\lambda) &= a_0 + a_1\lambda + ... + a_{n-1}\lambda^{n-1}, \\ B^*(\lambda) &= b_0 + b_1\lambda + ... + b_{m-1}\lambda^{m-1}. \end{aligned} \tag{2}$$

where $a_0, \dots, a_{n-1}$ and $b_0, \dots, b_{m-1}$ are the plant's parameters, some of them are unknown. Let us denote by $\boldsymbol{\theta}$ vector which consists of all the unknown parameters, and assume $\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^g$, where g is a number of unknown parameters.

Let us now rewrite the model (1) using matrices instead of polynomials A^*, B^*. For this purpose we introduce the following matrices: $A(\boldsymbol{\theta}) := \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}^n$,

$B(\boldsymbol{\theta}) := \begin{bmatrix} b_0 & b_2 & \cdots & b_{m-1} \\ 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \in \mathbb{R}^m \times \mathbb{R}^n$. One can easy check that for this matrix system (1) has became equal to the following uncertain model:

$$\begin{cases} \mathbf{x}_{t+1} = A(\boldsymbol{\theta})\mathbf{x}_t + B(\boldsymbol{\theta})\mathbf{u}_t + \mathbf{w}_t, \\ \boldsymbol{\theta} \in \Theta, \mathbf{w}_t \in W \end{cases} \tag{3}$$

where $\mathbf{x}_t = \begin{bmatrix} x_t & x_{t-1} & \cdots & x_{t-n+1} \end{bmatrix}^\mathsf{T} \in \mathbb{R}^n$, $\mathbf{u}_t = \begin{bmatrix} u_t & u_{t-1} & \cdots & u_{t-m+1} \end{bmatrix}^\mathsf{T} \in \mathbb{R}^m$, $\mathbf{w}_t = \begin{bmatrix} w_t & 0 & \cdots & 0 \end{bmatrix}^\mathsf{T} \in \mathbb{R}^n$.

The problem is to regulate the system to the predefined origin taking the following assumptions into account.

A. Assumptions

First of all, it is not required Θ to be convex. This is important for applying LSCR approach further, because an output confidence set of LSCR (that will be considered as a new value of Θ) is non-convex in general.

Now let us briefly describe Assumptions 1-4 defined in [3]. At first, it is required $A(\theta), B(\theta)$ to be stabilizable for any $\theta \in \Theta$.

Then, because of in the real systems under control there are some predefined constrains, we are also add them into the model. And we suppose that this constrains do not depend on time instant. Let us define two convex sets of constraints, under X and under U: $C_X := \{x \in \mathbb{R}^n : f_X(x, \theta) \preceq 0\}$, $C_U := \{u \in \mathbb{R}^m : f_U(u, \theta) \preceq 0\}$, where functions $f_X : \mathbb{R}^n \times \Theta \to \mathbb{R}^r$, $f_X : \mathbb{R}^m \times \Theta \to \mathbb{R}^s$ are convex in x and u.

Finally, because of external disturbances the equilibrium is not attainable. In other words, we can require regulation to a neighborhood of the origin, described by a terminal set. For this purpose we will recall Assumption 4 in [3] to define a terminal set. Define a convex set $C_f := \{x \in \mathbb{R}^n : f_C(x) \preceq 0\}$, where $f_C : \mathbb{R}^n \to \mathbb{R}^q$. And let us also suppose that there is $u = Kf$, such that $A(\theta)x + B(\theta)Kx + W \in C_f$ for $\forall x \in C_f, \forall \theta \in \Theta, \forall W$. Finally, $f_X(x, \theta) \preceq 0, f_U(Kx, \theta) \preceq 0, \forall x \in C_X, \forall \theta \in \Theta$.

In [3] a control is looking like $\hat{\mathbf{u}}_t = K\mathbf{x}_t + \mathbf{v}_t$, where K is defined above, and $\mathbf{v}_t$ is an additionally constructed control required by regularization to a terminal set. In this paper we construct a new control $\mathbf{u}_t := \hat{\mathbf{u}}_t + \boldsymbol{\Delta}_t = K\mathbf{x}_t + \mathbf{v}_t + \boldsymbol{\Delta}_t$, where $\boldsymbol{\Delta}_t$ is a vector constructed from identically distributed random values.

III. Estimation of Model Parameters

Model Predictive Control optimizes a finite time-horizon basing on knowledge about past and present. In other words, we minimize some predefined cost function for N steps forward to build a control. And then obtained control is applied for only current time instant. After a new value of x_t was obtained, we repeat this step, and so on.

It is important that parameters of the model can be changed within the set Θ at some time instant. And because of that our purpose is not only to estimate parameters of the model (i.e. shrinking Θ), but also to detect changes of the parameters. To be more specific, new $\Theta_t \subset \Theta$ will be computed for each $t = 1, 2, \dots, T$.

The proposed algorithm is iterative and consist of the following steps:

1) Initialize the initial set: $\Theta_0 = \Theta$. After that repeat the following Steps 2-3:
2) For each $t = 1, 2, \dots, T$ compute a new control $\hat{\mathbf{u}}_t = K\mathbf{x}_t + \mathbf{v}_t$ like described in III-A on the previous confidence region $\Theta_{t/M}$ constructed in Step 3) below.
3) For each $t = M, 2M, \dots, kM, \dots$ compute a new confidence region Θ_t using III-B. Let's note horizon N is independent of M, and usually $N < M$.

We denoted by M a number of time instants between two successive recomputing of Θ_t. In other words, we apply some special control at each time instant within $[t; t+M-1]$ and after that compute new Θ_t. Further, N is a horizon length for MPC, it should be not very large because of computational complexity. On the other hand, big value of M is good, it makes estimation to be more accurate.

A. MPC-based Set of Controls for LSCR

Let us define the following cost function to be minimized:

$$J(\mathbf{x}_t, \mathbf{V}_t, \boldsymbol{\omega}_t) := \max_{i=1,\dots,S} \left(\sum_{k=0}^{N-1} d(\mathbf{x}_{k|t}^{(i)}, C_f) + \sum_{k=0}^{N-1} \mathbf{v}_{k|t}^T \mathbf{v}_{k|t} \right), \tag{4}$$

where $d(x, C_f) := \min_{y \in C_f} \|x - y\|$, N is a selected horizon, S is a number of random selected scenarios, $\mathbf{v}_{k|t}$ is a control applied at time instant

$t+k$, $\mathbf{V}_t := \begin{bmatrix}\mathbf{v}_{0|t} & \mathbf{v}_{1|t} & \cdots & \mathbf{v}_{N-1|t}\end{bmatrix}^{\mathsf{T}}$, $\boldsymbol{\omega}_t := \begin{bmatrix}\boldsymbol{\theta}_t^{(1)} & \boldsymbol{\theta}_t^{(2)} & \cdots & \boldsymbol{\theta}_t^{(S)}\end{bmatrix}^{\mathsf{T}}$ denotes the "multisample" of scenarios at time instant t, and $\mathbf{x}_{k|t}^{(i)}$ is a system state at time $t+k$ assuming that i-th scenario is selected (see more detailed description in equations (5) and (6) in [3]). To be more specific, we have to minimize a cost function over Θ, and we use Scenario Approach [8] to do this. The main idea behind this method is to extract a bunch of random "scenarios", and solve this modified problem only for the selected scenarios to get approximate solution of the initial problem.

Now our purpose is to select a control to minimize J under assumptions stated above. Since we want to minimize J, we can consider an equivalent problem of minimizing z_t under condition $J(\mathbf{x}_t, \mathbf{V}_t, \boldsymbol{\omega}_t) \preceq z_t$ to apply Scenario Approach [8]. This transformation was made in the problem (7) in [3]:

$$P(\mathbf{x}_t, \boldsymbol{\omega}_t) : \min_{\mathbf{V}_t, z_t} z_t \tag{5}$$

subject to

$$\begin{cases} J(\mathbf{x}_t, \mathbf{V}_t, \boldsymbol{\omega}_t) \preceq z_t \\ f_X(\mathbf{x}_{k|t}^{(i)}, \boldsymbol{\theta}) \preceq 0, \forall i = 1, \ldots, S, \forall k = 0, \ldots, N-1 \\ f_U(\mathbf{u}_{k|t}^{(i)}, \boldsymbol{\theta}) \preceq 0, \forall i = 1, \ldots, S, \forall k = 0, \ldots, N-1 \\ f_C(\mathbf{x}_{k|t}^{(i)}) \preceq 0, \forall i = 1, \ldots, S, \forall k = 0, \ldots, N-1, \end{cases} \tag{6}$$

where $\mathbf{u}_{k|t}^{(i)} = K\mathbf{x}_{k|t}^{(i)} + \mathbf{v}_{k|t}$.

Let $\mathbf{V}_t^* := \begin{bmatrix}\mathbf{v}_{0|t}^* & \mathbf{v}_{1|t}^* & \cdots & \mathbf{v}_{N-1|t}^*\end{bmatrix}^{\mathsf{T}}$ denote a minimum point of J, and in this case $\mathbf{v}_t = \mathbf{v}_{0|t}^*$. Finally, we compute a new control like $\hat{\mathbf{u}}_t = K\mathbf{x}_t + \mathbf{v}_t$ (see equation (2) in [3]).

B. LSCR-based estimation of parameters of uncertainty

The method, proposed in [7], [12], [13], is used to estimate parameters of a dynamic plant, described by an autoregressive moving average model with additive external noise. It computes data-based confidence region for unknown parameters of the plant with predefined probability.

This method computes a confidence region for the parameters by applying N random controls and analysing difference between real and expected outputs as described in Section 6 in [7]. It supposes that each of this controls looks like $u_t = \overline{u}_t + \Delta_t$, where $\overline{u}_t$ is an adjustable feedback, while Δ_t is random identically distributed sequence (see Section 3 in [7]).

Now let us recall that in III-A we obtained the control in the form $\hat{\mathbf{u}}_t = K\mathbf{x}_t + \mathbf{v}_t$. If we use this control as an adjustable feedback $\overline{u}$, then we can define a new control as

$$\mathbf{u}_t = K\mathbf{x}_t + \mathbf{v}_t + \boldsymbol{\Delta}_t, \tag{7}$$

where $\boldsymbol{\Delta}_t := \begin{bmatrix}\Delta_t & \Delta_{t-1} & \cdots & \Delta_{t-N+1}\end{bmatrix}^{\mathsf{T}}$ and

$$\Delta_i = \begin{cases} \frac{1}{R} & , \textit{with probability } \frac{1}{2} \\ -\frac{1}{R} & , \textit{with probability } \frac{1}{2}, \end{cases} \tag{8}$$

where $R \in \mathbb{R}$, $R > 0$.

It is clear that this disturbed control does not minimizes (4). In the next section we will talking about how much disturbed cost function J differs from the origin one.

IV. Selection of Additional Control

Let us consider the control (7). As was stated in the previous section, it is a disturbed version of $\hat{\mathbf{u}}_t = K\mathbf{x}_t + \mathbf{v}_t$. It leads us to the following question. Once we computed $\mathbf{v}_t$ and disturbed it by $\boldsymbol{\Delta}_t$, i.e. obtained a new control $\mathbf{v}_t + \boldsymbol{\Delta}_t$, how large is a difference $|J(\mathbf{x}_t, \mathbf{V}_t^*, \boldsymbol{\omega}_t) - J(\mathbf{x}_t, \mathbf{V}_t^* + \mathbf{T}_t, \boldsymbol{\omega}_t)|$? Here $\mathbf{T}_t := \begin{bmatrix}\boldsymbol{\Delta}_{0|t} & \boldsymbol{\Delta}_{1|t} & \cdots & \boldsymbol{\Delta}_{N-1|t}\end{bmatrix}^{\mathsf{T}}$, where $\boldsymbol{\Delta}_{i|t}$ is a perturbation of corresponding $\mathbf{v}_{i|t}^*$. The answer to this question is presented in the current section.

From a practical point of view, the question could be stated in the following way. Given some $\epsilon \geq 0$, how to select R to guarantee $|J(\mathbf{V}_t^*) - J(\mathbf{V}_t^* + \mathbf{T}_t)| \leq \epsilon$? Here we do not use $\mathbf{x}_t$ and $\boldsymbol{\omega}_t$ to be shorter.

Let us note that theoretically one can select R as big as possible, but practically we cannot disturb a control with an arbitrary small value. On the other hand, we want to be close to the minimum of the cost function as much as possible.

Let us consider the cost function J at a given scenario i^* on which J is minimal:

$$J(\mathbf{V}_t^*) = \sum_{k=0}^{N-1} d(\mathbf{x}_{k|t}, C_f) + \sum_{k=0}^{N-1} \mathbf{v}_{k|t}^{*T}\mathbf{v}_{k|t}^*, \tag{9}$$

here and further $\mathbf{x}_{k|t} := \mathbf{x}_{k|t}^{(i^*)}$.

Similarly, we obtain the following equation for the disturbed control $\mathbf{V}_t^* + \boldsymbol{\Delta}_t$:

$$J(\mathbf{V}_t^* + \mathbf{T}_t) = \sum_{k=0}^{N-1} d(\overline{\mathbf{x}}_{k|t}, C_f) + \sum_{k=0}^{N-1} (\mathbf{v}_{k|t}^* + \boldsymbol{\Delta}_{k|t})^T(\mathbf{v}_{k|t}^* + \boldsymbol{\Delta}_{k|t}), \tag{10}$$

where $\overline{\mathbf{x}}_{k|t}$ is a system state corresponding to the control $\mathbf{v}_{k|t}^* + \boldsymbol{\Delta}_{k|t}$.

Now let us estimate $|\mathbf{x}_{k|t} - \overline{\mathbf{x}}_{k|t}|$ to sum up them afterwards to obtain an estimation of $|\sum_{k=0}^{N-1} d(\mathbf{x}_k, C_f) - \sum_{k=0}^{N-1} d(\overline{\mathbf{x}}_k, C_f)|$.

$$\begin{cases} \overline{\mathbf{x}}_{1|t} - \mathbf{x}_{1|t} = (A\mathbf{x}_{0|t} + B(\mathbf{u}_{0|t} + \boldsymbol{\Delta}_{0|t}) + \mathbf{w}_t) - \\ (A\mathbf{x}_{0|t} + B\mathbf{u}_{0|t} + \mathbf{w}_t) = B\boldsymbol{\Delta}_{0|t}, \\ \overline{\mathbf{x}}_{2|t} - \mathbf{x}_{2|t} = (A\overline{\mathbf{x}}_{1|t} + B(\mathbf{u}_{1|t} + \boldsymbol{\Delta}_{1|t}) + \mathbf{w}_{t+1}) - \\ (A\mathbf{x}_{1|t} + B\mathbf{u}_{1|t} + \mathbf{w}_{t+1}) = AB\boldsymbol{\Delta}_{0|t} + B\boldsymbol{\Delta}_{1|t}, \\ \ldots \\ \overline{\mathbf{x}}_{N-1|t} - \mathbf{x}_{N-1|t} = \sum_{i=0}^{N-2} A^i B\boldsymbol{\Delta}_{N-i|t}. \end{cases} \tag{11}$$

Now it is clear that $\overline{\mathbf{x}}_{i|t} - \mathbf{x}_{i|t} = \sum_{j=0}^{i-1} A^j B\boldsymbol{\Delta}_{i-j|t}$, $\forall i \in 1, 2, \ldots, N-1$. Using this fact, we derive the following estimation:

$$\begin{aligned}
d(\overline{\mathbf{x}}_{i|t}, \mathbf{x}_{i|t}) = \left\|\overline{\mathbf{x}}_{i|t} - \mathbf{x}_{i|t}\right\| &= \left\|\sum_{j=0}^{i-1} A^j B \mathbf{\Delta}_{i-j|t}\right\| \\
&\leq \sum_{j=0}^{i-1} \left\|A^j B \mathbf{\Delta}_{i-j|t}\right\| \\
&\leq \sum_{j=0}^{i-1} \left\|A^j\right\|_{op} \left\|B \mathbf{\Delta}_{i-j|t}\right\| \\
&\leq \sum_{j=0}^{i-1} \|A\|_{op}^j \left\|B \mathbf{\Delta}_{i-j|t}\right\| \\
&\leq \frac{\|A\|_{op}^N - 1}{\|A\|_{op} - 1} \|B\|_{op} \mathbf{\Delta}_{max|t},
\end{aligned}$$

where $\mathbf{\Delta}_{max|t} = \max_{i=0,\ldots,N-1} \left\|\mathbf{\Delta}_{i|t}\right\|$; $\|A\|_{op} := inf\{c \geq 0 : \|Ax\| \leq c\|x\|, \forall x \in X\}$ is an operator norm induced by the vector norm used in (4). We also used the inequality $\|RS\|_{op} \leq \|R\|_{op}\|S\|_{op}$ here, which is true for an operator norm and quadratic matrix R, S.

Then, using triangle inequality we derive:

$$\begin{aligned}
\left|\sum_{i=0}^{N-1} d(\overline{\mathbf{x}}_{i|t}, C_f) - \sum_{i=0}^{N-1} d(\mathbf{x}_{i|t}, C_f)\right| &\\
\leq \sum_{i=0}^{N-1} d(\overline{\mathbf{x}}_{i|t}, \mathbf{x}_{i|t}) &\\
\leq \sum_{i=0}^{N-1} \frac{\|A\|_{op}^N - 1}{\|A\|_{op} - 1} \|B\|_{op} \mathbf{\Delta}_{max|t} &\\
\leq N \frac{\|A\|_{op}^N - 1}{\|A\|_{op} - 1} \|B\|_{op} \mathbf{\Delta}_{max|t}. &
\end{aligned} \tag{12}$$

Now we will estimate the difference of the rest. We have:

$$\begin{aligned}
&\left\|\sum_{k=0}^{N-1} \mathbf{v}_{k|t}^{*T} \mathbf{v}_{k|t}^* - \sum_{k=0}^{N-1} (\mathbf{v}_{k|t}^* + \mathbf{\Delta}_{k|t})^T (\mathbf{v}_{k|t}^* + \mathbf{\Delta}_{k|t})\right\| \\
&\leq \sum_{k=0}^{N-1} \left\|(\mathbf{v}_{k|t}^{*T} \mathbf{v}_{k|t}^* - (\mathbf{v}_{k|t}^* + \mathbf{\Delta}_{k|t})^T (\mathbf{v}_{k|t}^* + \mathbf{\Delta}_{k|t}))\right\| \\
&\leq \sum_{k=0}^{N-1} (\left\|\mathbf{v}_{k|t}^*\right\| + \left\|\mathbf{\Delta}_{k|t}\right\|)^2 \\
&\leq N(\mathbf{v}_{max|t}^* + \mathbf{\Delta}_{max|t})^2,
\end{aligned} \tag{13}$$

where $\mathbf{v}_{max|t}^* := \max_{i=0,\ldots,N-1} \left\|\mathbf{v}_{i|t}^*\right\|$.

Finally, using (12) and (13) we can formulate the following theorem.

Theorem *1:* Let $\{\mathbf{x}_{i|t}\}_{i=0}^{N-1}, \{\mathbf{v}_{i|t}^*\}_{i=0}^{N-1}$ be the solution of (5) obtained at time t. Let also $A, B, N, \mathbf{T}_t, \mathbf{\Delta}_{max|t}, \mathbf{v}_{max|t}^*$ be defined as above. Then:

$$\begin{aligned}
&|J(\mathbf{V}_t^*) - J(\mathbf{V}_t^* + \mathbf{T}_t)| \\
&\leq N\left(\frac{\|A\|_{op}^N - 1}{\|A\|_{op} - 1} \|B\|_{op} \mathbf{\Delta}_{max|t} + (\mathbf{v}_{max|t}^* + \mathbf{\Delta}_{max|t})^2\right).
\end{aligned} \tag{14}$$

Remark 1: Let us note that (14) does not guarantee that $\overline{\mathbf{x}}_{k|t}, \overline{\mathbf{u}}_{k|t}$ satisfies all the constraints in (5). But on practice, we can initially estimate the constraints in such a way to satisfy them all.

Remark 2: An operator norm $\|A\|_{op}$ depends on a vector norm, and there are at least two common cases when it can be simply defined:

1) If $\|x\| := \sum_{i=1}^n |x_i|$ then the operator norm is $\|A\|_{op} := \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|$.
2) In case of usual Euclid norm of a vector, the operator norm is $\|A\|_{op} := \sqrt{\sum_{i,j=1}^n a_{ij}^2}$.

Remark 3: If we would know the upper bound of convergence time we can estimate convergence probability (estimation is slightly rough). Let us define $\hat{T}$ - the upper bound of convergence, $\hat{p}$ - the probability in LSCR method. The convergence probability is calculated by the following scheme:

1) Divide $\hat{T}$ by M to obtain the number of times LSCR method was applied.
2) Multiply $\frac{\hat{T}}{M}$ by $\hat{p}$ to obtain probability of finding suitable sets for unknown parameters.
3) Multiply $\frac{\hat{T}}{M}\hat{p}$ by value of convergence probability in [3].

V. Experiments

As we previously mentioned, there are several things of interest from a practical point of view. The first one is how the proposed approach works with a real model. And the second one is how the estimation of model parameters reacts to their changes.

We have implemented the proposed method in Python. Let's consider the following model:

$$\begin{cases} x_{t+1} + a_1(\theta)x_t = b_1(\theta)u_t + w_t \\ \theta \in \Theta_t, w_t \in W, \\ \\ -100 < u \leq 100, \\ 0 < x \leq 1000 \end{cases} \tag{15}$$

where $\Theta_t = [-1, 1]$, $a_1(\theta)$ and $b_1(\theta)$ are unknown parameters (initially $a_1 = 1.1$ and $b_1 = 0.7$), $x_0 = 30$ is an initial state, and $w_t \sim \mathcal{N}(1, 0.8)$ is a noise sequence. Also here we stated two constraints on u and x: $-100 < u \leq 100$, $0 < x \leq 1000$.

Further, let us satisfy Assumptions stated in II. Suppose $K = 0.6$ and terminal set C_f is $[-2; 2]$. Also let horizon N be equal to 3, $S = 10$, $M = 60$ and $\Delta \sim \text{Bern}\left(\frac{1}{2}\right)$.

Under this assumptions (4) is:

$$J(\mathbf{x}_t, \mathbf{V}_t, \boldsymbol{\theta}) := \max_{i=1,\ldots,10} \left(\sum_{k=0}^{2} d(\mathbf{x}_{k|t}^{(i)}, C_f) + \sum_{k=0}^{2} \mathbf{v}_{k|t}^T \mathbf{v}_{k|t}\right).$$

And (5) is:

$$P(\mathbf{x}_t, \boldsymbol{\omega}_t) : \min_{\mathbf{V}_t, z_t} z_t \tag{16}$$

subject to

$$J(\mathbf{x}_t, \mathbf{V}_t, \boldsymbol{\omega}_t) \preceq z_t.$$

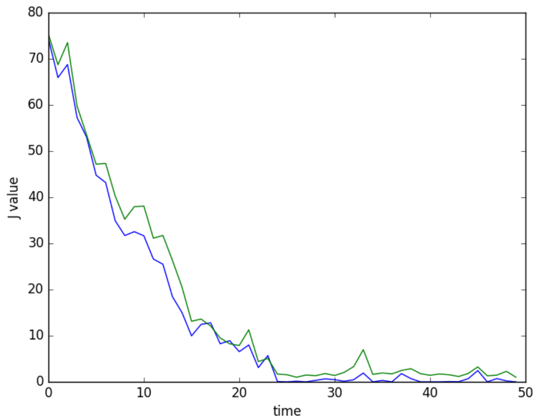

Fig. 1: The values of J on the disturbed and origin controls are denoted by green and blue lines respectively.

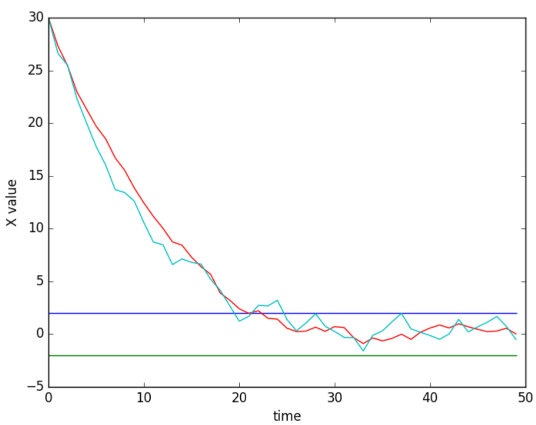

Fig. 2: Dark blue and green lines denote the terminal set. Red and blue lines denote controls $\hat{\mathbf{u}}_t = K\mathbf{x}_t + \mathbf{v}_t$ and $\mathbf{u}_t = K\mathbf{x}_t + \mathbf{v}_t + \mathbf{\Delta}_t$.

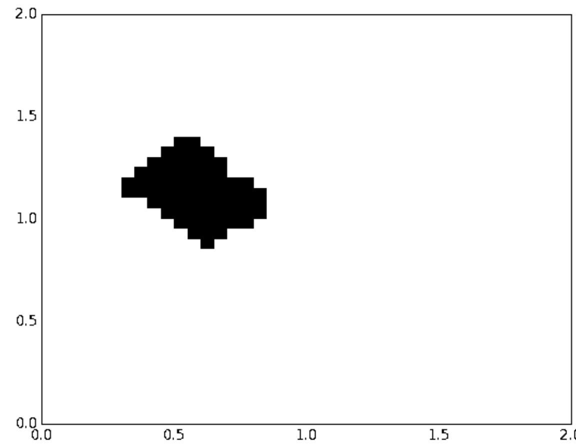

Fig. 3: Approximation of model parameters $a_1 = 1.1, b_1 = 0.7$.

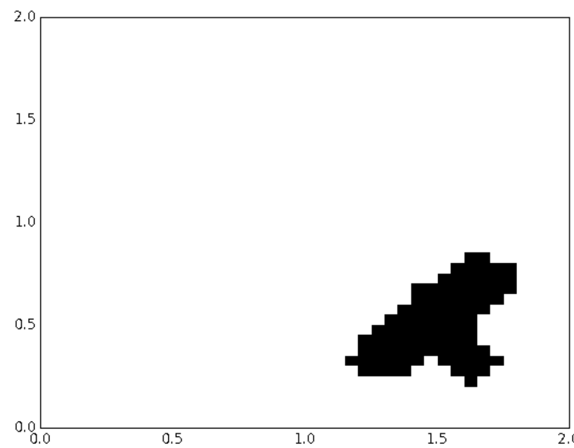

Fig. 4: Approximation of model parameters a_1 and b_1 after they were changed to $a_1 = 0.6, b_1 = 1.5$.

In order to check (14), we need some assumptions about matrices A, B and control v. Suppose that $\|A\|_{op} = \sqrt{a_0^2 + a_1^2} = \sqrt{1^2 + 1.1^2}$, $\|B\|_{op} = b_1(\theta) \leq 1$, $\Delta_{max} = 1$, and $v_{max} \leq 0.6$. This is empirically obtained constraints. If this conditions hold, then we have:

$$\begin{aligned} &|J(\mathbf{V}_t^*) - J(\mathbf{V}_t^* + \mathbf{T}_t)| \\ &\leq 3\left(\frac{(\sqrt{1^2+1.1^2})^3 - 1}{0.5 - 1} 1 * 1 + (0.6+1)^2\right) \approx 21.4. \end{aligned}$$

This result is shown on Fig.1. One can see that at each time instant the difference between green and blue lines is not greater that 21.4.

Another important point is system state convergence. It is demonstrated via Fig.2).

Now we will demonstrate how the approach works when parameters a_1, b_1 are changing. This task is related to Fault Detection and Isolation problem. In terms of the paper it means that a fault occurs when one of the model parameters moves out of Θ_t. "To catch" this fault and compute a new control one need to recompute Θ_t. We simulated a fault by changing parameters a_1, b_1 at time t_{fault}. Initially, $a_1 = 1.1$ and $b_1 = 0.7$ (see Fig.3). On Fig.4 one can see that Θ_t was successfully recomputed and contains new values of $a_1 = 0.6, b_1 = 1.5$.

After that we used Monte Carlo simulation to obtain more representable numerical results. So, let's recall which parameters of interest can vary - set of uncertainties, noise distribution, system constraints.

$$\begin{cases} x_{t+1} + a_1(\theta)x_t = b_1(\theta)u_t + w_t \\ \theta \in \Theta_t, w_t \in W, \\ \\ -a < u \leq a, \\ 0 < x \leq b \end{cases} \tag{17}$$

That is, we can choose different sets Θ_t, W, and values of parameters a and b.

In the first test we would like to figure out how the method behaves for different values of a_1, b_1. We chose 10 different diameters for Θ_0 to do this. Then 20 random sets

$diameter$	percent of fails
0.2	17%
0.5	17%
0.8	18%
1	21%
2	21%
3	21%
3.5	21%
4	22%
4.5	22%

TABLE 1: Number of fails for different diameters of Θ_0.

$a, b \in$	percent of fails
[0, 0.5] x [1,10]	26%
[0.5, 1] x [10, 15]	24%
[1, 1.5] x [15, 20]	22%
[1.5, 2] x [20, 25]	22%
[2, 2.5] x [25, 30]	22%
[2.5, 3] x [30, 35]	20%
[3, 3.5] x [35, 40]	20%
[3.5, 4] x [45, 50]	18%
[4, 4.5] x [50, 55]	17%

TABLE 2: Number of fails for different values of a and b.

from $\mathbb{R}^2$ of the specified diameter were generated. After that 100 different pairs of a_1 and b_1 were chosen from each set and used as input parameters of the algorithm. For instance, the first row in the Table 1 was obtained by generating 20 random sets of diameter 0.2, then we chose a_1, b_1 from the obtained set. All results are presented in the Table 1 . As one can see the actual percent of fails is about 20%. The lowest value (17%) is for the smallest diameter, but this fact is not theoretically confirmed.

In the second test we tried different values of the system constraints to understand how they impact on the algorithm's behaviour. For this purpose we chose 10 sets represented in the first column of the Table 2 and chose 100 random pairs a, b from each set to use them as input parameters of the algorithm.

The third test shows how fast LSCR make the set of uncertainty smaller. To demonstrate this we chose four different diameters of the sets a_1 and b_1 belongs to, and looking while diameter of the set on the current time instant does not become low or equal to this diameter. For instance, if we look at the cell $(20, 0.8)$ we see 9%. It means a_1 and b_1 will be inside some set with diameter 0.8 between 15 and 20 time instants with probability 0.09. Similarly, for the cell $(30, 0.2)$, a_1 and b_1 will be inside some set with diameter 0.2 between 30 and 35 time instants with probability 0.23.

VI. Conclusion

The problem of online parameter estimation of MPC model uncertainties is considered. A new method based on robust MPC and LSCR approach is suggested. It is shown that the estimation is suitable on practice and estimation of model parameters is accurate enough. Also this method was applied to fault detection problem, when some of model parameters were changed.

t / $diameter$	1	0.8	0.5	0.2
5	1%	1%	0%	0%
10	4%	4%	4%	2%
15	6%	7%	7%	3%
20	9%	9%	9%	6%
25	17%	15%	12%	14%
30	22%	18%	18%	19%
35	16%	20%	21%	23%
40	10%	11%	13%	12%
45	7%	8%	8%	8%
40	5%	5%	6%	6%

TABLE 3: The table represents "convergence speed" of the algorithm. The first column states for diameter of the set algorithm converges to. The first row represents time instants between which algorithm converged.

In future works we plan extend theoretical part of the paper. For this purpose we investigate asymptotic properties of the system (3) in the case of different changes for precise values of the parameters at some time instant. Such research will allow us to detect faults in system and further we will approbate our method to solve collision avoidance problem for autonomous mobile robots.

We also want to consider possible applications of this method for the medical diagnosis of biological processes in human body, including analysis of medical ultrasound data.

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