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Randomized Controls for Linear Plants and Confidence Regions for Parameters under External Arbitrary Noise

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Abstract— **This paper considers the possibilities of randomized controls for designing confidence regions for unknown parameters. The assumptions about external noise that affect a linear plant are reduced to a minimum: external noise can virtually be arbitrary, but, independently of it, the user must be able to add test perturbations through the input channel. Based on a finite set of observations, we suggest a new procedure which can be used in adaptive control schemes. It has been developed in the general framework of "counting of leave-out sign-dominant correlation regions" (LSCR), which is mostly being promoted by M. Campi et al. for identification problems. The procedure returns confidence regions which are guaranteed to contain true parameters with a user-chosen probability. The theoretical results are illustrated by an example of a nonminimum-phase second-order plant.**

I. INTRODUCTION

The real world is usually so complex that its model is often used with simplifications and many assumptions. To describe unknown relations, one can include a lot of uncertainties into a model. But it is difficult to understand their real nature and properties. Besides, it is important to determine the limits of an uncertainty in the model that can be tolerated by a specified application. Therefore, developing methods and procedures for assessment of a model quality is a central issue in system identification [1], [2].

Insufficient variety of input signals complicates the problem of identification. An opportunity for a control system to produce a special control (trial, test, probing) signal in the input channel can significantly alleviate the problem of reconstructing unknown parameters. Special randomized test signals in the input channel allow identification of the control plant parameters when we consider a plant model with almost arbitrary additive noise [3], [4]. The procedure suggested in [3], [4] is valid for any noise v_t and does not require a priori knowledge of its characteristics; noise may be not random or may be white or correlated, with zero-mean or bias; a signal-noise ratio may be high or low. The recovery of unknown parameter values is provided by the properties of randomized test signals which are added together with an intrinsic adaptive control signal from a closed loop. This approach follows from Feldbaum's concept of *dual control* [5]: *control must be not only directing, but also learning*.

Consider a dynamical system

$$y_t = G_\star(z^{-1})u_t + v_t \tag{1}$$

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with input u_t and output y_t shown in Fig. 1.

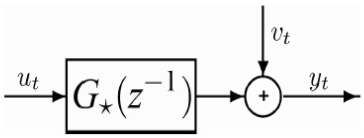

Fig. 1. Dynamical system.

Noise v_t describes all other sources, apart from u_t, which cause variation in y_t. z^{-1} is a delay operator: $z^{-1}u_t = u_{t-1}$. The transfer function $G_\star(z^{-1})$ belongs to a set of transfer functions $G(\theta, z^{-1})$, parameterized by $\theta \in \mathbb{R}^s$, i. e., $G_\star(z^{-1}) = G(\theta_\star, z^{-1})$ for some $\theta_\star$. The structure of the model class $G(\theta, z^{-1})$ is known, but $\theta_\star$ itself is unknown. The problem under consideration is to determine, based on a finite set of input and output data collected at time $t = 1, 2, \ldots, N$, a confidence region $\widehat{\Theta}$ for $\theta_\star$ with a specified probability chosen by a user. Moreover, $\widehat{\Theta}$ must be constructed without any a priori knowledge of the level, distribution, or correlation of the noise.

The standard approach to obtaining confidence regions is to use *an asymptotic theory* of system identification (see, e.g., [1], [4], [6], [7]). Although these results have been used successfully in many applications, asymptotic estimates are only reliable when the data volume N tends to infinity. When the number of points of data measurements is finite, the asymptotic theory may cause erroneous results even for large data sets. In the case of a small finite amount of data the one of possible approach is considered in [8].

In [4], [9], for the case of arbitrary noise (e. g., *unknown but bounded noise*), the randomization was used to develop an identification algorithm, which allowed for obtaining an asymptotically confidence region of an indefinitely small size. These results were extended to the case of time-varying parameters in [10], [11].

In this paper, we present a procedure which gives rigorously guaranteed nonasymptotic confidence regions for unknown parameters of a linear dynamical control plant which is disturbed by arbitrary noise. The procedure consists of simple input design steps followed by an algorithm named LSCR (Leave-out Sign-dominant Correlation Regions), which is mostly promoted by M. Campi and E. Weyer [12]. But the LSCR method is difficult to use directly when we consider identification problems in the context of adaptive control under arbitrary noise. In particular, the practical application of the LSCR method to systems with feedback, which was considered in [12] (Remark 3 on p. 2711), is only possible for an a priori chosen stationary control law. If the

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control plant is not stabilized and input and output variables increase infinitely, a linear model is usually not valid, from the practical point of view, and the regions obtained by the algorithms from [12], which were proved theoretically, may not be directly relevant to the initial statement. If the closed-loop regulator changes in time, depending on current observations, it implies that one of the main conditions of the applicability of the LSCR method from [12] is violated.

The main contribution of this paper in comparison with the results of [12] is based on a special parameterization of a plant model. This leads to smaller dimensions of confidence regions and opens up the possibility of using LSCR for a class of arbitrary feedbacks (in particular, for regulators with adjustable coefficients). Earlier, the same parameterization method was described in [3], [4], [10] for designing asymptotic estimation algorithms. The procedure considered in this paper can be applied to any finite number of observations. It was suggested in [13]. Here we establish weaker conditions for external noise and illustrate a possibility of applying the algorithm to a more difficult case of a nonminimum-phase second-order plant.

The paper is organized as follows: At the beginning, we give a preliminary example for illustrative purposes. Then, in Section III, we formulate a formal problem setting. Section IV provides the rules to form control inputs (control synthesis). Section V introduces the main assumptions and describes a special method of transfer function reparameterization. Section VI presents a procedure for constructing confidence regions. The main theoretical result is given in Section VII. The simulation results for a nonminimum-phase second-order plant are shown in Section VIII. At the end, we make conclusions.

II. Preliminary Example

To illustrate the main ideas of the LSCR method, following the logic of [12], we consider an example of a second-order scalar control plant, which is described by the following equation:

$$y_t - 2y_{t-1} + y_{t-2} = b_\star u_{t-1} + 1.6u_{t-2} + v_t, \tag{2}$$

where $t = 1,\dots,N$ is the time moments, $N = 15$, $y_0 = y_{-1} = u_{-1} = 0$, and $b_\star$ is unknown. There is no information about external noise v_t, but if it is random, it does not depend on u_t. The main difference between our example and the one considered in [12] consists in non-stable properties of plant (2).

The aim is to build, based on observations of inputs and outputs, a confidence interval $\widehat{\Theta}$ which contains $\theta_\star = b_\star$ with the probability of 80%. Note that such a problem with any probability of up to 100% always has a trivial solution — the whole real axis. Moreover, the problem is to construct such an algorithm of estimation $b_\star$ and generation of input data $u_0,\dots,u_{14}$ that the resulting confidence interval $\widehat{\Theta}$ characterizes $b_\star$ more accurately. For example, it can mean that the corresponding sets $\widehat{\Theta}$ "gather" closely to the point $b_\star$ as $N \to \infty$.

PROCEDURE

Generate a sequence of independent identically distributed random (i.i.d.) variables $\Delta_t, t = 0,\dots,14$ (we will call them randomized trial perturbations), which take the values ± 1 with the equal probability $\frac{1}{2}$.

Assuming that

$$u_t = \Delta_t,\ t = 0,\dots,14,$$

obtain the corresponding output values $y_t, t = 1,\dots,15$.

Using the observed data $\Delta_0,\dots,\Delta_{14}, u_{-1}, y_{-1},\dots,y_{15}$, for $t = 1,\dots,15$, and denoting observable variables $\psi_t = 2y_{t-1} - y_{t-2} + 1.6u_{t-2}$, we can consider the error

$$\varepsilon_t(b) = y_t - \psi_t - bu_{t-1}$$

as a linear function of b.

Denote

$$f_t(b) = \Delta_{t-1}\varepsilon_t(b) = (\Delta_{t-1}(y_t - \psi_t)) - (\Delta_{t-1}u_{t-1})\,b.$$

We want to compute the empirical estimates of the correlations $E[\Delta_{t-1}\varepsilon_t(b)]$ using $f_t(b)$. (Hereinafter $E[\cdot]$ is a symbol of mathematical expectation).

Note that from the definitions, we can derive

$$E[\Delta_{t-1}\varepsilon_t(b)] = (b_\star - b)E[\Delta_{t-1}^2] + E[\Delta_{t-1}v_t] = (b_\star - b).$$

Based on this fact, we *calculate* the series of empirical correlation estimates, using different subsets of the available data.

Exclude from the consideration the areas of possible values of b, in which the empirical estimates too often take only positive (or only negative) values.

Our aim is to build a confidence interval which contains $b_\star$ with a probability of at least 80%. Following [12], *choose* $q = 2$ and $M = 20$ such that $1 - 2q/M = 0.8$. Further, choosing $19 = M - 1$ random subsets of data, we *calculate* the empirical estimates

$$g_i(b) = \sum_{t=1}^{15} h_{i,t} \cdot f_t(b),\ i = 1,\dots,19,$$

where $h_{i,t}$ are i.i.d. Bernoulli variables, which take the values 0 or 1 with the equal probability $\frac{1}{2}$, i. e., $h_{i,t}$ determine whether $f_t(b)$ is used to calculate the i-th empirical correlation. (Of course, the estimates should be scaled, but we are not going to do this because below we will only use their sign).

Next, since it is very unlikely that all functions $g_i(b_\star)$ have the same sign exclude those b for which either all values $g_i(b)$ or all but one (less than $q = 2$) have the same sign. It directly follows from Theorem 1 [12] that the remainder interval is an 80% confidence area for $b_\star$.

A similar example with the plant

$$y_t = b_\star u_t + v_t$$

was considered in Introduction of [12]. The simulation with $b_\star = 1$ and i.i.d. sequence of normally distributed noise v_t with mean 0.5 and dispersion 0.1 (bias noise) resulted in the confidence interval $[0.874; 1.119]$. We note again that the

procedure for obtaining the confidence interval never used any information about noise (independence, mean value, or dispersion). The above characteristics of the noise were only used in the experiment to simulate noise itself.

Inclusion of additional dynamics terms into the left- and right-hand parts of (2) does not affect either the procedure behavior or the assessment of the confidence set. But as a result, the output variable increases significantly (see Fig. 2, line 1), which is bad from the practical point of view. This is due to the instability of the transfer function. In adaptive control problems with unknown but bounded noise, the output variable can be stabilized by using feedback with adjustable coefficients (see, for example, [3]). The same effect can be achieved in combination with some other methods of identification.

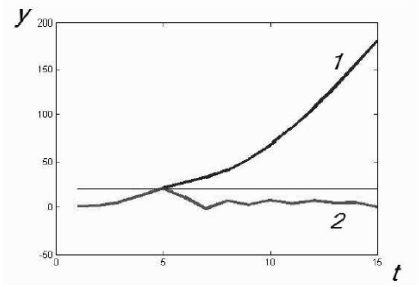

Fig. 2. Output variables y_t, $t = 1,2,\ldots,15$.

For example, if we choose a sufficiently high C_v (e. g., $C_v = 10$) and assume that $|v_t| \le C_v$, we can use the membership set approach. In [3], the authors proposed to use a stabilizing "*Stripe*" algorithm, which forms a sequence of estimates $\hat{b}_t$. The estimate $\hat{b}_t$ is used to build the stabilizing part $\bar{u}_t$ of control input, which is added to the randomized control Δ_t

$$u_t = \Delta_t + \bar{u}_t.$$

The stabilizing part $\bar{u}_t$ is obtained by the feedback law

$$\bar{u}_t = \begin{cases} \beta_t^{(1)} y_t + \beta_t^{(2)} y_{t-1} - \alpha_t u_{t-1}, & \hat{b}_t \neq 0, \\ 0, & \hat{b}_t = 0, \end{cases}$$

where coefficients $\alpha_t, \beta_t^{(1)}$ and $\beta_t^{(2)}$ are determined by the polynomial equation

$$(1-2\lambda+\lambda^2)(1+\alpha_t\lambda) - \lambda(\hat{b}_t + 1.6\lambda)(\beta_t^{(1)} + \beta_t^{(2)}\lambda) = 1.$$

The control plant with unknown parameter $b_\star$ may be a nonminimum-phase plant (input nonstable) since $|b_\star| < 1.6$; but the above feedback is input stable for the plant with parameters $\hat{b}_t$ (see, e. g., [4]). In simulations, we use a stabilizing "*Stripe*" algorithm in the form described in [4], [10]. This algorithm turns on when the output variable exceeds the bounded level of 20. As a result, it stabilizes the behavior of the output (see Fig. 2, line 2) and input variables.

In this case, the property $E[\Delta_{t-1}\varepsilon_t(b)] = (b_\star - b)$ is still satisfied, since $E[\Delta_{t-1}\bar{u}_{t-1}] = 0$, and the LSCR procedure is operable. The functions $g_i(b)$, $i = 1,2,\ldots,19$, and the 80% confidence interval $\widehat{\Theta} = [0.850; 1.045]$ obtained in the simulation are shown in Fig. 3.

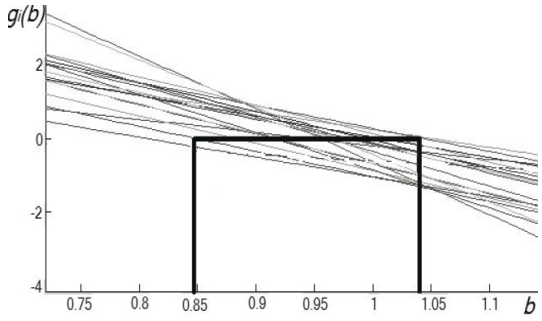

Fig. 3. Functions $g_i(b)$, $i = 1,2,\ldots,19$, and 80% confidence interval $\widehat{\Theta} = [0.850; 1.045]$ for $b_\star = 1.0$.

Although we use the level of C_v in the synthesis of the stabilizing feedback, the design of the confidence region obtained by the LSCR algorithm does not depend on C_v. Even if we choose an understated level C_v, the suggested algorithm still works, whereas the methods of identification of the set membership approach will lead to wrong results.

The case of unknown coefficient $a_\star$ instead of $b_\star$ is more complicated. We cannot use the procedure proposed in [12] directly for estimating the two-dimensional parameter $(a_\star, b_\star)^{\mathrm{T}}$ because it also leads to a dramatic increase of the output variable. However, the possibility to use the feedback with adjustable coefficients is limited by the condition of independence of input randomized trial perturbations and noise, which is not satisfied. An example of an upgraded procedure for this case is discussed in [13].

III. Problem Statement

The procedure discussed further intended to identify the unknown parameters of a dynamic scalar linear control plant, which is described by an autoregressive moving average model, is based on reparameterization of the mathematical model of a plant. Instead of the natural parameters of the plant—dynamic coefficients—it is convenient to use some other parameters, which are in one-to-one correspondence with them. Such reparameterization is a result of rewriting the plant's equation in moving average model form, which makes it possible to use the LSCR procedure for building the confidence region, even in the cases if an adaptive algorithm is used in the feedback channel.

We assume that a control plant has scalar input and output and it is described by Equation (1) in discrete time with $G_\star(z^{-1}) = B_\star(z^{-1})/A_\star(z^{-1})$, where

$$A_\star(\lambda) = 1 + a_\star^{(1)}\lambda + \cdots + a_\star^{(n_a)}\lambda^{n_a},$$

$$B_\star(\lambda) = b_\star^{(l)}\lambda^l + b_\star^{(l+1)}\lambda^{l+1} + \cdots + b_\star^{(n_b)}\lambda^{n_b},$$

the natural numbers n_a, n_b are the output and input (control) model orders; l is a delay in control, $1 \le l \le n_b$; $a_\star^{(1)}, \ldots, a_\star^{(n_a)}, b_\star^{(l)}, \ldots, b_\star^{(n_b)}$ are the plant parameters, a part of which is unknown.

It is required to define, with a given probability, an area of reliability for unknown coefficients of plant (1) by the observations of outputs $\{y_t\}$ on a finite interval of time $t = 1,2,\ldots,N$, and known inputs (controls) $\{u_t\}$, which can be chosen.

IV. Control Actions with Randomized Test Signals

Let $s \le n_a + n_b - l + 1$ be a positive integer number. (It is usually equal to the quantity of unknown parameters of plant (1)). And let $N = s \cdot N_\Delta$ be with some integer N_Δ.

Let us choose a sequence of independent random variables symmetrically distributed around zero (a randomized test perturbation) $\Delta_0, \Delta_1, \dots, \Delta_{N_\Delta - 1}$ and add them to the input channel once per every s time moments (at the beginning of each time interval) in order to "enrich" the variety of observations.

To be more precise, we will build controls $\{u_t\}_{t=0}^{N-l}$ by the rule

$$u_{sn+i-l} = \begin{cases} \Delta_n + \bar{u}_{sn-l}, & i = 0, \\ \bar{u}_{sn+i-l}, & i = 1,2,\dots,s-1, \end{cases} \quad n = 0,\dots,N_\Delta - 1,$$

where "intrinsic" controls $\{\bar{u}_t\}$ are determined by the adjustable feedback law

$$\bar{u}_t = \mathscr{U}_t(y_t, y_{t-1}, \dots, \bar{u}_{t-1}, \dots),\ t \ge 0,\ \bar{u}_{-k} = 0,\ k > 0.$$

The type and characteristics of feedback depend on specific practical problems. In particular, it is possible to use a trivial law of "intrinsic" feedback: $\bar{u}_t = 0$, $t = 0,1,\dots,N-l$.

V. Main Assumptions and Reparameterization of the Transfer Function

Main assumption

A1. The user can choose Δ_n and this choice does not affect the external noise $v_{sn}, \dots, v_{s(n+1)-1}$. (In the mathematical sense, Δ_n does not depend on $\{v_t\}_{t=1}^{s(n+1)-1}$.)

Note that no assumptions are made about the noise v_t and the upper limits of the noise amplitudes. If the noise is random, there are no assumptions about the zero-mean or any autocorrelation properties.

For time sn, $n = 0,\dots,N_\Delta - 1$, we can denote $\bar{v}_{sn} = v_{sn} + (1 - A_\star(z^{-1}))y_{sn} + (B_\star(z^{-1}) - b_\star^l z^{-l})u_{sn}$ and rewrite Equation (1) in the following form:

$$y_{sn} = \Delta_n \theta_\star^{(1)} + \theta_\star^{(1)} \bar{u}_{sn-l} + \bar{v}_{sn},$$

where $\theta_\star^{(1)} = b_\star^{(l)}$. This equation shows a direct relation between observation y_{sn} and test signal Δ_n which does not depend on the "new" noise $\bar{v}_{sn}$.

Similarly, we rewrite Equation (1) for the rest of time $sn + k - 1$, $k = 2,\dots,s$, sequentially excluding the variables $y_{sn+k-1}, \dots, y_{sn}$ from the left-hand side of the equation, using the same equation (1) for early time instants

$$y_{sn+k-1} = \Delta_n \theta_\star^{(k)} + \sum_{i=0}^{k-1} \theta_\star^{(k-i)} \bar{u}_{sn-l+i} + \bar{v}_{sn+k-1}, \tag{3}$$

where $\theta_\star^{(k-i)}$, $i = 0,\dots,k-1$ are the corresponding coefficients of the remaining right-hand side terms with $\bar{u}_{sn-l+i}$.

In [3] and [4], the authors suggest forming new parameters as s-vector $\theta_\star$ of coefficients $\theta_\star^{(k)}$ obtained in (3). They also give conditions for the invertibility of such a reparameterization procedure.

The next formula follows immediately from the above definition $\theta_\star = \mathbb{A}^{-1}\mathbb{B}$, where $s \times s$ matrix $\mathbb{A}$ and s-vector $\mathbb{B}$ are

$$\mathbb{A} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ a_\star^{(1)} & 1 & \dots & 0 & 0 \\ a_\star^{(2)} & a_\star^{(1)} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & a_\star^{(n_a)} & \dots & a_\star^{(1)} \quad 1 \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} b_\star^{(l)} \\ \vdots \\ b_\star^{(n_b)} \\ \vdots \\ 0 \end{pmatrix}.$$

Consider the conditions of the existence of a corresponding inverse function.

Assumptions

A2. Let s be a positive integer and $\Theta \subset \mathbb{R}^s$ such that a set of the plant's unknown parameters are uniquely determined by some function $\tau(\theta) : \Theta \to \mathbb{R}^{n_a + n_b}$, from the above-defined vector $\theta_\star$.

By Lemma 2.2 on p. 117 from [4], Assumption **A2** holds for $s = n_a + n_b - l + 1$ when the plant's orders n_a, n_b are known and the following assumption is satisfied.

A3. The polynomials $z^{n_a} A_\star(z^{-1})$ and $z^{n_b} B_\star(z^{-1})$ are mutually prime.

In [4] there is an algorithm for the inverse function $\tau(\theta)$.

In practice, usually only part of plant parameters are unknown. Sometimes, unknown parameters correspond to the low degrees of z^{-1}, which are smaller than some $\bar{n}_a$ and $\bar{n}_b$, respectively. In this case, we can choose $s = \bar{n}_a + \bar{n}_b - l + 1$, which is significantly less than $n_a + n_b - l + 1$. Moreover, the "new" noise $\bar{v}_{sn+k-1}$ in (3) can be divided into two parts: nonmeasurable $\tilde{v}_{sn+k-1}$ and measurable ψ_{sn+k-1}. The latter is determined by observable inputs and outputs with known coefficients (see the example below).

Example. Consider the second-order plant

$$y_t + a_\star^{(1)} y_{t-1} + y_{t-2} = b_\star^{(1)} u_{t-1} + 1.6 u_{t-2} + v_t, \tag{4}$$

$t = 1,2,\dots,N$, with unknown coefficients $a_\star^{(1)}$ and $b_\star^{(1)} \ne 0$.

Denote $\tau_\star = (a_\star^{(1)}, b_\star^{(1)})^{\mathrm{T}}$. Let $s = 2$ and vector $\theta_\star$ of the "new" parameters be

$$\theta_\star = \begin{pmatrix} b_\star^{(1)} \\ 1.6 - a_\star^{(1)} b_\star^{(1)} \end{pmatrix} \in \mathbb{R}^2.$$

In this case, if $\theta^{(1)} \ne 0$ then Assumption **A2** and the inverse function $\tau(\theta)$ is

$$\tau(\theta) = \begin{pmatrix} \frac{1.6 - \theta^{(2)}}{\theta^{(1)}} \\ \theta^{(1)} \end{pmatrix}.$$

Equations (3) have the following forms:

$$y_{2n} = \Delta_n \theta_\star^{(1)} + \theta_\star^{(1)} \bar{u}_{2n-1} + \psi_{2n} + \tilde{v}_{2n},$$

$$y_{2n+1} = \Delta_n \theta_\star^{(2)} + \theta_\star^{(2)} \bar{u}_{2n-1} + \theta_\star^{(1)} \bar{u}_{2n} + \psi_{2n+1} + \tilde{v}_{2n+1},$$

where $\psi_{2n+k} = 1.6\bar{u}_{2n-2+k} - y_{2n-2+k}$, $k = 0,1$, $\tilde{v}_{2n} = v_{2n} - a_\star^{(1)} y_{2n-1}$, $\tilde{v}_{2n+1} = v_{2n+1} + a_\star^{(1)}(a_\star^{(1)} y_{2n-1} + y_{2n-2} - 1.6\bar{u}_{2n-2} - v_{2n})$.

VI. Procedure for Constructing Confidence Regions

1) Using observational data, we can write predictors as a function of θ

$$\hat{y}_{sn+k-1}(\theta) = \Delta_n \theta^{(k)} + \sum_{i=0}^{k-1} \theta^{(k-i)} \bar{u}_{sn+k-l-i}, \qquad (5)$$

$$n = 0, \ldots, N_\Delta - 1,\ k = 1, \ldots, s.$$

2) We can calculate the prediction error

$$\varepsilon_t(\theta) = y_t - \hat{y}_t(\theta),\ t = 1, \ldots, N.$$

3) According to the observed data, we form a set of functions of θ

$$f_{sn+k-1}(\theta) = \Delta_n \varepsilon_{sn+k-1}(\theta),\ n = 0, \ldots, N_\Delta - 1,\ k = 1, \ldots, s.$$

4) Choose a positive integer $M > 2s$ and construct M different binary stochastic strings (of zeros and ones) $(h_{i,1}, \ldots, h_{i,N})$, $i = 0, 1, \ldots, M-1$, as follows: $h_{0,j} = 0$, $j = 1, \ldots, N$, all the other elements $h_{i,j}$ take the values of zero or one with the equal probability $\frac{1}{2}$. We calculate for $k = 1, \ldots, s$

$$g_i^{(k)}(\theta) = \sum_{n=0}^{N_\Delta - 1} h_{i,ns+k} \cdot f_{ns+k-1}(\theta),\ i = 0, \ldots, M-1.$$

5) Choose q from the interval $[1; M/2s]$. For $k = 1, \ldots, s$, construct a region $\widehat{\Theta}^{(k)}$ such that at least q of the $g_i^{(k)}(\theta)$ functions are strictly higher than 0 and at least q are strictly lower than 0.

We define the confidence set by the formula

$$\widehat{\Theta} = \bigcap_{k=1}^{s} \widehat{\Theta}^{(k)}. \qquad (6)$$

Remarks. 1. The procedure described above is similar to the one suggested in [12], but it has two significant differences from it. First, we consider a confidence set $\widehat{\Theta}$ in state space $\mathbb{R}^s$ instead of $\mathbb{R}^{n_a+n_b}$. The confidence regions $\widehat{\Theta}^{(k)}$, $k = 1, \ldots, s$, are the subsets of $\mathbb{R}^k$ instead of $\widehat{\Theta}^{(k)} \subset \mathbb{R}^{n_a+n_b}$. Second, randomized trial perturbations are included through the input channel only once per every s time instants instead of permanent perturbations in [12].

2. If we can divide the "new" noise $\bar{v}_{sn+k-1}$ in (3) into two parts—$\tilde{v}_{sn+k-1}$ and ψ_{sn+k-1}— where the first part is nonmeasurable, whereas the second is determined by observable inputs and outputs with known coefficients, then, in the above-described procedure, we can use stronger predictors instead of (5)

$$\hat{y}_{sn+k-1}(\theta) = \Delta_n \theta^{(k)} + \sum_{i=0}^{k-1} \theta^{(k-i)} \bar{u}_{sn+k-l-i} + \psi_{sn+k-1}.$$

VII. Main Result

The probability that $\theta_\star$ belongs to each of $\widehat{\Theta}^{(k)}$, $k = 1, 2, \ldots, s$, is given in the following theorem.

Theorem 1: Let Assumption **A1** be satisfied. Consider $k \in \{1, 2, \ldots, s\}$ and assume that $Prob(g_i^{(k)}(\theta_\star) = 0) = 0$. **Then**,

$$Prob\{\theta_\star \in \widehat{\Theta}^{(k)}\} = 1 - 2q/M, \qquad (7)$$

where $\widehat{\Theta}^{(k)}, q$ and M are from steps 5 and 4 of the above-described procedure.

Proof: See Appendix.

Remark. The number of data N does not enter directly into the bound (7) given in Theorem 1. The dependence on N is implicitly included into the formula (7) by M which can not be greater than the number of subsets of cardinality more than $2s$ in the set of N elements.

The next corollary follows directly from Theorem 1.

Corollary 2: Under the conditions of Theorem 1

$$Prob\{\theta_\star \in \widehat{\Theta}\} \geq 1 - 2sq/M, \qquad (8)$$

where $\widehat{\Theta}$ is taken from (6).

Note that, as it was pointed out in [12], the value of the probability in (7) is accurate, but not the lower limit. Inequality in (8) is obtained owing to the fact that the events $\{\theta_\star \notin \widehat{\Theta}^{(k)}\}$, $k = 1, \ldots, s$ may overlap.

From the above, it is easy to derive the main result of this paper.

Theorem 3: Let Assumptions **A1**,**A2** be satisfied and assume that $Prob(g_i^{(k)}(\theta_\star) = 0) = 0$. **Then**, the set $\tau(\widehat{\Theta})$ is the confidence set for unknown parameters of plant (1) with a confidence level of no less than $1 - 2sq/M$.

Note that earlier, under similar assumptions, in [3], [4], the authors showed a possibility of forming an estimation algorithm, based on the control strategy described in Section IV, which provides asymptotical convergence of estimates to true values of unknown parameters of plant (1).

VIII. Example

We return to control plant (4) from Section V with $N = 960$ and unknown parameters $a_\star$ and $b_\star$.

Define functions $f_t(\theta)$:

$$f_{2n}(\theta) = \Delta_n(y_{2n} - \Delta_n \theta^{(1)} - \theta^{(1)} \bar{u}_{2n-1} - \psi_{2n}),\ n = 0, \ldots, N_\Delta - 1,$$

$$f_{2n+1}(\theta) = \Delta_n(y_{2sn+1} - \Delta_n \theta^{(2)} - \theta^{(2)} \bar{u}_{2n-1} - \theta^{(1)} \bar{u}_{2n} - \psi_{2n+1}).$$

Let us choose $M = 480$ and $q = 6$ and calculate the empirical correlations

$$g_i^{(k)}(\theta) = \sum_{n=0}^{499} h_{i,2n+k} \cdot f_{2n+k-1}(\theta),\ i = 1, \ldots, 479,\ k = 1, 2.$$

For $k = 1, 2$, we construct the regions $\widehat{\Theta}^{(k)}$ which only include the values of θ for which no less than 6 of the functions $g_i^{(k)}(\theta)$, $i = 1, \ldots, 479$, are strictly higher than zero and no less than 6 of them are strictly lower than zero.

By virtue of Theorem 3, the vector of true parameters with a probability of more than $95\% = 1 - 2 \cdot 2 \cdot 6/480$ belongs to the confidence set $\tau(\widehat{\Theta}) = \tau(\widehat{\Theta}^{(1)} \bigcap \widehat{\Theta}^{(2)})$.

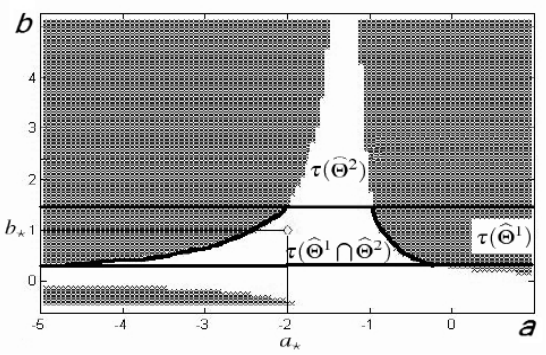

Fig. 4. Confidence set $\tau(\widehat{\Theta})$.

Fig. 4 shows the regions $\tau(\widehat{\Theta})$, $\tau(\widehat{\Theta}^{(1)})$, and $\tau(\widehat{\Theta}^{(2)})$ obtained from the simulation with true values $a_\star = -2$ and $b_\star = 1$, and characteristics of noise and stabilizing feedback like those in Section II.

IX. Conclusions and Acknowledgments

This paper presents a procedure for constructing a confidence set of unknown parameters of a linear scalar control plant, based on a finite set of data measurements. The procedure is defined as an intersection of confidence regions which contain true parameters with specified guaranteed probability.

From the theoretical point of view, an important feature of the suggested procedure is that it operates without any significant assumptions about the external noise. It is also of vital importance from the practical point of view since in practical applications, it is difficult to obtain a priori knowledge about the noise characteristics. The resulting confidence set is not conservative because it gives a rather good description of the uncertainties in the model. The results of this paper are relevant to the problems of constructing robust control systems.

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Appendix

The following preliminary Proposition 1, which is similar to Proposition 1 from [12], p. 2716, is instrumental to the proof of Theorem 1.

Proposition 1: Fix $k \in [1,\dots,s]$. Let H be a stochastic $M \times N_\Delta$ matrix with elements $h_{i,ns+k}$, $i = 0,1,\dots,M-1$, $n = 0,\dots,N_\Delta - 1$, from step 4 of the algorithms in Section VI, and let $\eta = col(\eta_1,\dots,\eta_{N_\Delta})$ be a vector independent of H, consisting of mutually uncorrelated random variables symmetrically distributed around zero. Given an $i \in [0,M-1]$, let H_i be the $M \times N$ matrix, whose rows are equal to the i-th row of H. **Then**, $H\eta$ and $(H - H_i)\eta$ have the same M-dimensional distribution provided that the i-th element of $(H - H_i)\eta$ (which is 0) is repositioned as the first element of the vector.

Proof of Theorem 1. The general scheme of the proof is the same as the similar proof of Theorem 1 in [12]. The only difference is in the stage when Proposition 1 is applied to prove that each variable $g_i^k(\theta_\star)$ has the same probability $1/M$ to be in the generic rth position (i. e., there are exactly $r-1$ other variables that are lower than the variable under consideration).

In our case, denote $\eta_n := \Delta_{n-1}\varepsilon_{(n-1)s+k-1}(\theta_\star)$. For the correlation between η_i and η_j, $i > j$, we derive successively

$$E[\eta_i\eta_j] = E[\Delta_{i-1}\varepsilon_{(i-1)s+k-1}(\theta_\star)\Delta_{j-1}\varepsilon_{(j-1)s+k-1}(\theta_\star)] =$$
$$= E[\Delta_{i-1}]E[\varepsilon_{(i-1)s+k-1}(\theta_\star)\Delta_{j-1}\varepsilon_{(j-1)s+k-1}(\theta_\star)] = 0$$

by the virtue of Assumption **A1** and the symmetry property of test perturbation distribution: $E[\Delta_{i-1}] = 0$. Hence, the variables $\eta_1,\dots,\eta_{N_\Delta}$ are mutually uncorrelated.

Take a variable $g_{\bar{i}}^{(k)}(\theta_\star)$ which is in the r-th position. The inequality

$$g_i^{(k)}(\theta_\star) - g_{\bar{i}}^{(k)}(\theta_\star) = \sum_{n=0}^{N_\Delta}(h_{i,ns+k} - h_{\bar{i},ns+k})\eta_n < 0$$

holds for $r-1$ selection of $i \in [0,M-1]$. It is the same as asking whether the $r-1$ entries of $(H - H_{\bar{i}})\eta$ are negative. From Proposition 1, we have that $(H - H_{\bar{i}})\eta$ has the same distribution as $H\eta$ and, therefore, $Prob\{$"$r-1$ entries of $(H - H_{\bar{i}})\eta$ are negative"$\} = Prob\{$"$r-1$ entries of $H\eta$ are negative"$\}$, and it does not depend on $\bar{i}$.

The rest of the proof is similar to that given in [12]. ■

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