Combined Procedure with Randomized Controls for the Parameters' Confidence Region of Linear Plant under External Arbitrary Noise

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Abstract— The new algorithm is proposed for the estimating of linear plant's unknown parameters in the case of observations with arbitrary external noises. It is based on adding of randomized inputs (test perturbations) through the feedback channel. The assumptions about the noise are reduced to a minimum: it can virtually be arbitrary but independently of it the user must be able to add test perturbations. We combine the previous result about asymptotic properties of randomized control strategy with the new one which is followed by a nonasymptotic approach of LSCR (Leave-out Sign-dominant Correlation Regions) method. The new algorithm gives confidence regions for series of finite sets of observations. These regions shrink to the true values of an unknown parameters when number of observations tents to infinity while the algorithm complexity does not increases.

I. INTRODUCTION

A pervasive problem in data mining is the need to make meaningful inference from a limited amount of data. Such inference usually involves an estimation process and an uncertainty calculation (e. g., confidence region). For many estimators (such as least squares, maximum likelihood, maximum a posterior, etc.) exist an asymptotic theory that provides the basis for determining probabilities and confidence regions in the case of large samples. However, except for relatively simple cases, it is generally not possible to determine this uncertainty information in the smallsample setting. The new approach was proposed in [1] based on LSCR (Leave-out Sign-dominant Correlation Regions) method. If the number of observation increases infinitely then the resulting confidence intervals have good asymptotic properties in the sense that it shrink to the true point under some general conditions. But the complexity of the algorithm increases significantly when number of observations tends to infinity. More early in [2] there was proposed a simple recurrent algorithm with good asymptotic properties which is based on the similar randomized inputs adding in the feedback channel. The main contribution of this paper is to combine these two approaches.

For a linear control plant with almost arbitrary additive noises in the observations the identification is possible with randomized test signals as part of control actions. Noise does not necessarily possess any useful statistical properties and does not need to be random at all. The reconstruction of unknown values of parameters is provided based on the properties of a test signal which is mixed with a control signal. The introduction of a test signal in a control channel can deteriorate the control performance. However, in an appropriate decision about the intensity of a test signal the output process will be indistinguishable from an optimal process through time (if the intensity of a test signal is diminished rapidly with time it is not necessary that the identification process is complete).

The identification investigation techniques with test signals was used in [3] and subsequently extended in [4] to closed control systems. In these works an assumption of the a-priori stability of a plant was made, a noise was assumed to be a white noise process, and in addition, a relatively limiting constraint was placed on the noisy control. In [2], [5] for the non a-priori stable case there were suggested the algorithm when special randomized test signals in the input channel allow to identify asymptotically the control plant unknown parameters under almost arbitrary additive noise in a plant model. The procedure is valid for any noise v_t and does not require a-priori knowledge of its characteristics; noise may be not random or may be white or correlated random, with zero-mean or bias; a signal-noise ratio may be high or low. The recovery of unknown parameter values is provided by the properties of randomized test signals which are added together with an intrinsic adaptive control signal from a closed loop. This approach follows from Feldbaum's concept of dual control [6]: control must be not only directing but also learning. Recently similar randomized control strategies were put forward in [7].

In [5], [8] for the case of an arbitrary noise (e. g., *unknown but bounded noise*) the randomization was used to develop an identification algorithm which allows to obtain an asymptotically confidence region of an indefinitely small size. These results were extended to the case of time-varying parameters in [9], [10]. The information about the maximum possible amplitude of the noise has only been used in the formulas for estimating the rate of convergence, i. e., this knowledge is not required for operability of an identification algorithm.

The identification method discussed below is based on the reparametrization of the mathematical model of a plant (instead of coefficients of the plant as its initial parameters, some alternative parameters are convenient to use, which are in an one-to-one correspondence to the initial parameters). This enables the plant to be written in the form which is not too different from a "linear observation scheme". Then justified recurrent algorithms such as stochastic approximation algorithms can be applied for estimating unknown values of

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the parameters. In this paper we combine the ideas of former asymptotic results from [2], [5] and new procedure from [11], [12] which gives rigorously guaranteed non-asymptotic confidence regions for unknown parameters of a linear dynamical control plant which is disturbed by arbitrary noise. The control strategy considered in [11], [12] is different from the control strategy of [1]. It includes randomized items in the input channel only one time per some interval when the control strategy of [1] has randomized part of inputs on each iteration. This fact allows to prove in [12] more weak assumptions about independence of external noise and randomized part of control inputs.

The paper is organized as follows: At the beginning we give a preliminary example for illustrative purposes. Then, in Section III, we formulate a formal problem setting. Section IV provides the rules to form control inputs (control synthesis). Section V introduces the main assumptions and describes a special method of linear plant model reparameterization. Stochastic approximation algorithm is considered in Section VI. Next Section VII summarizes the result of [12] about properties of confidence regions in the case of a finite number of observations. The new algorithm and the main theoretical result of this paper are in Section VIII. At the end, we discuss our future plans.

II. PRELIMINARY EXAMPLE

Is it possible to get smart estimates under arbitrary external noise?

For example, let's consider the simple problem of an unknown parameter θ_{\star} estimating from the observations:

$$y_t = \boldsymbol{\theta}_\star \cdot \boldsymbol{u}_t + \boldsymbol{v}_t, \tag{1}$$

where we are able

- to chose the inputs (control actions) $u_t, t \in [1..N]$,
- to measure the outputs y_t (see Fig. 1).

Here and further we will use notation [i..j] for the set of integers from the integer *i* to the integer *j*.



Fig. 1. The model of observations.

The problem is *to find or estimate* the unknown parameter $\theta_{\star} \in \mathbb{R}$ by the sequence of inputs and outputs $\{u_t, y_t\}$ without any restrictions for the sequence of external noises $\{v_t\}$.

Does not it seem absurd such a statement of the problem?

From the deterministic point of view, yes, of course! Nobody does not know deterministic algorithms provided some sense in answers (other than a meaningless solution — the entire real axis). For the fixed number of observations and for any proposed answer as a value or a finite interval one can always suggest such v_t that the following observation will be wrong for the proposed answer. The algorithm of a sequential estimation of an unknown parameter θ_{\star} of (1) consists of two steps:

- 1) Input (control actions) u_t selection.
- 2) Estimation of the parameter θ_{\star} based on the data obtained u_t, y_t (for example, calculation of an estimate $\hat{\theta}_t$ or a set $\hat{\Theta}_t$ containing θ_{\star}).

If in addition to the problem setting we would be to assume a random (probabilistic) nature of the noise v_t then we could be use the strong law of large numbers. Under its conditions we can to talk about estimating an unknown parameter θ_{\star} by averaging of the observation data: $\hat{\theta}_t = \frac{1}{t} \sum_{i=1}^{t} y_i$. The simulation results with the true parameter $\theta_{\star} = 3$ and the observations which were made with an independent and uniformly distributed on the interval [-0.5, 0.5] noise v_t are given in Table 1. Row 5 indicates the proximity of the estimate $\hat{\theta}_7 = 2.99$ to the true parameter $\theta_{\star} = 3$.

Table 1.							
t	1	2	3	4	5	6	7
<i>u</i> _t	1	1	1	1	1	1	1
	$v_t = \operatorname{rand}() - 0.5$						
y _t	2.9	2.8	3.2	3.3	2.6	3.4	2.7
$\widehat{\theta}_t$	2.9	2.85	2.97	3.05	2.96	3.03	2.99
$v_t = \operatorname{rand}() - 0.5 + m, \ m = 1$							
y _t	3.9	3.8	4.2	4.3	3.6	3.9	4.2
$\widehat{\theta}_t$	3.9	3.85	3.97	4.05	3.96	4.03	3.99

If the observations were carried out with the random noise too but with the unknown mean value (expectation) $m = E\{v_t\}$ (for example, m = 1, Table 1, row 6) then the simulation results (Table 1, row 8) shows that the algorithm failed: $\hat{\theta}_7 = 3.99$. This value is substantially exceeds the true parameter $\theta_{\star} = 3$. The discrepancy is similar to the uncertainty level m = 1. (Here and further $E\{\cdot\}$ is a symbol of the mathematical expectation).

Despite the seeming absurdity of attempts to estimate an unknown parameter θ_{\star} under arbitrary external noises, from the practical needs it is often still have to decide.

Consider the following rule of a random input selection for the first step

$$u_t = \begin{cases} +1, \text{ with probability } \frac{1}{2}, \\ -1, \text{ with probability } \frac{1}{2}. \end{cases}$$
(2)

At the second step we calculate a value

$$\tilde{y}_t = u_t \cdot y_t$$

by the known values (u_t, y_t) . For the "new" sequence of observations $\{\tilde{y}_t\}$ we have a model which is similar to (1)

$$\tilde{y}_t = \theta_\star \cdot \tilde{u}_t + \tilde{v}_t$$

where $\tilde{u}_t = u_t^2$ and $\tilde{v}_t = u_t \cdot v_t$.

Let's suppose, as in the simulation before, that v_t is a random noise but with unknown expectation. If v_t is an external noise it is natural to assume that it does not depend on the our randomized input (control) at the first step. Hence we have

$$E\{\tilde{v}_t\} = E\{u_t \cdot v_t\} = E\{u_t\} \cdot E\{v_t\} = 0 \cdot m = 0,$$

i. e. in the "new model" the "hard (ill poses)" observation problem of estimating an unknown parameter θ_{\star} of (1) is converted by using the random selection rule for inputs (controls) in the first step to the "standard" problem of estimating of an unknown parameter θ_{\star} observed with an independent zero-mean noise.

In Table 2 we summarize the corresponding results of the simulation.

Table 2.							
t	1	2	3	4	5	6	7
u_t	-1	1	-1	1	1	1	-1
	$v_t = rand() - 0.5 + m, m = 1$						
<i>y</i> _t	-2.1	3.8	-1.8	4.3	3.6	4.4	-2.3
\tilde{u}_t	1	1	1	1	1	1	1
\tilde{y}_t	2.1	3.8	1.8	4.3	3.6	4.4	2.3
$\widehat{\theta}_t$	2.1	2.95	2.57	3.00	3.12	3.33	3.19

The comparison of results from Table 2, row 7 with previous one from Table 1, row 8 shows that new estimates are substantially better but the quality of evaluations turned out lower than in the more relevant results from Table 2, row 5 because "new errors" \tilde{v}_t have a bigger variance than v_t .

The probability of making a wrong decision can be estimated asymptotically using the assessment of the correspondence mean rate of convergence in [8] and Chebyshev's inequality. For every *t* and for any $\varepsilon > 0$ if $E\{v_j^2\} \le \sigma_v^2$, $j \in [1..t]$ then we obtain

$$Prob\{|\widehat{\theta}_t - \theta_\star| \ge \varepsilon\} \le \frac{1}{t} \frac{\sigma_\nu^2}{\varepsilon^2} + o\left(\frac{1}{t}\right).$$

Following the method described by M. Campi in [14] for the finite number of observations (N = 7) with an arbitrary external noise v_t a new rigorous mathematical result of a guaranteed set of possible values of an unknown parameter θ_{\star} can be obtained:

- 1) Let be M = 8 and select randomly seven (= M 1) different groups of four indexes T_1, \ldots, T_7 .
- 2) Compute seven partial sums $\tilde{s}_i = \sum_{i \in T_i} \tilde{y}_i$, $i \in [1..7]$.
- 3) Build the confidence interval

$$\widehat{\Theta} = [\min_{i \in [1..7]} \widetilde{s}_i; \max_{i \in [1..7]} \widetilde{s}_i].$$

It contains θ_{\star} with the probability p = 75% (= 1 - 2 · 1/*M*).

For the sample data $\{(u_t, y_t)\}$ from Table 2 we obtain by the above described method:

Table 3.				
i	T_i	\tilde{s}_i		
1	$\{2, 3, 4, 5\}$	3.375		
2	$\{1, 3, 4, 6\}$	3.15		
3	$\{2, 3, 5, 6\}$	3.4		
4	$\{1, 2, 6, 7\}$	3.15		
5	$\{1, 4, 5, 7\}$	3.075		
6	$\{2, 3, 5, 7\}$	2.875		
7	$\{1, 4, 6, 7\}$	3.275		

Hence, the unknown parameter θ_{\star} belongs the interval $\widehat{\Theta} = [2.875; 3.4]$ with the probability p = 75%.

Under arbitrary external noises the randomization in the input data selection process allows to get a quite reasonable result for the unknown parameter estimation problem which seems absurd and cannot be handled by any deterministic algorithm!

Remark. An alternative probabilistic approach is a Bayesian estimation when the noise v_t probability is attributed a-priori to a nature with a probability distribution Q. But Bayesian and randomized approaches are quite different from the practical point of view. In the Bayesian approach the probability Q describes the probability of a value of v_t in comparison with other, i. e. it is a part of the model of the problem. In contrast, the probability P in the randomized approach is artificially selected. P exists only in our algorithm and it has a known nature, and therefore the traditional problem of "a bad model" is not happen as it can often happen with the Q in the Bayesian approach.

III. PROBLEM STATEMENT

Consider a dynamic scalar linear control plant which is described in a discrete time by an autoregressive moving average model

$$A_{\star}(z^{-1})y_t = B_{\star}(z^{-1})u_t + v_t \tag{3}$$

with scalar inputs u_t , outputs y_t , and noises v_t .

In Equation (3) z^{-1} is a delay operator: $z^{-1}y_t = y_{t-1}, z^{-1}u_t = u_{t-1}$, the polynomials $A_{\star}(\lambda)$ and $B_{\star}(\lambda)$ have forms

$$A_{\star}(\lambda) = 1 + a_{\star}^{(1)}\lambda + \dots + a_{\star}^{(n_a)}\lambda^{n_a},$$
$$B_{\star}(\lambda) = b_{\star}^{(l)}\lambda^l + b_{\star}^{(l+1)}\lambda^{l+1} + \dots + b_{\star}^{(n_b)}\lambda^{n_b},$$

where the positive integers n_a, n_b are known output and input (control) model orders; l is a delay in control, $1 \le l \le n_b$; $\tau = (a_{\star}^{(1)}, \ldots, a_{\star}^{(n_a)}, b_{\star}^{(l)}, \ldots, b_{\star}^{(n_b)})^{\mathrm{T}}$ is a vector of plant parameters a part of which is unknown. Here and further we will use superscripts in brackets as coefficients' indexes.

Noise v_t describes all other sources apart from $\varphi_t = (-y_{t-1}, \dots, -y_{t-n_a}, u_{t-l}, \dots, u_{t-n_b})^{\mathrm{T}}$ which cause variation in y_t .

It is required to define, with a given probability, an area of reliability for unknown coefficients of plant (3) by the observations of outputs $\{y_t\}$ and known inputs (controls) $\{u_t\}$ which can be chosen.

Procedures discussed further intended to identify the plant' unknown parameters are based on reparameterization of the plant mathematical model. Instead of the natural plant parameters—dynamic coefficients—it is convenient to use some other parameters which are in one-to-one correspondence with them. Such reparameterization is a result of rewriting the plant's equation (3) in the moving average model form which makes it possible to use the stochastic approximation and LSCR procedures for building the confidence region, even in the cases when an adaptive algorithm is used in the feedback channel.

IV. CONTROL ACTIONS WITH RANDOMIZED TEST SIGNALS

Let $s \le n_a + n_b - l + 1$ be a positive integer. (It is usually equal to the amount of plant (3) unknown parameters). And let be the number of observations $N = s \cdot K \cdot N_{\Delta}$ with some positive integers K and N_{Δ} .

Let us choose a sequence of independent random variables symmetrically distributed around zero (a randomized test perturbation) $\Delta_0, \Delta_1, \dots, \Delta_{KN_{\Lambda}-1}$:

$$E\{\Delta_n\}=E\{\Delta_n^3\}=0, E\{\Delta_n^2\}=\sigma_{\Delta}^2, E\{\Delta_n^4\}\leq M_4,$$

and add them to the input channel with some gain coefficients β_k , $k \in [1..K]$ once per every *s* time moments (at the beginning of each time interval [skn - l..skn + s - l - 1)] where $n \in [1..N_{\Delta}]$) in order to "enrich" the variety of observations.

To be more precise, we will build controls $\{u_t\}$ by the rule

$$u_{sn+i-l} = \begin{cases} \beta_{n \div N_{\Delta}} \Delta_n + \bar{u}_{sn-l}, \ i = 0, \\ \bar{u}_{sn+i-l}, \ i \in [1..s-1] \text{ or } i \in [-s+1..-1], \end{cases}$$

where $n \in [1..KN_{\Delta}]$, and "own (intrinsic)" controls $\{\bar{u}_t\}$ are determined by the adjustable feedback law

$$\bar{u}_t = \mathscr{U}_t(y_t, y_{t-1}, \dots, \bar{u}_{t-1}, \dots), \, \bar{u}_{-t} = 0, \, t \ge 0$$

The type and characteristics of a feedback depend on practical problems specifics. In particular, it is possible to use a trivial law of "intrinsic" feedback: $\bar{u}_t = 0, t \in [1..N - l]$, or in [2] it is proposed to use the stabilized regulator

$$C(z^{-1}, \tilde{\tau}_t)\bar{u}_t = D(z^{-1}, \tilde{\tau}_t)y_t \tag{4}$$

with parameters $\tilde{\tau}_t = \hat{\tau}_{t-s}$ which are tuning by the "Strip"-algorithm

$$\widehat{\tau}_{t} = \widehat{\tau}_{t-1} - \frac{(\boldsymbol{\varphi}_{t}^{\mathrm{T}} \widehat{\tau}_{t-1} - \boldsymbol{y}_{t}) \mathbf{1}_{\{|\boldsymbol{\varphi}_{t}^{\mathrm{T}} \widehat{\tau}_{t-1} - \boldsymbol{y}_{t}| - 2C_{v} - \delta \|\boldsymbol{\varphi}_{t}\| > 0\}}{\|\boldsymbol{\varphi}_{t}\|^{2}} \boldsymbol{\varphi}_{t}, \quad (5)$$

where $\mathbf{1}_{\{\cdot\}}$ is the characteristic function of a set, C_{ν} is a choosing sufficiently large noise bound $\delta > 0$ is a small constant, and a feedback regulator (4) is determined by such polynomials $C(\lambda, \tau)$ and $D(\lambda, \tau)$ that $A(\lambda, \tau)C(\lambda, \tau) - B(\lambda, \tau)D(\lambda, \tau)$ is a stable polynomial.

V. MAIN ASSUMPTIONS AND REPARAMETERIZATION OF THE PLANT MODEL

Main assumption

A1. The user can choose Δ_n and this choice does not affect the external noise $v_{sn}, \ldots, v_{s(n+1)-1}$. (In the mathematical sense, Δ_n does not depend on $\{v_t\}_{t=1}^{s(n+1)-1}$.)

Note that we do not make any assumptions about the statistical structure of noise v_t . It can be non-random. If it is random, there are no assumptions about the zero-mean or any autocorrelation properties.

For time instants $sn, n \in [1..KN_{\Delta}]$, we can denote $\bar{v}_{sn} = v_{sn} + (1 - A_{\star}(z^{-1}))y_{sn} + (B_{\star}(z^{-1}) - b_{\star}^{(l)}z^{-l})u_{sn}$ and rewrite Equation (3) in the following form:

$$y_{sn} = \beta_{n \div N_{\Delta}} \Delta_n \theta_{\star}^{(1)} + \theta_{\star}^{(1)} \bar{u}_{sn-l} + \bar{v}_{sn},$$

where $\theta_{\star}^{(1)} = b_{\star}^{(l)}$. This equation shows a direct relation between observation y_{sn} and test signal Δ_n which does not depend on the "new" noise \bar{v}_{sn} .

Similarly, we can rewrite Equation (3) for the rest of time sn + i - 1, $i \in [2..s]$, $n \in [1..KN_{\Delta} - 1]$, sequentially excluding the variables $y_{sn+i-1}, \ldots, y_{sn}$ from the left-hand side of Equation (3) using the same equation for more early time instants

$$y_{sn+i-1} = \beta_{n \div N_{\Delta}} \Delta_n \theta_{\star}^{(i)} + \sum_{j=0}^{i-1} \theta_{\star}^{(i-j)} \bar{u}_{sn-l+j} + \bar{v}_{sn+i-1}.$$
 (6)

Here $\theta_{\star}^{(i-j)}$, $j \in [0..i-1]$ are the corresponding coefficients of the remaining right-hand side terms with $\bar{u}_{sn-l+j}, \bar{v}_{sn+1} =$ $v_{sn+1} - a_{\star}^{(1)} \bar{v}_{sn} + (1 - A_{\star}(z^{-1}) + a_{\star}^{(1)}z^{-1})y_{sn+1} + (B_{\star}(z^{-1}) - b_{\star}^{(l)}z^{-l} - b_{\star}^{(l+1)}z^{-l-1})u_{sn} + \theta_{\star}^{(1)}\bar{u}_{sn-l+1}$, and $\bar{v}_{sn+i}, i \in [2..s]$, are determined sequentially in a similar way.

In [2] and [5], the authors suggest forming new model parameters as *s*-vector θ_{\star} of coefficients $\theta_{\star}^{(i)}$ obtained in (6). They also give conditions for the invertibility of a such reparameterization procedure.

The next formula follows immediately from the above definition $\theta_{\star} = \mathbb{A}^{-1}\mathbb{B}$, where $s \times s$ matrix \mathbb{A} and *s*-vector \mathbb{B} are

$$\mathbb{A} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ a_{\star}^{(1)} & 1 & \dots & 0 & 0 \\ a_{\star}^{(2)} & a_{\star}^{(1)} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & a_{\star}^{(n_{a})} \dots & a_{\star}^{(1)} & 1 \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} b_{\star}^{(l)} \\ \vdots \\ b_{\star}^{(n_{b})} \\ \vdots \\ 0 \end{pmatrix}.$$

How can we make the choice of an integer parameter s? The most natural way is to choose it by a such way which guarantees the existence of an inverse function corresponding to the new model parameters θ .

Assumptions

A2. Let *s* be a positive integer such that a set of the plant's unknown parameters are uniquely determined by some function $\tau(\theta)$ from the above-defined vector θ_{\star} .

By Lemma 2.2 on p. 117 from [5] Assumption A2 holds for $s = n_a + n_b - l + 1$ when the plant's orders n_a, n_b are known and the following assumption is satisfied.

A3. The polynomials $z^{n_a}A_{\star}(z^{-1})$ and $z^{n_b}B_{\star}(z^{-1})$ are mutually prime.

In [5] there is also the algorithm for the inverse function $\tau(\theta)$ construction.

Usually in a practice, only a part of plant parameters are unknown. Sometimes, unknown parameters correspond to the low degrees of z^{-1} which are smaller than some \bar{n}_a and \bar{n}_b , respectively. In this case, we can choose $s = \bar{n}_a + \bar{n}_b - l + 1$, which is significantly less than $n_a + n_b - l + 1$. Moreover, the "new" noise \bar{v}_{sn+i-1} in (6) can be cutted into two parts: nonmeasurable \tilde{v}_{sn+i-1} and measurable ψ_{sn+i-1} . The second measurable part is determined by observable inputs and outputs with known coefficients (see the example below).

Example. Consider the second-order plant

$$y_t + a_{\star}^{(1)} y_{t-1} + y_{t-2} = b_{\star}^{(1)} u_{t-1} + 1.6 u_{t-2} + v_t, \tag{7}$$

with unknown coefficients $a_{\star}^{(1)}$ and $b_{\star}^{(1)} \neq 0$. Denote

$$au_\star = egin{pmatrix} a^{(1)}_\star \ b^{(1)}_\star \end{pmatrix}.$$

Let be s = 2 and vector θ_{\star} of the "new" parameters is

$$\theta_{\star} = \begin{pmatrix} b_{\star}^{(1)} \\ 1.6 - a_{\star}^{(1)} b_{\star}^{(1)} \end{pmatrix} \in \mathbb{R}^{2}$$

In this case, the inverse function $\tau(\theta)$ is

$$\tau(\boldsymbol{\theta}) = \begin{pmatrix} \frac{1.6 - \boldsymbol{\theta}^{(2)}}{\boldsymbol{\theta}^{(1)}} \\ \boldsymbol{\theta}^{(1)} \end{pmatrix}.$$

Equations (6) have the following forms:

$$y_{2n} = \beta_{n \div N_{\Delta}} \Delta_n \theta_{\star}^{(1)} + \theta_{\star}^{(1)} \bar{u}_{2n-1} + \psi_{2n} + \tilde{v}_{2n},$$
$$y_{2n+1} = \beta_{n \div N_{\Delta}} \Delta_n \theta_{\star}^2 + \theta_{\star}^2 \bar{u}_{2n-1} + \theta_{\star}^1 \bar{u}_{2n} + \psi_{2n+1} + \tilde{v}_{2n+1}$$

where $\psi_{2n+i} = 1.6\bar{u}_{2n-2+i} - y_{2n-2+i}$, i = 0, 1, $\tilde{v}_{2n} = v_{2n} - a_{\star}^{(1)}y_{2n-1}$, $\tilde{v}_{2n+1} = v_{2n+1} + a_{\star}^{(1)}(a_{\star}^{(1)}y_{2n-1} + y_{2n-2} - 1.6\bar{u}_{2n-2} - v_{2n})$.

VI. STOCHASTIC APPROXIMATION ALGORITHM

Consider the estimation algorithm

$$\begin{cases} \bar{\tau}_t = \tau(\widehat{\theta}_{n-1}), \, t \in [s(n-1)+1..sn], \, n \in [1..KN_{\Delta}], \\ \widehat{\theta}_n^{(i)} = \widehat{\theta}_{n-1}^{(i)} - \frac{1}{n} \Delta_n(u_{sn-l} \widehat{\theta}_{n-1}^{(i)} - y_{sn+i-1}), \, i \in [1..s]. \end{cases}$$
(8)

Let be for a sufficiently large $\bar{R} > 0$ the "own" control $\{\bar{u}_t\}$ is built by the feedback controller (4) with adjustable parameters $\bar{\tau}_t$ when inputs and outputs are bounded and otherwise with estimates $\hat{\tau}_t$ which are formed by "Strip"-algorithm (5)

$$\tilde{\tau}_t = \begin{cases} \bar{\tau}_t, & \text{if } |y_t| + |u_{t-1}| < \bar{R}, \\ \hat{\tau}_{t-s} & - \text{otherwise.} \end{cases}$$
(9)

Theorem 1. If the conditions A1-A2 and

$$2\sigma_{\Delta}^2 > 1, E\{v_t^2\} \le \sigma_v^2, \sum_{n=1}^{\infty} \frac{\beta_{n+N_{\Delta}}^{-1}}{n} = \infty, \sum_{n=1}^{\infty} \frac{\beta_{n+N_{\Delta}}^{-2}}{n^2} < \infty$$

are satisfied **then** for an arbitrary initial vector $\hat{\theta}_0 \in \mathbb{R}^s$ the algorithm (8) ensures the estimates $\{\hat{\theta}_n\}$ such that for an arbitrary $\rho > 0$ the following limit bounds are valid in the mean square sense:

$$E\{|\widehat{\theta}_{n}^{(i)}-\theta_{\star}^{(i)}|^{2}\} \leq \frac{\beta_{n \div N_{\Delta}}^{-1}}{n} \frac{\rho C_{\overline{u}}^{2}+\sigma_{i,\overline{v}}^{2}}{2\sigma_{\Delta}^{2}-1} + o\left(\frac{\beta_{n \div N_{\Delta}}^{-1}}{n}\right), \quad (10)$$

 $i \in [1..s], \ C_{\bar{u}} = \sup_t \bar{u}_t, \ \sigma_{i,\bar{v}}^2 = \sigma_v^2 + (C_{\bar{u}}^2 + \sigma_\Delta^2) \sum_{j=1}^{i-1} |\theta_\star^{(j)}|^2.$ *Remark.* If $\beta_k \to 0$ as $k \to \infty$ and $C_{\bar{u}} < \infty$ it follows that

a randomized test signal vanishes with time (e. g., we can choose $\beta_k = 1/\ln(k)$). That is why an adaptive system can be synthesized with the described identification algorithm since with time their output becomes indistinguishable from the output of an optimal system synthesized for known control plant parameters.

The proof of Theorem 1. By virtue the representation (6) for any $i \in [1..s]$ the algorithm (8) is strictly the same as algorithms (2) in [8] or (2.2) in [5] for the linear regression model

$$y_{sn+i-1} = u_{sn-l} \boldsymbol{\theta}_{\star}^{(i)} + \bar{v}_{sn+i-1}.$$

To prove Theorem 1 it is possible to use the corresponding result of Theorem 2 from [8] for a partial case $\beta_k = 1$ or Theorem 5 on p. 111 from [8] for the general case since all their conditions hold under conditions of Theorem 1.

For the considered case we have $|E\{u_{sn-l}\}| \leq C_{\bar{u}}$, $|E\{u_{sn-l}^2\}| \leq C_{\bar{u}}^2 + \sigma_{\Delta}^2$, and $E\{\bar{v}_{sn+i-1}^2\} \leq \sigma_v^2 + (C_{\bar{u}}^2 + \sigma_{\Delta}^2)\sum_{j=1}^{i-1}|\theta_{\star}^{(j)}|^2$. These assessments allow to derive the result of Theorem 1.

VII. PROCEDURE FOR CONSTRUCTING CONFIDENCE REGIONS FOR FINITE NUMBER OF OBSERVATIONS

The previous result has an asymptotic nature. For the finite number of observations we can use the following procedure.

- 1) For each $k \in [1..K]$ consider the finite time interval $[k's..k's+sN_{\Delta}-1]$ where $k' = (k-1)N_{\Delta}$.
- 2) Using the data of observation, for $n \in [1..N_{\Delta}]$, $i \in [1..s]$ we can write sN_{Δ} predictors as a function of θ

$$\hat{y}_{k's+sn+i-1}(\boldsymbol{\theta}) = \beta_k \Delta_{k'+n} \boldsymbol{\theta}^{(i)} + \sum_{j=0}^{i-1} \boldsymbol{\theta}^{(i-j)} \bar{u}_{k's+sn+i-l-j}.$$

3) We can calculate the prediction error

$$\varepsilon_t(\theta) = y_t - \hat{y}_t(\theta), t \in [k's..k's + sN_\Delta - 1].$$

4) According to the observed data, for $n \in [1..N_{\Delta}]$, $i \in [1..s]$ we form a set of sN_{Δ} functions of θ

$$f_{k's+sn+i-1}(\boldsymbol{\theta}) = \Delta_{k'+n} \varepsilon_{k's+sn+i-1}(\boldsymbol{\theta}).$$

5) Choose a positive integer M > 2s and for $j \in [0..M-1]$ construct M different binary stochastic strings (of zeros and ones) $(h_{j,1}, \ldots, h_{j,sN_{\Delta}})$ as follows: $h_{0,i} = 0, i \in [1..N]$, all the other elements $h_{j,i}$ take the values of zero or one with the equal probability $\frac{1}{2}$. We calculate

$$g_{k,j}^{(i)}(\theta) = \sum_{n=0}^{N_{\Delta}-1} h_{j,ns+i} \cdot f_{k's+ns+i-1}(\theta), \ i \in [1..s].$$

6) Choose q from the interval [1; M/2s]. For $i \in [1..s]$ construct regions $\widehat{\Theta}_k^{(i)}$ such that at least q of the $g_{k,j}^{(i)}(\theta)$ functions are strictly higher than 0 and at least q functions are strictly lower than 0.

We define the confidence set by the formula

$$\widehat{\Theta}_k = \bigcap_{i=1}^s \widehat{\Theta}_k^{(i)}.$$
(11)

Remarks. 1. The procedure described above is similar to the one suggested in [1] but it has two significant differences from it. First, we consider a confidence set $\widehat{\Theta}_k$ in state space \mathbb{R}^s instead of $\mathbb{R}^{n_a+n_b}$ as in [1]. It is better since often we have $s < n_a + n_b$. Moreover, the considered confidence regions $\widehat{\Theta}_k^{(i)}$, $i \in [1..s]$ are the subsets of \mathbb{R}^i instead the case of [1]

when $\widehat{\Theta}_{k}^{(i)} \subset \mathbb{R}^{n_{a}+n_{b}}$. Second, randomized trial perturbations are included through the input channel only once per every *s* time instants instead of permanent perturbations in [1]. This is better from the control point of view. We do not disturb the plant so often.

2. If we can cut the "new" noise $\bar{v}_{k's+sn+i-1}$ in (6) into two parts— $\tilde{v}_{k's+sn+k-1}$ and $\psi_{k's+sn+k-1}$ — where the first part is nonmeasurable but whereas the second one is determined by observable inputs and outputs with known coefficients then we can use more stronger predictors $\hat{y}_{k's+sn+i-1}(\theta) = \beta_k \Delta_{k'+n} \theta^i + \sum_{j=0}^{i-1} \theta^{(i-j)} \bar{u}_{k's+sn+i-l-j} + \psi_{k's+sn+i-1}$ in the above-described procedure.

The probability that θ_{\star} belongs to each of $\widehat{\Theta}_{k}^{(i)}$, $i \in [1..s]$ is given in the following theorem.

Theorem 2: Let condition A1 be satisfied. Consider $i \in [1..s]$ and assume that $Prob(g_{k,j}^{(i)}(\theta_{\star}) = 0) = 0$. Then

$$Prob\{\theta_{\star} \in \widehat{\Theta}_{k}^{(i)}\} = 1 - 2q/M \tag{12}$$

where M, q and $\widehat{\Theta}_k^{(i)}$ are from steps 5 and 6 of the abovedescribed procedure.

Proof: See [12].

The next corollary follows directly from Theorem 2.

Corollary 3: Under the conditions of Theorem 2

$$Prob\{\theta_{\star} \in \Theta_k\} \ge 1 - 2sq/M \tag{13}$$

where $\widehat{\Theta}_k$ is taken from (11).

Note that, as it was pointed out in [1], the value of the probability in (12) is accurate but not the lower limit. Inequality in (13) is obtained by virtue to the fact that the events $\{\theta_* \notin \widehat{\Theta}_k^{(i)}\}, i \in [1..s]$ may overlap.

From the above, it is easy to derive.

Theorem 4: Let conditions A1–A2 be satisfied and assume that $Prob(g_{k,j}^{(i)}(\theta_{\star}) = 0) = 0$. Then the set $\tau(\widehat{\Theta})$ is the confidence set for unknown parameters of plant (3) with a confidence level of no less than 1 - 2sq/M.

VIII. MAIN RESULT

Now we are ready to formulate the main result. *Algorithm*

1) For k = 1, 2, ..., K.

- 2) To generate the sequence $\{\widehat{\theta}_n\}$ by the algorithm (8).
- 3) To choose $\varepsilon > 0$ and to build the parallelepiped

$$\bar{\Theta}_{k} = \prod_{i=1}^{s} \{ \boldsymbol{\theta}^{(i)} : |\widehat{\boldsymbol{\theta}}_{kN_{\Delta}}^{(i)} - \boldsymbol{\theta}^{(i)}| \le \varepsilon \}.$$
(14)

- 4) To choose q and M and to compute the region $\widehat{\Theta}_k$ by the algorithm (11).
- 5) We define the confidence set as the intersection $\widetilde{\Theta}_k = \widehat{\Theta}_k \cap \overline{\Theta}_k$.

Theorem 5: Let the conditions of Theorems 1 and 2 hold. **Then** $Prob\{\theta_{\star} \in \widetilde{\Theta}_k\} \geq$

$$\geq (1 - 2sq/M) \left(1 - \frac{\beta_k}{kN_\Delta} \frac{\rho C_{\bar{u}}^2 + \sigma_{s,\bar{v}}^2}{\varepsilon^2 (2\sigma_\Delta^2 - 1)} \right)^s + o\left(\frac{\beta_k}{kN_\Delta}\right).$$
(15)

Proof: The result of Theorem 5 is immediately follows by applying the Chebushev's inequality to the result of Theorem 1 and by Corollary 3.

IX. CONCLUSION

In the future work we plan to use above theoretical results in our practical project: multiagents group of UAV [?]. For the UAV control syntheses it is important to develop algorithms of a flight optimization. One of possibilities is to use above described randomized algorithms [?]. Other way is to accumulate energy and increase the flight range by using the thermal updrafts which are formed in the lower atmosphere due to disruption of warm air from the surface when it is heated by sunlight [16].

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