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ADAPTIVE CONTROL WITH TEST SIGNALS[1]

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1. Introduction

The severity of the identification problem is commonly caused by the insufficient variability of an input signal. In control systems test control signals can be fed to the input of a control plant, which alleviates the problem of the reconstruction of unknown parameters of the plant. For example, under the assumption that noise lacks a harmonic signal arriving at the input of a linear stable stationary plant is transformed into a harmonic output signal once the transient is completed. The amplitude of this signal is proportional to the value of the plant's transfer function at a frequency of the harmonic signal. On this basis when varying a frequency one can construct the plant transfer function, that is, in essence, the signal can be identified. In a similar manner, the plant's unit impulse response can be reconstructed for input impulses (step functions).

With test signals as control signals the identification of a plant is possible for additive noise acting on it, too. Noise does not necessarily possess any useful stochastic properties and does not need to be stochastic at all. The reconstruction of unknown values of parameters is provided with properties of a test signal, which is mixed with a control signal. The introduction of a test signal in a control channel can deteriorate the control performance. However, in an appropriate decision about the intensity of a test signal the output process will be indistinguishable from an optimal process through time (if the intensity of a test signal is diminished rapidly with time it is not necessary that the identification process is complete).

The identification method discussed below is based on the reparametrization of the mathematical model of a plant (instead of coefficients of the plant as its initial parameters, some alternative parameters are convenient to use, which are in an one-to-one correspondence to the initial parameters). This enables the plant to be written in the form which is not too different from a 'linear observation scheme'. Then justified recurrent algorithms such as stochastic approximation algorithms can be applied for estimating unknown values of the parameters.

The identification investigation techniques with test signals was first used in [1] and subsequently extended in [2] to closed control systems. In these works an assumption was made of the *a priori* stability of a plant, a disturbance was assumed to be a white noise process, and in addition, a relatively limiting constraint was placed on the noisy control. As shown in [3], this constraint can be satisfied for feedbacks in the special form only and when parameters of the plant are known with sufficient certainty; furthermore, because

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noise in the control channel is a stationary white noise process the suboptimal control alone was discussed. In the works [4]–[5] the method is so modified that the assumption of the *a priori* stability of a plant need not be made, while external noise (acting on a plant additively) is assumed to be noncorrelated to a test signal (which is 'noise' in the observation channel) and its second moments be bounded. Modifications of the method for minimizing the number of parameters estimated are suggested in the works [6]–[9]. In the works [8]–[11] general properties of algorithms of the stochastic approximation type with input test signals are studied.

This paper sums up investigations on the identification method with test signals. In particular, for a linear plant acted on by additive bounded noise conditions of the consistency of estimates of the unknown coefficients obtained by this method are given. Notice that such noise is not of necessity random and does not necessarily possess useful statistic properties providing a means for traditional mathematical statistic methods such as the method of least squares, the likelihood maximum method, *etc.*. The degree of convergence of estimates is refined, as well as the asymptotic normality of estimates is established. The possibilities of the method are exemplified by the adaptive minimax control problem.

2. Adaptive control with test signals

a) *The setting of the adaptive control problem* . Let a control plant with scalar inputs and outputs be described by an equation of the form:

$$a(\nabla, \tau_*)y_t + b(\nabla, \tau_*)u_t = v_t, \tag{1}$$

in which ∇ is the shift translation operator ($\nabla y_t = y_{t-1}$);

$$a(\lambda, \tau_*) = 1 + \lambda a_1 + \cdots + \lambda^n a_n, \qquad b(\lambda, \tau_*) = \lambda^k b_k + \lambda^{k+1} b_{k+1} + \cdots + \lambda^m b_m; \tag{2}$$

the positive integer k is a signal time delay, $1 \leq k \leq m$, and

$$\tau_* = \mathrm{col}\,(a_1, a_2, \ldots, a_n, b_k, b_{k+1}, \ldots, b_m) \tag{3}$$

is a vector of unknown coefficients of the control (1) (for simplicity all coefficients of (1) are taken to be unknown). It is assumed that $\tau_* \in \mathcal{T}$, where $\mathcal{T} \subseteq \mathbf{R}^p$ is a known convex closed set of possible values of the vector (3), $p = n + m - k + 1$. It is assumed that noise $v = \{v_t,\ t \in \mathbf{N}\}$ is arbitrary except that it satisfies the condition

$$|v_t| \leq C_v \tag{4}$$

with C_v as a known level of noise. In particular, noise v does not need to be random.

We assume that for every value of the vector $\tau \in \mathcal{T}$ there exists a feedback of the form:

$$\alpha(\nabla, \tau)u_t + \beta(\nabla, \tau)y_t = 0, \tag{5}$$

which not only stabilizes the control plant (1) when $\tau_* = \tau$, but it also ensures an appropriate control performance such as, *e.g.*, the control optimality with respect to some criterion). Let coefficients of the polynomial

$$\alpha(\lambda,\tau) = 1 + \lambda\alpha_1(\tau) + \cdots \lambda^p\alpha_p(\tau), \qquad \beta(\lambda,\tau) = \beta_0(\tau) + \lambda\beta_1(\tau) + \cdots \lambda^p\beta_p(\tau) \tag{6}$$

be known and continuous functions in the set $\mathcal{T}$. We are reminded that for $\tau_* = \tau$ the stabilization of the feedback (the controller) (5) is equivalent to the characteristic polynomial $g(\lambda,\tau) = a(\lambda,\tau)\alpha(\lambda,\tau) - b(\lambda,\tau)\beta(\lambda,\tau)$ of the closed system (1), (5) being stable, *i.e.*, ensuring the satisfaction of the inequality

$$\sup_{t\in\mathbf{N}} (|y_t| + |u_t|) < \infty. \tag{7}$$

When the vector (3) is unknown, it is natural to use the feedback

$$\alpha(\nabla,\tau_t)u_t + \beta(\nabla,\tau_t)y_t = 0, \tag{8}$$

where τ_t is an estimate of the vector τ_* at the instant t (assuming that $\tau_t \in \mathcal{T}$). If the algorithm for obtaining the estimates $\tau_t = \tau_t(y^t, u^{t-1})$ ensures their consistency, *i.e.*,

$$\lim_{t\to\infty} \tau_t = \tau_*, \tag{9}$$

then the control formed by the feedback (8) becomes indistinguishible from the control formed by the feedback (5) as $t \to \infty$. In other words, if the control performance criterion does nor depend on transients in a closed control system then the controller (8) ensures a control performance identical with that of the controller(5) synthesized for a particular value of the vector τ of coefficients. In that event the control formed by the controller (8) with tuned parameters is said to be *adaptive*.

b) *The stabilizing modified 'Strip' algorithm.* We now investigate more fully the adaptive stabilization problem, in which the control goal is the satisfaction of the inequality (7). Various algorithms are suggested for solving this sufficiently simple problem, which make the satisfaction of (7) possible in circumstances where coefficients of (1) are unknown. We describe the *stabilizing* modified 'Strip' algorithm [12], pointing out that this algorithm is combined with an arbitrary *identifying* algorithm intended for forming consistent estimates (see (9)), and further the techniques for combining these methods will be considered.

The equation (1) can be transformed into the equation

$$y_t + \Phi_{t-1}^*\tau_* = v_t, \tag{10}$$

where

$$\Phi_{t-1} = \mathrm{col}\,(y_{t-1}, y_{t-2}, \cdots, y_{t-n}, u_{t-k}, u_{t-k-1}, \cdots, u_{t-m}). \tag{11}$$

By virtue of (4) the inequalities

$$|y_t + \Phi_{t-1}^*\hat{\tau}| \le 2C_v + \epsilon|\Phi_{t-1}|, \qquad t \in \mathbf{N}, \qquad |\Phi| = \sqrt{\Phi^*\Phi}, \tag{12}$$

are soluble with respect to $\hat\tau$ for any $\epsilon \geq 0$ (*e.g.*, the vector $\hat\tau = \tau_*$ satisfies these inequalities). The 'goal' inequalities (12) generate the algorithm

$$\hat\tau_{t+1} = \mathrm{Pr}_{\mathcal{T}}\Big(\hat\tau_t - \frac{\eta_t \mathbf{1}(|\eta_t| - 2C_v - \epsilon|\Phi_{t-1}|)}{|\Phi_{t-1}|^2}\Phi_{t-1}\Big), \qquad \eta_t = y_t + \Phi_{t-1}^*\tau_t = \Phi_{t-1}^*(\tau_t - \tau_*) + v_t, \tag{13}$$

where $\mathbf{1}(\cdot)$ is the Heaviside function, $\mathrm{Pr}_{\mathcal{T}}$ is a projector into the set $\mathcal{T}$ (which correlates an arbitrary vector $\hat\tau \in \mathbf{R}^p$ with the vector closest to it from $\mathcal{T}$).

Given an arbitrary initial condition $\hat\tau_1$ the algorithm (13) converges in the finite number of steps for any $\epsilon > 0$ (see. [12], Theorem 2.1.8). This implies that there exists a finite moment $t_* = t_*(\hat\tau_1, v, \epsilon)$ of time such that $\hat\tau_t = \hat\tau_{t_*}$ for $t \geq t_*$. It is not necessary that the equality $\hat\tau_{t_*} = \tau_*$ be satisfied but from (13) it follows that the following inequalities are satisfied:

$$|v_t'| \leq 2C_v + \epsilon|\Phi_{t-1}| \qquad v_t' = \Phi_{t-1}^*(\hat\tau_* - \tau_*) + v_t, \qquad t \geq t_*. \tag{14}$$

Then for $t \geq t_*$ the control system (1), (8) can be transformed into the system

$$a(\nabla, \hat\tau_{t_*})y_t + b(\nabla, \hat\tau_{t_*})u_t = v_t', \qquad \alpha(\nabla, \hat\tau_{t_*})u_t + \beta(\nabla, \hat\tau_{t_*})y_t = 0. \tag{15}$$

Because of (14) the system (15) is dissipative for a sufficiently small $\epsilon > 0$, which can easily be shown by direct Lyapunov's method, and hence for the system (1), (8), (13) the inequality (7) is satisfied.

c) *The integration of the stabilizing algorithm into an identifying algorithm.* Let

$$\tau_t = \tau_t(y^t, u^{t-1}, \tau^{t-1}) \tag{16}$$

be some so called identifying algorithm for forming estimates τ_t of the unknown vector of the parameter τ_*. The proof of the consistency of estimates formed by the identifying algorithm is generally based on the assumption that with this approach the inequalities (7) are satisfied. If the modified 'Strip' algorithm is integrated into such an identifying algorithm these inequalities are satisfied immediately.

Let $\mathbf{D}_R = \{\mathrm{col}\,(y, u) : |y|^2 + |u^2| \leq R\}$ be a circle of radius R in the plane (y, u). The rule for forming the control $u = \{u_t,\ t \in \mathbf{N}\}$ is taken to be

$$u_t = \bar u_t + \bar w_t, \qquad \alpha(\nabla, \tilde\tau_t)\bar u_t + \beta(\nabla, \tilde\tau_t)y_t = 0. \tag{17}$$

Here $\bar u$ is formed by a controller with tuned parameters $\tilde\tau_t$,

$$\tilde\tau_t = \begin{cases} \tau_t & \text{if} \quad \mathrm{col}\,(y_t, u_{t-1}) \in \mathbf{D}_R \;\&\; \tau_t \in \mathcal{T}, \\ \hat\tau_t & \text{otherwise,} \end{cases} \tag{18}$$

and $\bar w_t$ are test signals, which later will be introduced in a special way. For a sufficiently large R such a control ensures the satisfaction of (7) if an arbitrary identifying algorithm is implemented. Moreover, if the identifying algorithm ensures the consistency of estimates τ_t then the egresses of the variables y_t, u_{t-1} from the circle $\mathbf{D}_R$ are limited in number.

We emphasize that in many instances the problem can be solved by application of an identifying algorithm alone without using a stabilizing algorithm.

d) *The introduction of an estimation parameter.* When constructing an identifying algorithm, it is usual to take into account properties of noise v in (1). Therefore for white noise the LSM (or its recurrent modifications) is widely applied; if noise is finitely correlated the instrumental variable method is used, and the likelihood maximum method is frequently used if distribution densities of a time series v are known (see, *e.g.*, [13]). In the problem considered the methods mentioned are unusable, because noise v does not necessarily possess useful statistic properties (moreover, it is not of necessity random). In that event unknown coefficients of (1) can be reconstructed for sufficiently general assumptions of noise v if random test signals are introduced into the control channel. One way to do this is through the identifying algorithm based on the reparametrization of (1), which is as follows.

For an arbitrary fixed $l \in \mathbf{N}$ the equation

$$F^{(l)}(\lambda)a(\lambda,\tau) + \lambda^{l+1}G^{(l)}(\lambda) = 1$$

in polynomials $F^{(l)}(\cdot)$, $G^{(l)}(\cdot)$ is uniquely soluble if $\deg F^{(l)}(\cdot) \leq l$. Coefficients $F_{l'}^{(l)}$ of the polynomial $F^{(l)}(\cdot)$ can be determined from a linear system, for which the matrix of coefficients is lower triangular, and $F_0^l = 1$. Letting polynomials $F^{(l)}(\cdot) = F^{(l)}(\cdot,\tau)$, $G^{(l)}(\cdot) = G^{(l)}(\cdot,\tau)$ be known for each $\tau \in \mathcal{T}$ and acting by the operator $F^{(l)}(\nabla) = F^{(l)}(\nabla,\tau_*)$ on the both sides of (1), we obtain

$$y_t = G^{(l)}(\nabla,\tau_*)y_{t-l-1} + F^{(l)}(\nabla,\tau_*)b(\nabla,\tau_*)u_t + F^{(l)}(\nabla,\tau_*)v_t. \tag{19}$$

We break down the set $\mathbf{N}$ of natural numbers into non-intersecting subsets ('discrete intervals') $\mathbf{N}_s = \{ps, ps+1, \ldots, (s+1)p-1,\ s \in \mathbf{N}\}$, $p = n+m-k+1$. On the interval numbered s in view of (19) the equation (1) can be expressed as

$$\begin{aligned} y_{ps+l} &= G^{(l)}(\nabla,\tau_*)y_{ps-1} + F^{(l)}(\nabla,\tau_*)b(\nabla,\tau_*)u_{ps+l} + F^{(l)}(\nabla,\tau_*)v_{ps+l}, \\ l &= 0,1,\ldots,p-1. \end{aligned} \tag{20}$$

Denoting the coefficients multiplied by u_{ps+l} of (20) by $\theta^{(l)} = \theta^{(l)}(\tau_*)$, $l = 0,1,\ldots,p-1$, we obtain

$$\theta^{(l)} = \theta^{(l)}(\tau_*) = \sum_{l'=0}^{l} F_{l'}^{(l)}(\tau_*)b_{l-l'+k} = F^{(l)}(\nabla,\tau_*)b_{l+k}(\tau_*), \quad l = 0,1,\ldots,p-1, \tag{21}$$

where it is assumed that $b_{l'}(\tau_*) = 0$ for $l' \neq k,\ k+1, \ldots,\ m$. These quantities are selected for parameters instead of the initial parameters (3). Newly obtained parameters possess the following property, which is almost evident:

Lemma 1 *The quantities (21) satisfy the relations:*

$$\begin{aligned} a(\nabla,\tau_*)\theta^{(l)} &= b_{k+l}, \quad l = 0,1,\ldots,p-1, \qquad p = n+m-k+1, \\ \theta^{(l)} &= 0, \quad l < 0, \qquad b_s = 0, \quad s > m. \end{aligned} \tag{22}$$

We introduce a vector parameter θ of dimension $p = m+n-k+1$ by the formula:

$$\theta = \theta(\tau) = \mathrm{col}\,(\theta^{(0)}(\tau), \theta^{(1)}(\tau), \ldots, \theta^{(p-1)}(\tau)), \tag{23}$$

where the components $\theta^{(l)}(\tau)$ are determined from the linear system (22) when $\tau_* = \tau$. The relation (23) is the mapping of the set $\mathcal{T} \subset \mathbf{R}^p$ to some subset of vectors θ of dimension p. The inverse transformation can be defined as follows. Letting $\theta = \mathrm{col}\,(\theta^{(0)}, \theta^{(1)}, \ldots, \theta^{(p-1)})$ be an arbitrary vector from $\mathbf{R}^p$, $p = n+m-k+1$, a collection of $n+1$ vectors $\theta^{[l]}$ can be defined thus:

$$\theta^{[l]} = \mathrm{col}\,(\theta^{(l)}, \theta^{(l-1)}, \ldots, \theta^{(l-n)}) \qquad (\theta^{(l)} = 0, \quad l < 0), \qquad l = m-k, m-k+1, \ldots, m-k+n. \tag{24}$$

Let the vector $\bar{a} = \mathrm{col}\,(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$ be found from the condition

$$\bar{a} = \operatorname*{argmin}_{a \in \mathbf{R}^n} \left| \| \, \theta^{[m+n-k-1]} \quad \theta^{[m+n-k-2]} \quad \cdots \quad \theta^{[m-k]} \, \| \, \bar{a} + \theta^{[m+n-k]} \right|, \tag{25}$$

where $\mathrm{argmin}_a f(a)$ is an arbitrary vector a ensuring the least value of the function $f(\cdot)$. We define the vector $\bar{b} = \mathrm{col}\,(\bar{b}_k, \bar{v}_{k-1}, \ldots, \bar{b}_m)$ using the relations

$$\bar{b}_{k+l} = \bar{a}(\nabla)\theta^{(l)}, \quad l = 0, 1, \ldots, m, \tag{26}$$

where $\bar{a}(\lambda) = 1 + \lambda \bar{a}_1 + \lambda^2 \bar{a}_2 + \cdots + \lambda^n \bar{a}_n$. Then $\bar{\tau} = \bar{\tau}(\theta) = \mathrm{col}\,(\bar{a}, \bar{b})$, and the function $\bar{\tau}(\cdot)$ is given on the whole $\mathbf{R}^p$. If the vector $\theta = \theta(\tau_*)$ is defined by the relations (22), then the equality $\bar{\tau}(\theta(\tau_*)) = \tau_*$ need no be satisfied. However, if the vectors $\theta^{[l]}$, $l = m-k, m-k+1, \ldots, m-k+n-1$, (see (24)) are linearly independent then the unique pair of vectors, $(\bar{a}, \bar{b})$, is determined by the formulae (25), (26); this pair coincides with the pair (a, b), which implies $\bar{\tau}_* = \tau_*$. The following assertion establishes a link between the vectors τ and θ.

Lemma 2 ([7], Lemma 5.5.1). *Letting the polynomials $a(\cdot, \tau_*)$, $b(\cdot, \tau_*)$ be mutually non-concellable, we arrive at linearly independent p-vectors $\theta^{[l]}$, $l = m-k, m-k+1, \ldots, m-k+n-1$, (see (24), (22))*

e) *A test signal.* Let us assume that a test signal $\bar{w}$ specially introduced into the control channel (see (17)) has the form

$$\begin{aligned} \bar{w} &= \{\bar{w}_t, \; t \in \mathbf{N}\}, \quad \bar{w}_t = 0, \quad t \neq ps, \\ \bar{w}_{ps} &= \sum_{s'=0}^{\infty} w_{s'} \delta_{s,s'}, \quad w_s = \frac{e_s}{\sqrt{1 + \ln\{s\}}}, \qquad s \in \mathbf{N}. \end{aligned} \tag{27}$$

Here $e = \{e_s, \; s \in \mathbf{N}\}$ is a scalar time series consisting of random quantities which are independent and possess the following properties:

$$\mathbf{E} e_s = \mathbf{E} e_s^3 = 0, \qquad \mathbf{E} e_s^2 = \sigma^2, \qquad \mathbf{E} e_s^4 = \bar{\sigma}^2, \qquad |e_s| \leq C_e, \tag{28}$$

where σ^2, $\bar{\sigma}^2$, C_e are positive constants. If noise v and (or) initial data in (1) are random then the time series e is assumed to be independent of them (we recall that in (1) the quantities y_t, $t \le 0$, and u_t, $t < 0$ serve as initial data, while in a controller initial data are zero).

f) *An identification algorithm.* Considering (27), we rewrite the relations (20) on the 'interval' of time, $\mathbf{N}_s = \{ps, ps+1, \ldots, (s+1)p-1,\ s \in \mathbf{N}\}$, $p = m+n-k$, in the form:

$$Y_s = \theta(\tau_*) w_s + f_s(\theta(\tau_*)) + \phi_s, \tag{29}$$

where $w = \{w_s,\ s \in \mathbf{N}\}$ is a test signal (see (27)),

$$\begin{aligned} Y_s &= \mathrm{col}\,(y_{ps+1}, y_{ps+2}, \ldots, y_{ps+p}), \qquad f_s(\theta_*) = \mathrm{col}\,(f_s^{(0)}(\theta_*), f_s^{(1)}(\theta_*), \ldots, f_s^{(p-1)}(\theta_*)), \\ f_s^{(l)}(\theta_*) &= F^{(l)}(\nabla) b(\nabla) \bar{u}_{ps+l} + G^{(l)}(\nabla, \tau_*) y_{ps-1} = \sum_{l'=k}^{l+k} \theta_*{}^{(l'-k)} \bar{u}_{ps+l-l'} + G^{(l)}(\nabla, \tau_*) y_{ps-1}, \\ \phi_s &= \mathrm{col}\,(\phi_s^{(0)}, \phi_s^{(1)}, \ldots, \phi_s^{(p-1)}), \qquad \phi_s^{(l)} = F^{(l)}(\nabla, \tau_*) v_{ps+l}, \qquad l = 0, 1, \ldots, p-1, \end{aligned} \tag{30}$$

and $\theta_* = \theta(\tau_*)$ is a vector of the parameters (23). The relation (29) looks like a 'linear observation scheme' with respect to the vector parameter $\theta(\tau_*)$, in which random vectors w_s, f_s are stochastically 'almost non-correlated'. This suggests the form of the algorithm for estimating the unknown vector $\theta(\tau_*)$. To this end the estimation (identification) algorithm can be taken as

$$\theta_{s+1} = \theta_s + \gamma \frac{1 + \ln\{s\}}{s} \Big(Y_s - \theta_s w_s - f_s(\theta_s) \Big) w_s, \qquad s \in \mathbf{N}, \tag{31}$$

where γ is a positive number and w_s is test signals (see (27)).

f) *The consistency of estimates.* We formulate the following assertion of the consistency of estimates obtained by the algorithm (31).

Theorem 1 *It is assumed that the following conditions are satisfied:*

- *for the unknown parameter $\tau_* \in \mathcal{T}$ the polynomials $a(\cdot, \tau_*)$, $b(\cdot, \tau_*)$ are mutually non-concellable;*
- *the test signal $\bar{w} = \{\bar{w}_t,\ t = 0, 1, \ldots\}$ is given by the formulae (27), (28);*
- *noise $v = \{v_t,\ t \in \mathbf{N}\}$ satisfies the condition (4). Should this noise be random it is then stochastically independent of the test signal $\bar{w} = \{\bar{w}_t,\ t \in \mathbf{N}\}$, as well as the initial data in (1) provided they are random;*
- *the control u is determined by (17);*
- *the inequality $2\gamma\sigma^2 \ge 1$ is satisfied.*

Then for an arbitrary initial condition $\theta_0 \in \mathbf{R}^p$ the algorithm (31) combined with the algorithm (13) ensures the estimates $\theta_s = \mathrm{col}\,(\theta_s^{(0)}, \theta_s^{(1)}, \ldots, \theta_s^{(p-1)})$ such that for an arbitrary $\delta > 0$ the following limit relations are valid with probability 1 and in the mean square sense:

$$\lim_{s\to\infty} s^{1-\delta}|\theta_s - \theta(\tau_*)|^2 = 0, \qquad \lim_{t\to\infty} t^{1-\delta}|\tau_t - \tau_*|^2 = 0. \tag{32}$$

Here $\tau_t = \bar{\tau}(\theta_s)$, $ps \le t < ps + p$, where the function $\bar{\tau}(\cdot)$ is determined by (24), (26).

Remark. From (27) it follows that $\lim_{t\to\infty} \bar{w}_t = 0$ with probability 1, implying that a test signal vanishes with time. That is why adaptive systems can be synthesized with the identification algorithm described in such a way that with time their output becomes indistinguishable from the output of an optimal system synthesized for a known parameter of a control plant.

3. The asymptotic normality of estimates

With 'external' noise v from (1) taken as a quasi-stationary signal, more fine properties of estimates obtained by identifying algorithm can be revealed. More precisely, letting a be some nonzero real vector of p dimensionality, the time series $V = \{V_t,\ t \in \mathbf{N}\}$, $V_t \in \mathbf{R}^p$, will be said to be *a-quasi-stationary* provided that there exists the limit $R = R(a) = \lim_{t\to\infty} \mathbf{E}|a^* V_t|^2$ which differs from zero. To be sure, if the process V is deterministic then $\mathbf{E}|a^*V_t|^2 = |a^*V_t|^2$. Being quasi-stationary for an arbitrary nonzero vector $a \in \mathbf{R}^p$, a time series V is said to be *quasi-stationary*. For such a series the quantity $R(a)$ is a quadratic form of the vector $a \in \mathbf{R}^p$. A non-negative $(p \times p)$ matrix of $R(a)$ is called by the *covariance of a quasi-stationary time series* $V = \{V_t,\ t \in \mathbf{N}\}$ and denoted by $R(a) = a^* R_V a = \lim_{t\to\infty} \mathbf{E} V_t V_t^*$. If the time series V is stationary, which means that $\mathbf{E} V_t V_t^* = R_V$, then it is a-quasi-stationary for $a^* R_V a \neq 0$. If the matrix R_V is non-degenerate then the series V is quasi-stationary in the sense of the above definition. In that event Theorem 1 may be strengthened as follows.

Theorem 2 *Let us assume that under the conditions of Theorem 1 the time series $V = \{V_t,\ t \in \mathbf{N}\}$, $V_t = \mathrm{col}\,(v_t, v_{t-1}, \ldots, v_{t-p+1})$ is a-quasi-stationary and that a step parameter γ of the algorithm satisfies the inequalities $1 < 2\gamma\sigma^2 \le 2$.*

Then for an arbitrary initial condidtion $\theta_0 \in \mathbf{R}^p$ the algorithm (31) combined with the algorithm (13) ensures the estimates θ_s, for which the randon quantities

$$\Delta\theta_{s+1} = \sqrt{\frac{s}{\ln\{s\}}}(\theta_* - \theta_{s+1})^* F^{-1} a, \tag{33}$$

have asymptotically (when $s \to \infty$) the normal distribution,

$$\Delta\theta_{s+1} \sim \mathrm{N}(0, \frac{\gamma^2\sigma^2}{2\gamma\sigma^2 - 1} a^* R_V a). \tag{34}$$

Here F is a $(p\times p)$ matrix defined through coefficients of the polynomials $F^{(l)}(\cdot)$ by the formula

$$F=\left\|\begin{array}{ccccc} 1 & 0 & 0 & \dots & 0\\ F_1^{(1)} & 1 & 0 & \dots & 0\\ F_2^{(2)} & F_1^{(2)} & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ F_p^{(p)} & F_{p-1}^{(p)} & F_{p-2}^{(p)} & \dots & 1 \end{array}\right\|. \tag{35}$$

Corollary. Let $\tau_t=\bar\tau(\theta_s)$, $ps\le t<ps+p$, where in view of (26), (25) $\bar\tau(\theta)=\mathrm{col}\,(\bar a,\bar b)$. Then by Lemmas 1, 2 and the conditions of Theorem 2 for sufficiently large t the following relation is valid:

$$\tau_t-\tau_*=\nabla\bar\tau(\theta_*)(\theta_s-\theta_*)+o\left(\frac{1}{t}\right),\qquad ps\le t<ps+p,$$

where $\nabla\bar\tau(\theta_*)$ is a $(p\times p)$ matrix which is the Freshe derivative of the function $\bar\tau(\cdot)$ at the point θ_*. Hence by virtue of (35) and for a non-degenerate matrix R_V it is stated that

$$\sqrt{\frac{t}{\ln\{t\}}}(\tau_*-\tau_t)\sim\mathrm{N}(0,\nabla\bar\tau(\theta_*)F^*R_VF\nabla\bar\tau(\theta_*)^*).$$

4. Appendix

a) *The proof of Theorem 1.* By virtue of 31 the quantities $\epsilon_s^{(\delta)}=s^{\frac{1-\delta}{2}}(\theta_*-\theta_s)$ are related by the following formula:

$$\epsilon_{s+1}^{(\delta)}\approx(1-\frac{2\gamma e_s^2-1+\delta}{2s})\epsilon_s^{(\delta)}+\frac{\gamma\sqrt{\ln\{s\}}}{s^{\frac{1+\delta}{2}}}\zeta_s,\qquad \zeta_s=(f_s(\theta_*-\theta_s)+\phi_s)e_s. \tag{36}$$

From this point on, the sign $\approx$ is taken to mean that the equality is satisfied up to values of the highest order of smallness when $s\to\infty$ and can be disregarded. Under the conditions of Theorem 1 the following inequality is valid:

$$\mathbf{E}(|\epsilon_{s+1}^{(\delta)}|^2|y^{ps},u^{ps-1})\approx(1-\frac{2\gamma\sigma^2-1+\delta}{s})|\epsilon_s^{(\delta)}|^2+\gamma^2\frac{2\ln\{s\}}{s^{1+\delta}}\mathbf{E}(|\zeta_s|^2|y^{ps},u^{ps-1}). \tag{37}$$

Since

$$\sum_{s=1}^{\infty}\frac{\ln\{s\}}{s^{1+\delta}}<\infty,\qquad\sum_{s=1}^{\infty}\frac{2\gamma\sigma^2-1+\delta}{s}=\infty$$

then by the familiar Doob theorem on the convergence of semi-martingals the limit equalities (32) are valid. □

b) *The proof of Theorem 2.* When $\delta=1$, the relations (36) can be rewritten as

$$\begin{aligned}\sqrt{\frac{s}{\ln\{s\}}}(\theta_*-\theta_{s+1})^*a &\approx \sqrt{\frac{s}{\ln\{s\}}}\prod_{k'=1}^{s}(1-\frac{\gamma e_{k'}^2}{k'})(\theta_*-\theta_1)^*a+\sum_{k=2}^{s}\zeta_{sk},\\ \zeta_{sk} &= -\gamma\sqrt{\frac{s}{\ln\{s\}}}\prod_{k'=k}^{s}(1-\frac{\gamma e_{k'}^2}{k'})\frac{\sqrt{\ln\{k\}}}{k}\phi_{k-1}^*ae_{k-1}.\end{aligned} \tag{38}$$

Owing to the independence of random values e_s^2 we have $\mathbf{E}\prod_{k'=1}^{s}(1-\frac{\gamma e_{k'}^2}{k'})^2 = \prod_{k'=1}^{s}(1-\frac{\gamma\sigma^2}{k'})^2 \approx (\frac{s}{k})^{-2\gamma\sigma^2}$; therefore the first summand in the right hand side of (38) vanishes in the mean square sense as $s\to\infty$, and can be disregarded for large s. The rest summand represents the sum of centred 'small' independent random quantities, resulting in

$$\mathbf{E}\zeta_{sk}^2 = \gamma^2\sigma^2\frac{s}{\ln\{s\}}\frac{\ln\{k\}}{k^2}\prod_{k'=k}^{s}(1-\frac{\gamma\sigma^2}{k'})^2\mathbf{E}(\phi_{k-1}^* a)^2 \approx \gamma^2\sigma^2\frac{s^{-2\gamma\sigma^2+1}}{\ln\{s\}}k^{2\gamma\sigma^2-2}\ln\{k\}a^* R_V a.$$

Hence it follows that under the conditions of Theorem 2 the following limit equality is satisfied:

$$\lim_{s\to\infty}\mathbf{E}\zeta_{sk}^2 = 0, \tag{39}$$

where the limit is uniform in $k\in\mathbf{N}$. Since

$$\mathbf{E}|\zeta_{sk}^3| \le \frac{(\ln\{k\})^{3/2}}{k^3}\prod_{k'=k}^{s}\left(1-\frac{\gamma\sigma^2}{k'}\right)^3 \approx C\frac{s^{-3\gamma\sigma^2+3/2}}{(\ln\{s\})^{3/2}}k^{3\gamma\sigma^2-3}(\ln\{k\})^{3/2}$$

where $C=C(a)$ is a constant, then the following Lyapunov condition is satisfied:

$$\lim_{s\to\infty}\sum_{k=2}^{s}\mathbf{E}|\zeta_{sk}^3| = 0. \tag{40}$$

The conditions (39), (40) imply that the conditions of the central limit theorem are fulfilled, according to which the random quantities $\Delta\theta_{s+1}^{(b)} = \sum_{k=2}^{s}\zeta_{sk}$ have asymptotically the normal distribution (as $s\to\infty$). By virtue of (38) we obtain the desired expression (33) by straightforward calculation. $\Box$

References

[1] Saridis G.M., G.Stein. A new algorithm for linear system identification. *IEEE Trans. Automat. Control*, AC-13(4):592–584, 1968.

[2] Saridis G.M., R.N. Lobbia. Parameter identification and control of linear discrete time system. *IEEE Trans. Automat. Control*, AC-17(1):52–60, 1972.

[3] Saridis G.M., R.N. Lobbia. Comment on 'Parameter identification and control of linear discrete time system'. *IEEE Trans. Automat. Control*, AC-20(3), 1975.

[4] Agafonov S. A stochastic approximation algorithm with noise at an input in the problem of adaptive control of a linear plant. VINITI, No 5682-81, Leningrad, 1981.

[5] Agafonov S., V. Fomin. Identification of control plants with test signals *VINITI*, No 4226-82, Leningrad, 1982.

[6] Granitchin O., V. Fomin. Adaptive control with test signals. *Automat Remote*, (2):100–112, 1986

[7] V. Fomin. *Methods of control of linear discrete plants*. Leningrad State Univ. Publ., Leningrad, 1985.

[8] Granitchin O. A stochastic approximation procedure with test signals. *Automat Remote*, (2): 97–104, 1992.

[9] Granitchin O. Stochastic approximation procedure with perturbation on an input and dependent observation disturbances. *Vestnik Leningrad Univ.*, 1(4), No 22:27–31. 1989.

[10] Granitchin O. Stochastic approximation under dependent noises. In *Approximation, Probability and Related Fields*, Plenum, USA, pages 247–271, 1994.

[11] Granitchin O., A. Portnov. Asymptotic optimality stochastic approximation algorithms with perturbation on an input. In *Proceedings of 2-nd Workshop on Simulation*, pages 322–327, St.–Petersburg, 1996.

[12] Fomin V., V. Fradkov, and V. Yakubovich. *Adaptive Control of Dynamic Systems*. Nauka, Moscow, 1981.

[13] Fomin V. *Recurrent Estimation and Adaptive Filtration*. Nauka, Moscow, 1984.