PERIODIC WAVELETS[1]

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Abstract. We give definitions of Multiresolution analysis and wavelet decomposition for a wide range of quasi-Banach spaces of periodic distributions. Elementary properties of such MRA are investigated. Economical algorithms of wavelet decomposition and reconstruction are represented.

§0. Introduction

We consider a general approach to definition of Multiresolution analysis (MRA) $\{V^j\}_{j=0}^{\infty}$ of a broad range of quasi-Banach spaces of periodic distributions (generalized functions). In spite of the fact that most of the known MRA of $\mathbb{L}^2$ satisfy our definition, it does not coincide with known ones (see, for example, [10], [15], [17], [18]) even for the space $\mathbb{L}^2$. Our constraints for MRA are more severe. The result of these constraints is possibility to recover MRA by any of the spaces V^j. Periodizations of practically all classical MRA of the space $\mathbb{L}^2$ (see [4], [9]) possess such property. We know only two examples of MRA, introduced by Y.Meyer [9, p.22] and C.K.Chui and H.N. Mhaskar [2], which do not possess this property and, consequently, are not MRA from our point of view.

In §1 we introduce a special class $\mathfrak{H}$ of quasi-Banach spaces. For these spaces we define the notion of MRA. We note that most of classical spaces of distributions (such as Lebesgue and Hardy spaces, the Besov—Lizorkin—Triebel spaces, the Orlicz spaces, the space of Borel measures, and many others) belongs to $\mathfrak{H}$.

Because each of the spaces $\{V^j\}$ are invariant with respect to the translate by $2\pi/2^j$, then it is clear that bases of eigen functions of an operator of the translate are of fundamental importance in the study of these spaces. Using a representation in these bases we obtain a description of properties of the spaces V^j. In particular, we find a full description of all possible MRA of $X \in \mathfrak{H}$.

In §2 we define wavelet spaces for fixed MRA of $X \in \mathfrak{H}$. Wavelet decomposition of the space X is generated by dual MRA of the spaces of bounded convolution operators acting from the space X to $\mathbb{L}^\infty$.

Two numerical algorithms of wavelet decomposition and reconstruction are considered in §3. The first of them is based on the transfer to the frequency domain. For a function $f \in V^j$ this algorithm is of complexity $O(j2^j)$. The second one is applicable only for a case, analogous to the case of compactly supported (generally speaking, nonorthogonal) wavelets for MRA of function spaces on a real line. The complexity of such algorithm is equal to $O(\sigma 2^j)$, where σ is a size of the support.

In §4 we show that in every MRA there exist many bases of translates with finite masks of refinement equations. For such bases we construct algorithms of wavelet expansion and recovery those are realized as convolutions with finite windows of a size at most a size of the mask.

§1. Multiresolution analysis of spaces of periodic functions and distributions.

We denote by $\mathbb{D}$ the space of 2π-periodic distributions. We recall some principal properties of distributions. Full constructing a theory of periodic distributions can be found, for example, in [7]. Distributions can be considered either as bounded linear functionals on the space $\mathbb{C}^\infty$ of infinitely differentiable functions or as formal trigonometric series

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx},$$

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where the sequence $\hat{f}(n) \overset{\text{def}}{=} c_n$ of Fourier coefficients has at most power rate of growth. Convolution of distribution f and g is defined to be $f*g\,(x) \overset{\text{def}}{=} \sum_{n=-\infty}^{+\infty} \hat{f}(n)\hat{g}(n)\exp(inx)$. We denote by f_r the convolution of f with the Poisson kernel $\mathcal{P}_r$,

$$f_r(x) = f * \mathcal{P}_r\,(x) = \sum_{n=-\infty}^{+\infty} c_n r^{|n|} e^{inx}.$$

Let $X \subset \mathbb{D}$ be a quasi-Banach space. We denote by $M(X, \mathbb{L}^\infty)$ the space of bounded convolution operators (to be more exact, the space of kernels of such operators) acting from X to $\mathbb{L}^\infty$.

Definition 1. We denote by $\mathfrak{H}$ a collection of quasi-Banach spaces X with the quasi-norm $\|\bullet \mid X\|$ satisfying the following conditions:

a) $\mathbb{C}^\infty \subset X \subset \mathbb{D}$;

b) $\|f \mid X\| = \sup\limits_{r<1} \|f_r \mid X\|$ for any $f \in X$;

c) X is invariant with respect to the changes of variable $x \mapsto \pm x + \tau$ and $x \mapsto 2x$, as well as with respect to $x \mapsto x/2$ as soon as a function is π-periodic;

d) X is invariant with respect to pointwise multiplication by the functions $\exp(inx)$, $n \in \mathbb{Z}$;

e) $f \in X$ if and only if $\bar{f} \in X$.

In what follows we study only problems of convergence but not its rate. So we can assume that a norm of all spaces $X \in \mathfrak{H}$ is invariant with respect to translates of argument and with respect to complex conjugation. In the opposite case, in view of conditions c) and e) we can do the appropriate renormalization.

In [11] and [12] we have introduced the class $\mathcal{H}$ of spaces of distributions which is very close to the class $\mathfrak{H}$. At least the both classes have many common properties. We formulate some properties of the spaces of $\mathfrak{H}$.

Let $X \in \mathfrak{H}$, then we denote by $\overset{\circ}{X}$ (or X°) the completion of the space $\mathbb{C}^\infty$ with respect to the quasi-norm of the space X;

$$\overset{\square}{X} \overset{\text{def}}{=} \left\{ f \in \mathbb{D} \mid \sup_{r<1} \|f_r \mid X\| < \infty \right\};$$

$$[X] \overset{\text{def}}{=} \left\{ Y \in \mathfrak{H} \mid \overset{\circ}{X} \subset Y \subset \overset{\square}{X} \right\}.$$

It is not hard to show that the operation $[\bullet]$ partitions $\mathfrak{H}$ into classes of equivalence. As it usually is, we denote by X^* the space of continuous linear functionals on X.

In what follows we shall denote by $\langle f, g\rangle$ a value of linear functional $g \in X^*$ at the distribution $f \in X$. It is well known that in the case $X = \mathbb{C}^\infty$ we can suppose

$$\langle f, g\rangle \overset{\text{def}}{=} f(\bullet) * \bar{g}(-\bullet)(0) = \sum_{n=-\infty}^{+\infty} \hat{f}(n)\bar{\hat{g}}(n), \tag{1.1}$$

where the series are absolutely convergent. Now we prove that the same formula is valid for an arbitrary space $X = \overset{\circ}{X}$. For this case the corresponding series in (1.1) are not absolutely convergent, however we can sum it by Abel — Poisson method.

First, we show that $\|f - f_r \mid X\| \to 0$ as $r \to 1$. Let $\varepsilon > 0$ be an arbitrary number. Then there exists $h \in \mathbb{C}^\infty$ such that $\|f - h \mid X\| < \varepsilon/3$. Because $h_r \to h$ in topology of the space $\mathbb{C}^\infty$ as $r \to 1$, then for r close enough to 1 we have $\|h_r - h \mid X\| < \varepsilon/3$. Hence,

$$\begin{aligned}\|f - f_r \mid X\| &\le C(\|f - h \mid X\| + \|h - h_r \mid X\| + \|h_r - f_r \mid X\|) \\ &\le C\left(\varepsilon/3 + \varepsilon/3 + \sup_{r<1} \|(h-f) * \mathcal{P}_r \mid X\|\right) = C\varepsilon,\end{aligned}$$

where, generally speaking, $C \ne 1$ because X is a quasi-Banach space but not a Banach one.

Thus, $\langle f, g\rangle = \lim_{r\to 1}\langle f_r, g\rangle$ as we wished.

Obviously, in the similar way for $f \in \overset{\circ}{X}$ we can prove that $\|f(\bullet) - f(\bullet + \tau) \mid X\| \to 0$ as $\tau \to 0$. In particular, it follows from this that when $g \in X^{\circ *}$ the function $f * g$ is continuous.

Property 1. *Let $X \in \mathfrak{H}$, $Y \in [X]$. Then $M(Y, \mathbb{L}^\infty) = (\overset{\circ}{X})^* \in \mathfrak{H}$.*

Remark. It is easy to prove, if $X \in \mathfrak{H}$, then a set of compact convolution operators, acting from X to $\mathbb{L}^\infty$, coincides with $(\overset{\circ}{X})^{*\circ}$.

Proof of Property 1. Let $f \in X$, $T_\tau[f](\bullet) = \overline{f(\tau - \bullet)}$, $g \in (\overset{\circ}{X})^*$, then

$$\begin{aligned}\|f * g\|_\infty &= \sup_{r<1} \|f * g * \mathcal{P}_r\|_\infty = \sup_{r<1}\sup_x \Big|\sum_{n\in\mathbb{Z}} \hat{f}_r(n)\hat{g}(n)e^{inx}\Big| = \sup_{r<1}\sup_x \Big|\sum_{n\in\mathbb{Z}} \bar{\hat{f}}_r(n)\bar{\hat{g}}(n)e^{-inx}\Big| \\ &= \sup_{r<1}\sup_x |\langle T_x[f_r], g\rangle| \le \sup_{r<1} \| f_r \mid X\| \cdot \| g \mid (\overset{\circ}{X})^* \| \le \| f \mid X\| \cdot \| g \mid (\overset{\circ}{X})^*\|.\end{aligned}$$

Conversely, let $g \in M(X, \mathbb{L}^\infty)$, $h \in \mathbb{C}^\infty$. Then we have

$$\begin{aligned}|\langle h, g\rangle| &= \left|\sum_{n\in\mathbb{Z}} \hat{h}(n)\bar{\hat{g}}(n)\right| = \left|\sum_{n\in\mathbb{Z}} \bar{\hat{h}}(n)\hat{g}(n)\right| = |\bar{h}(-\bullet) * g(\bullet)(0)| \\ &\le \|\bar{h}(-\bullet) * g(\bullet)\|_\infty \le \| h \mid X\| \cdot \| g \mid M(X, \mathbb{L}^\infty\|.\end{aligned}$$

Hence, in view of density $\mathbb{C}^\infty$ in $\overset{\circ}{X}$ we obtain

$$\| g \mid \overset{\circ}{X} \| \le \| g \mid M(X, \mathbb{L}^\infty)\|.$$

Now we prove that $M(X, \mathbb{L}^\infty) \in \mathfrak{H}$. The imbeddings $\mathbb{C}^\infty \subset M(X, \mathbb{L}^\infty) \subset \mathbb{D}$ are obvious. Let us prove property b) of spaces of $\mathfrak{H}$.

On the one hand,

$$\begin{aligned}\| g \mid M(X, \mathbb{L}^\infty)\| &= \sup\left\{\|f * g\|_\infty \;\Big|\; f \in \overset{\circ}{X}, \| f \mid X\| \le 1\right\} \\ &\ge \sup_{r<1}\sup\left\{\|f_r * g\|_\infty \;\Big|\; f \in \overset{\circ}{X}, \| f \mid X\| \le 1\right\} \\ &= \sup_{r<1}\sup\left\{\|f * g_r\|_\infty \;\Big|\; f \in \overset{\circ}{X}, \| f \mid X\| \le 1\right\} = \sup_{r<1} \| g_r \mid M(X, \mathbb{L}^\infty\|. \qquad (1.2)\end{aligned}$$

On the other hand, for any $\varepsilon > 0$ there exists $f \in \overset{\circ}{X}$ such that $\| f \mid X\| = 1$ and

$$\|f * g\|_\infty \ge \| g \mid M(X, \mathbb{L}^\infty\| - \varepsilon.$$

For r close enough to 1 we have

$$\|f * g\|_\infty \le \|f_r * g\|_\infty + \|(f - f_r) * g\|_\infty \le \|f * g_r\|_\infty + \varepsilon \le \| g_r \mid M(X, \mathbb{L}^\infty)\| + \varepsilon.$$

Joining the both inequalities, we obtain

$$\| g \mid M(X, \mathbb{L}^\infty)\| \le \sup_{r<1} \| g_r \mid M(X, \mathbb{L}^\infty)\| + 2\varepsilon.$$

It follows from this and from (1.2) that condition b) holds.

We show validity of condition c). Let $g \in M(X, \mathbb{L}^\infty)$, then

$$\begin{aligned}\| g(\tau - \bullet) \mid M(X, \mathbb{L}^\infty)\| &= \sup\left\{\|f(\tau - \bullet) * g(\bullet)\|_\infty \;\Big|\; f \in \overset{\circ}{X}, \|f \mid X\| = 1\right\} \\ &\le \| g \mid M(X, \mathbb{L}^\infty)\| \cdot \| f(\tau - \bullet) \mid X\| = \| g \mid X\|.\end{aligned}$$

Invariance of the space $M(X,\mathbb{L}^\infty)$ with respect to the change of a variable $x\mapsto x+\tau$ is proved in the similar way.

Invariance with respect to the change of a variable $x\mapsto 2x$ the following inequalities prove:

$$\begin{aligned}
&\|\, g(2\bullet) \mid M(X,\mathbb{L}^\infty)\,\| = \sup\left\{\|f(\bullet)*g(2\bullet)\|_\infty \;\Big|\; \|\, f\mid X\,\|\le 1\right\}\\
&\quad= \sup\left\{\Big|\sum_{n\in\mathbb{Z}}\hat f(2n)\hat g(n)\Big| \;\Bigg|\; \|\, f\mid X\,\|\le 1\right\} = \sup\left\{\|\big(f(\bullet/2)+f(\bullet/2+\pi)\big)*g(\bullet)\|_\infty/2 \;\Big|\; \|\, f\mid X\,\|\le 1\right\}\\
&\quad\le \sup\Big\{\|\,\big(f(\bullet/2)+f(\bullet/2+\pi)\big)\mid X\,\|\cdot\|\, g(\bullet)\mid M(X,\mathbb{L}^\infty)\,\| \;\Big|\; \|\, f\mid X\,\|\le 1\Big\}\le C_1C_2\|\, g\mid M(X,\mathbb{L}^\infty)\,\|,
\end{aligned}$$

where C_1 is a norm of an operator of the change of a variable $x\mapsto x/2$ in the space X, C_2 is a constant in the triangle inequality for the quasi-Banach space X (in particular, if X is a Banach space, then $C_2=1$).

Let us estimate a value $\|\, g(\bullet/2)\mid M(X,\mathbb{L}^\infty)\,\|$, if $g\in M(X,\mathbb{L}^\infty)$ is π-periodic:

$$\begin{aligned}
&\|\, g(\bullet/2)\mid M(X,\mathbb{L}^\infty)\,\| = \sup\left\{\|f(\bullet)*g(\bullet/2)\|_\infty \;\Big|\; \|\, f\mid X\,\|\le 1\right\}\\
&\qquad\qquad = \sup\left\{\|f(2\bullet)*g(\bullet)\|_\infty \;\Big|\; \|\, f\mid X\,\|\le 1\right\} = C_3\|\, g(\bullet)\mid M(X,\mathbb{L}^\infty)\,\|,
\end{aligned}$$

where C_3 is a norm of an operator of the change of a variable $x\mapsto 2x$ in the space X. Thus, property c) holds.

Let B_n, $n\in\mathbb{Z}$ be norms of operators of multiplication of functions of X by $\exp(inx)$, $g\in M(X,\mathbb{L}^\infty)$. Then

$$\begin{aligned}
&\|f(\bullet)*\big(e^{in\bullet}g(\bullet)\big)\|_\infty = \|\big(e^{-in\bullet}f(\bullet)\big)*g(\bullet)\|_\infty\\
&\qquad\le \|\, e^{-in\bullet}f(\bullet)\mid X\,\|\cdot\|\, g(\bullet)\mid M(X,\mathbb{L}^\infty)\,\|\le B_{-n}\|\, f\mid X\,\|\cdot\|\, g\mid M(X,\mathbb{L}^\infty)\,\|.
\end{aligned}$$

It follows from this that for the space $(\overset{\circ}{X})^*$ condition d) holds. Fulfillment for it condition e) is obvious.

Property 2. *Let $X\in\mathfrak{H}$ be a Banach space. Then*

$$M(M(X,\mathbb{L}^\infty),\mathbb{L}^\infty) = M(M(\overset{\square}{X},\mathbb{L}^\infty),\mathbb{L}^\infty) = X^{\circ*\circ*} = \overset{\square}{X}.$$

Remark. If X is a quasi-Banach but not a Banach one, then Property 2 fails. It follows from the fact that the space $X^{\circ*\circ*}$ is a Banach space.

Proof of Property 2. Obviously, we need to prove only the relation $X^{\circ*\circ*}=\overset{\square}{X}$. Let $Y=X^{\circ*}$, $Z=Y^{\circ*}$, $h\in\mathbb{D}$. Taking into account that for any Banach space B and $b\in B$ we have

$$\|\, b\mid B\,\| = \sup\big\{\langle b,c\rangle \;\big|\; \|\, c\mid B^*\,\|\le 1\big\},$$

from Property 1 we obtain

$$\begin{aligned}
\|\, h\mid Z\,\| &= \sup_{r<1}\|\, h_r\mid Z\,\| = \sup_{r<1}\sup\left\{\langle g,h_r\rangle \;\Big|\; g\in\overset{\circ}{Y},\|\, g\mid Y\,\|\le 1\right\}\\
&= \sup_{r<1}\sup\left\{\langle g_r,h\rangle \;\Big|\; g\in\overset{\circ}{Y},\|\, g\mid Y\,\|\le 1\right\} = \sup_{r<1}\sup\left\{\langle g,h_r\rangle \mid g\in Y,\|\, g\mid Y\,\|\le 1\right\}\\
&= \sup_{r<1}\|\, h_r\mid X\,\| = \|\, h\mid X\,\|.
\end{aligned}$$

It completes the proof of Property 2.

Definition 2. We say that the sequence $\{V^j\}_{j=0}^{\infty}$ of linear subspaces of $\overset{\circ}{X} \in \mathfrak{H}$ forms a multiresolution analysis (MRA) of the space $X \in \mathfrak{H}$, if it satisfies the conditions:
1) a) $V^0 \subset V^1 \subset \ldots \subset V^j \subset \ldots \subset X$;
   b) $\dim V^j = 2^j$, $j = 0, 1, 2, \ldots$;
2) $\bigcup_{j\geq 0} V^j$ is dense in $\overset{\circ}{X}$;
3) a) V^0 consists of constants;
   b) if $f(x) \in V^j$, then $f(2x) \in V^{j+1}$, $j = 0, 1, 2, \ldots$;
   c) if $f(x) \in V^{j+1}$, then $f(x/2) + f(x/2 + \pi) \in V^j$ $j = 0, 1, 2, \ldots$;
   d) every function $f \in V^{j+1}$ can be represented in the form $f(x) = f_1(x) + f_2(x + 2\pi \cdot 2^{-j-1}) + f_3(2x)$, where $f_1, f_2, f_3 \in V^j$, $j = 1, 2, \ldots$

Remark 1. We define MRA not only for $\mathbb{L}^2$ but for an arbitrary $X \in \mathfrak{H}$ as well. As we have mentioned in the Introduction this definition of MRA is distinct from the known definitions even for $X = \mathbb{L}^2$. First, we do not require the spaces V^j to be a linear span of the translates of a single function. Second, we do not require the spaces V^j to be invariant with respect to a shift of an argument by $2\pi 2^{-j}$. However, we shall show that MRA, which we have defined, possesses these two properties. Main distinguishing feature of our definition is that it guarantees the unique reconstruction of MRA by each of the spaces V^j $(j > 0)$. In the definitions of MRA from [10], [15] Property 3d) is absent and in [17] and [18] none of the conditions 3) is required. In these all cases it is not possible to guarantee the uniqueness of recovery. In the first three MRAs of the space $\mathbb{L}^2$ is studded and only in [18] MRA of the spaces $\mathbb{L}^p$ and $\mathbb{C}$ are considered.

Remark 2. It can be easy proved that property 3b) follows from the others. So it can be rejected from Definition 2.

Unless otherwise specified, we suppose $N = 2^n$, $L = 2^l$, $J = 2^j$ and so on.

Let $\varphi^l \in X$, and superscript l indicates belonging of the function to V^l, we denote by $\vec{\varphi}^l$ the vector-function $\vec{\varphi}^l(x) = \left(\varphi^l(x), \varphi^l(x - 2\pi/L), \ldots, \varphi^l(x - 2\pi(L-1)/L)\right)^T$. If its components are linear independent, we shall call such a vector *a basis of translates*. In what follows an understanding about this notation acts only for functions which are denoted by Greek letters.

Theorem 1. *Let $\{V^j\}_{j=0}^{\infty}$ be an arbitrary MRA of the space $X \in \mathfrak{H}$. Then there exists a set $\{\varphi^j\}_{j=0}^{\infty}$ of functions such that components of $\vec{\varphi}^l$ form a basis of the spaces V^l. Besides, for any j_1, j_2, $0 \leq j_1 < j_2$, we have*

$$\varphi^{j_1}(J_2 x/J_1) = \sum_{k=0}^{J_2/J_1-1} \varphi^{j_2}(x + 2\pi k J_1/J_2).$$

Corollary. *Let us fix some $j_0 > 0$. If under conditions and notation of Theorem 1 a refinement equation*

$$\varphi^j(x) = \sum_{n=0}^{2J_0-1} a_n \varphi^{j+1}(x - 2\pi n/2J) \tag{1.3}$$

holds for some $j = j_0$, then it takes place for any $j \leq j_0$. Besides, taking into account 2π-perodicity of φ^j, the refinement equation can be rewritten in the form

$$\varphi^j(x) = \sum_{n=0}^{2J-1} \left(\sum_{k=0}^{J_0/J-1} a_{n+2kJ} \right) \varphi^{j+1}(x - 2\pi n/2J).$$

A sequence of coefficients of a refinement equation is called its *mask*.

Before proving Theorem 1, we introduce a notation and formulate and prove some auxiliary results.

Lemma 1. *Let a sequence $\{V^j\}_{j=0}^{\infty}$ of spaces forms MRA of the space X. Then for every $j \geq 0$ there exists a basis $v_n^j(x)$, $n = 0, 1, \ldots, J-1$, satisfying for $m \in \mathbb{Z}$ the conditions:*
1) $v_0^j \equiv 1$, $j \geq 0$;
2) for any $k \neq n$, $0 \leq k, n < J$, $m \in \mathbb{Z}$, we have $\hat{v}_n^j(k + Jm) = 0$;

3) the recurrent relations

a) $\hat{v}_n^{j+1}(n+2Jm) = \hat{v}_n^j(n+2Jm)$ *for odd* n *(here and in what follows we suppose* $v_l^j \equiv v_{l+J}^j$*);*

b) $\hat{v}_n^{j+1}(n+2Jm) = \hat{v}_{n/2}^j(n/2+Jm)$ *for even* n

hold.

Remark 1. Obviously, items 1) — 3) of Lemma 1 gives a simple tool for univalent reconstruction of entire MRA from the known basis $\{v_n^{j_0}\}_{n=0}^{J_0-1}$ for a fixed $j_0 > 0$. In particulary, for the recovery MRA it is necessary to know only the function v_1^1.

Remark 2. Obviously, formulae of items 3a) and 3b) give the same results for all n except when $n = J$.

The bases v_n^j, existence of which follows from Lemma 1, for the case of spaces of periodic splines were studied by many authors (see [8], [13],[14], [16]). In [14] V. A. Zheludev used these bases to construct operational calculus on the space of periodic splines and effective algorithms of spline approximation. Functions of these bases V. A. Zheludev have called *orthsplines.* In [13] he also used orthsplines for constructing theory of spline wavelets and algorithms of wavelet decomposition and reconstruction.

Let us formulate some consequences of Lemma 1.

Corollary 1. *The spaces* $\{V^j\}_{j=0}^{\infty}$ *are invariant with respect to a shift of an argument by* $2\pi/J$.

Corollary 2. *Any function* v_n^j, $j > 0$, $0 < n < J$ *can be represented in the form of the series*

$$v_n^j(x) = \sum_{m\in\mathbb{Z}} c_{n+Jm}^j \exp(i(n+Jm)x), \tag{1.4}$$

with nonzero coefficients.

The statement of Corollary 1 follows from the fact that the basis functions v_n^j are eigen functions of the operator of $2\pi/J$-shift of an argument.

Validity of (1.4) follows from item 2) of Lemma 1. As for the fact that all coefficients in (1.4) are distinct from zero, it follows from item 3) of Lemma 1. Indeed, if we assume that $c_{n_0+Nm_0}^j = 0$, then we obtain from item 3) that a harmonic with number $n_0 + Jm_0$ is absent in the spaces V^k for $k \geq j$. It contradicts to item 2) of Definition 2.

Proof of Lemma 1. Let us take $v_0^j \equiv 1$, $j \geq 0$. We conduct the proof by induction.

First, we show the existence of the basis $\{v_n^j\}$ for $j = 1$. As above v_1^1 we can take an arbitrary function such that v_0^1 and v_1^1 form a basis of V^1. Obviously, we can suppose $\hat{v}_1^1(0) = 0$. Since an operator $A : f(x) \mapsto f(x/2) + f(x/2+\pi)$ rejects odd harmonics and doubles even ones (deviding their frequency in two), the function v_1^1 has not even harmonics. In the opposite case $A(v_1^1) \notin V^0$ that contradicts property 3c) of MRA.

We assume that the existence of the bases $\left\{\{v_n^k\}_{n=0}^{K-1}\right\}_{k=0}^{j}$ has been proved. Let us prove that the collection $\{v_n^{j+1}\}_{n=0}^{2J-1}$ of function, determined by recurrent formulae of items 3a) and 3b) of Lemma 1, is the desired basis.

Let us estimate a dimension of a linear space Y of functions which can be represented in the form $f(x) = f_1(x) + f_2(x + 2\pi/2J) + f_3(2x)$, where $f_1, f_2, f_3 \in V^j$. Obviously, for any $n = 1, 2, \dots, J-1$ linear spans of the function pairs $\{v_n^j(x), v_n^j(x + 2\pi/2J)\}$ and $\{v_n^{j+1}(x), v_{n+J}^{j+1}(x)\}$ coincides. On the other hand we have $v_n^j(2x) = v_{2n}^{j+1}(x)$. Hence, a value $\dim Y$ does not exceed a number of elements of collection $\{v_n^{j+1}\}$, i.e., $2J$. At the same time, according to items 1b) and 3d) of Definition 2, we have $\dim V^{j+1} = 2J$, $V^{j+1} \subset Y$. Thus, $Y = V^{j+1}$, and the functions $\{v_n^{j+1}\}$ form a basis of V^{j+1}.

We introduce the notation $\vec{v}^j$ for the vector $(v_0^j, \dots, v_{J-1}^j)^T$.

Lemma 2. *Components of* $\vec{\varphi}^j$ *form a basis of* V^j *if and only if the function* φ^j *is a linear combination of elements of the vector* $\vec{v}^j$ *with nonzero coefficients.*

We recall that by the Wiener theorem a linear span of translates of a function $f \in \mathbb{L}(\mathbb{R})$ is dense in $\mathbb{L}(\mathbb{R})$ if and only if $\hat{f}(\omega) \neq 0$ for all $\omega \in \mathbb{R}$. In our case the space V^j is invariant with respect to the shift by $2\pi/J$, and elements of the basis $\vec{v}^j$ are eigenvectors of an operator of this shift. So coefficients of expansion of functions in the basis $\vec{v}^j$ can be considered as their spectrum. Thus, the statement of Lemma 2 is an analog of the Wiener theorem. That is why we shall call the vector-function $\vec{v}^j$ *the Wiener basis* of the space V^j. Obviously, in each of the spaces V^j there exists only one Wiener basis up to renormalization and rearrangement of its components. We shall use the natural enumeration of elements of Wiener bases when number of an element is determined by harmonics which are contained in its spectrum.

Proof of Lemma 2. For the fixed J we suppose $\omega_J = \exp(2\pi i/J)$ and F_J is the matrix of the Discrete Fourier Transform (DFT) (its elements are determined by the relation $F_{lk} = \omega_J^{-lk}, l = 0, 1, \dots, J-1; k = 0, 1, \dots, J-1$). The matrix of the Inverse Discrete Fourier Transform (IDFT) can be written in the form F_J^*/J. Here and in what follows * means an operation of simultaneous transposition and complex conjugation of a matrix or of a vector.

Let we have the representation $\varphi^j(x) = \sum_{n=0}^{J-1} a_n v_n^j(x)$. We introduce the vector-functions $\vec{u}^j \stackrel{\text{def}}{=} (a_0 v_0^j, a_1 v_1^j, \dots, a_{J-1} v_{J-1}^j)^T$, $\vec{\Omega}_n^j \stackrel{\text{def}}{=} (\exp(inx), \exp(in(x - 2\pi/J)), \dots, \exp(in(x - 2\pi(J-1)/J)))^T$, where $n = k + Jm$, $0 \le k < J$, $m \in \mathbb{Z}$. We have

$$F_J^* \vec{\Omega}_n^j / J = \exp(inx) F_J^* (1, \omega_J^{-k}, \dots, \omega_J^{-k(J-1)})^T / J = \exp(inx) \vec{e}_k,$$

where $\vec{e}_k = (0, \dots, 0, 1, 0, \dots, 0)^T$ with 1 at the kth position. Thus, we obtain

$$F_J^* \vec{\varphi}^j / J = \vec{u}^j. \tag{1.5}$$

Components of the right-hand side of (1.5) form a basis of V^j if and only if all a_n are not equal to zero. Since DFT (and IDFT) is a nonsingular transformation, the same property is necessary and sufficient the components of $\vec{\varphi}^j$ to be a basis. Lemma 2 is proved. Moreover, we have proved that the Wiener basis and the basis of translates are connected (up to multiplication the functions v_n^j by nonzero factors) by direct and inverse DFT.

Proof of Theorem 1. Taking into account that $f(x/2) + f(x/2 + \pi) = 2\sum_{n=-\infty}^{+\infty} \hat{f}(2n) \exp inx$, according to Lemma 2, any collection of functions, satisfying Theorem 1, can be constructed in the form

$$\varphi^j(x) = \sum_{n=0}^{J-1} c_n^j v_n^j(x),$$

where v_n^j satisfy the recurrent relations of Lemma 1, as follows.

We take an arbitrary $c_0^0 \neq 0$ and suppose $\varphi^0 \equiv c_0^0$. If we have chosen the numbers c_n^j, then we set $c_n^{j+1} = c_{n/2}^j/2$ for even n, and we take an arbitrary nonzero c_n^{j+1} for odd n.

According to Lemma 1, every MRA of the space X is generated by a single function v_1^1. To describe all possible MRA of the space X it suffices to find all *permissible* functions v_1^1.

Theorem 2. *A function $v_1^1 \in \overset{\circ}{X} \in \mathfrak{H}$ generates MRA of X if and only if*

$$v_1^1(x) = \sum_{n=-\infty}^{+\infty} a_n \exp(i(2n+1)x),$$

where all a_n are distinct from 0.

Proof of Theorem 2. Necessity follows from Corollary 2 of Lemma 1. Let us prove sufficiency. First, we prove that $v_n^j \in \overset{\circ}{X}$, $0 \le n < J$, $j > 0$. Indeed, the boundedness of the operator of item 3b) of Lemma 1 follows from item 3c) of Definition 1. Hence, if $\| f_m(x) - v_n^j(x) \mid X \| \to 0$ as $m \to \infty$ ($f_m \in \mathbb{C}^\infty$), then $\| f_m(2x) - v_{2n}^{j+1}(x) \mid X \| \to 0$, i.e., $v_{2n}^{j+1} \in \overset{\circ}{X}$. The operator from item 3a) is bounded, since for odd n the function v_n^{j+1} can be represented as a linear combination of v_n^j and its translate by $2\pi/2J$. Obviously, $v_n^{j+1} \in \overset{\circ}{X}$.

Let us prove that property 2) of MRA holds. Fulfillment of other properties is obvious.

Because the space $\mathbb{C}^\infty$ is dense in $\overset{\circ}{X}$, and trigonometric polynomials are dense in $\mathbb{C}^\infty$, it suffices to prove that a functions of the form $\exp(inx)$, $n = 0, \pm 1, \pm 2, \ldots$ can be approximated by functions of the spaces V^j with an arbitrary accuracy.

We take a sequence $h_j = 1 - \exp(-inx)v_n^j/\hat{v}_n^j(n)$, $j = j_0, j_0+1, \ldots$. Let us show that it tends to 0 with respect to the norm $\| \bullet \mid X \|$. We denote by $\omega(\delta, f, X)$ a modulus of continuity of the function $f \in X$:

$$\omega(\delta, f, X) \overset{\text{def}}{=} \sup_{h<\delta} \| f(\bullet + h) - f(\bullet) \mid X \|.$$

In view of Property 1 of spaces of $\mathfrak{H}$ we have $\| \bullet \mid X^{\circ *} \| = \| \bullet \mid M(X, \mathbb{L}^\infty) \|$ (at least after equivalent renormalization). It means that for any $f \in X$ $(= \overset{\circ}{X})$

$$\| f \mid X \| = \sup \left\{ \| f * g \mid \mathbb{L}^\infty \| \;\middle|\; \| g \mid X^* \| \le 1 \right\}. \tag{1.6}$$

At the same time for any g, $\| g \mid X \| \le 1$ we have

$$\| f * g \mid \mathbb{L}^\infty \| \le \| f \mid X \| \cdot \| g \mid X^* \| \le \| f \mid X \|,$$

in particular,

$$\omega(\delta, f * g, \mathbb{L}^\infty) \le \omega(\delta, f, X). \tag{1.7}$$

We introduce the notation $\omega(\delta) = \omega(\delta, h_{j_0}, X)$. Since $h_{j_0} \in X = \overset{\circ}{X}$, then $\omega(\delta) \to 0$ as $\delta \to 0$. The functions h_j is $(2\pi/J)$-periodic and can be represented in the form

$$h_j(\bullet) = \frac{J_0}{J} \sum_{n=0}^{J/J_0 - 1} h_{j_0}(\bullet + 2\pi n/J).$$

It follows from this and from (1.7) that

$$\omega(\delta, h_j * g, \mathbb{L}^\infty) \quad \le \quad \begin{cases} \omega(\delta, h_{j_0} * g, \mathbb{L}^\infty), & \delta \le 2\pi/J, \\ \omega(2\pi/J, h_{j_0} * g, \mathbb{L}^\infty), & \delta > 2\pi/J, \end{cases} \quad \le \quad \begin{cases} \omega(\delta), & \delta \le 2\pi/J, \\ \omega(2\pi/J), & \delta > 2\pi/J. \end{cases}$$

Hence, $\| h_j * g \mid \mathbb{L}^\infty \| \le \omega(2\pi/J) \to 0$ as $j \to \infty$. Taking into account (1.6), we obtain $\| h_j \mid X \| \to 0$. Consequently, according to item d) of Definition 1, we have

$$\| \exp(in\bullet) - v_n^j/\hat{v}_n^j(n) \mid X \| = \| \exp(in\bullet) h_j(\bullet) \mid X \| \to 0 \text{ as } j \to 0.$$

As we wished.

In conclusion of this section it should be be said that Definition 2 can be used not only for spaces from $\mathfrak{H}$ but for many topological linear spaces of distributions, for example, for $\mathbb{C}^\infty$ or $\mathbb{D}$.

As for to extend Definition 2 over polynomial MRA (Y.Meyer [9], C.K. Chui and H.N. Mhaskar [2]), it can be done by means of rejecting item 3d) from the definition and changig item 1b) for requirement that V^j has a basis of translates. However, it leads to violating some useful properties of MRA. So it seems these important MRA should be considered according to general principles but separately from the general theory.

§2. Wavelets

The traditional approach to definition of wavelet spaces consists in definition of wavelets space W^j as the orthogonal complement V^j to V^{j+1}. This method of wavelet decomposition of functions is based on approximation of a function $f \in \mathbb{L}^2$ by its orthogonal projections to the spaces $V^j \subset \mathbb{L}^2$. The concept "orthogonal projection" lose their meaning for an arbitrary space $X \in \mathfrak{H}$. Some aspects of non-orthogonal wavelet decomposition of the space $\mathbb{L}^2(\mathbb{R})$ were consideres in [4, Ch. 8].

Let $\{V^j\}_{j=0}^\infty$ be MRA of a space $X \in \mathfrak{H}$. The components of vector-functions $\{\vec{v}^j\}_{j=0}^\infty$ form the Wiener bases of V^j. If $\{P^j\}_{j=0}^\infty$ is MRA of the space $(\overset{\circ}{X})^*$ with the Wiener bases $\{\vec{p}^j\}_{j=0}^\infty$ and such that $\langle v_n^j, p_n^j \rangle \ne 0$,

$n = 0, 1, \ldots, J-1$, $j = 0, 1, 2, \ldots$, then we shall say that MRA $\{P^j\}_{j=0}^{\infty}$ determines *a wavelet projection* to MRA $\{V^j\}_{j=0}^{\infty}$. Since for $i \neq j$ supports of the spectra of the functions v_n^j and p_n^i is disjoint, we have $\langle v_n^j, p_n^i \rangle = 0$. So projecting $X \mapsto V^j$ is determined by the relation

$$f \mapsto \mathcal{V}^j[f] \stackrel{\text{def}}{=} \sum_{n=0}^{J-1} \frac{\langle f, p_n^j \rangle}{\langle v_n^j, p_n^j \rangle} v_n^j. \tag{2.1}$$

Wavelet spaces are defined to be

$$W^j = \{f \in V^{j+1} \mid \langle f, p_n^j \rangle = 0, n = 0, 1, \ldots, J-1\}.$$

It is easyly to see that in the spaces W^j there exists Wiener bases which can be calculated by formulae

$$\begin{aligned} w_n^j &= \langle v_{n+J}^{j+1}, p_n^j \rangle v_n^{j+1} - \langle v_n^{j+1}, p_n^j \rangle v_{n+J}^{j+1}, \quad n = 1, 2, \ldots, J-1, \\ w_J^j &= v_J^{j+1}. \end{aligned} \tag{2.2}$$

Thus, the spaces W^j are invariant with respect to a shift of an argument by $(2\pi/J)$. Besides, the relations

$$V^j = V^0 \oplus W^0 \oplus \cdots \oplus W^{j-1}, \quad j = 1, 2, \ldots,$$

where $\oplus$ means a direct (generally speaking, nonorthogonal) sum of linear spaces, hold.

Obviously, by Property 2 of the classes $\mathfrak{H}$ for Banach space X we have a dual situation. MRA $\{V^j\}_{j=0}^{\infty}$ settles a wavelet projection of the space $(\overset{\circ}{X})^*$ to the spaces $\{P^j\}_{j=0}^{\infty}$ and determines wavelet spaces $\{Q^j\}_{j=0}^{\infty}$. The same statement is also valid for a quasi-Banach space X. Indeed, if $\{V^j\}$ form MRA of the quasi-Banach space $X \subset Y \stackrel{\text{def}}{=} X^{\circ * \circ *}$, then v_1^1 has nonvanishing odd harmonics. Hence, by Theorem 2, $\{V^j\}$ form MRA of the Banach space Y. However, MRA of the quasi-Banach space X do not define all possible wavelet projections to MRA $\{P^j\}$ of the space $(\overset{\circ}{X})^*$.

Theorem 3. *Let $\{V^j\}_{j=0}^{\infty}$ be an arbitrary MRA of the space $X \in \mathfrak{H}$, MRA $\{P^j\}_{j=0}^{\infty}$ of the space $(\overset{\circ}{X})^*$ determines a wavelet projection, $\{W^j\}_{j=0}^{\infty}$ are the corresponding wavelet spaces. Then there exists a set $\{\psi^j\}_{j=0}^{\infty}$ of functions such that components of $\vec{\psi}^l$ form a basis of the spaces W^l. Besides, for any j_1, j_2, $0 \leq j_1 < j_2$ we have*

$$\psi^{j_1}(J_2 x/J_1) = \sum_{k=0}^{J_2/J_1 - 1} \psi^{j_2}(x + 2\pi k J_1/J_2).$$

We omit a proof of Theorem 3 because it is similar to the proof of Theorem 1.

Corollary. *Let us fix some $j_0 > 0$. If under conditions and notation of Theorem 3 a refinement equation*

$$\psi^j(x) = \sum_{n=0}^{2J_0 - 1} b_n \varphi^{j+1}(x - 2\pi n/2J) \tag{2.3}$$

holds for $j = j_0$, then it takes place for any $j \leq j_0$. Besides, taking into account 2π-perodicity of φ^j and ψ^j, the refinement equation can be rewritten in the form

$$\psi^j(x) = \sum_{n=0}^{2J-1} \left(\sum_{k=0}^{J_0/J - 1} b_{n+2kJ} \right) \varphi^{j+1}(x - 2\pi n/2J).$$

The principal question, connected with wavelet projections, is about possibility of approximation to any function with an arbitrary precision. We give answer this question for the simplest case $X = \ell^p$, $1 \leq p \leq \infty$, where

$$\| f \mid \ell^p \| \stackrel{\text{def}}{=} \left(\sum_{n=-\infty}^{+\infty} |\hat{f}|^p \right)^{1/p}.$$

In what follows, as it usually is, we suppose $q = p/(p-1)$ for $p > 1$; $q = \infty$ for $p = 1$.

Theorem 4. *Let $\{P_j\}_{j=0}^{\infty}$ determines a wavelet projection to MRA $\{V^j\}_{j=0}^{\infty}$ of ℓ^p, $1 \le p < \infty$. Then any distribution of ℓ^p is approximated by its projections to V^j with an arbitrary precision if and only if the collection $\{\{\Delta_{nj}\}_{n=0}^{J-1}\}_{j=0}^{\infty} \stackrel{\mathrm{def}}{=} \{\{\| p_n^j \mid \ell^q \| \cdot \| v_n^j \mid \ell^p \| / \langle v_n^j, p_n^j \rangle\}_{n=0}^{J-1}\}_{j=0}^{\infty}$ of numbers is bounded.*

Remark. Since $(\ell^\infty)^\circ \stackrel{\mathrm{def}}{=} \mathbf{c}_0 \neq \ell^\infty$, Theorem 4 is only valid for $\mathbf{c}_0$ but not for ℓ^∞.

As we have mentioned, to determine wavelet spaces for the given MRA $\{V^j\}_{j=0}^{\infty}$ of X we have to settle a method of projecting to the spaces V^j. For each of MRA $\{V^j\}_{j=0}^{\infty}$ of the most important particular case $X = \mathbb{L}^2$ it makes sense to consider an orthogonal projection which is determined by relations $P^j = V^j$. According to Theorem 4, any function can be approximated by its orthogonal projections to the spaces V^j. Taking into account that an orthogonal projection gives the best approximation, this fact follows also from density of $\bigcup_{j=0}^{\infty} V^j$ in $\mathbb{L}^2$.

Proof of Theorem 4. Necessity. Let there exist sequences n_m and j_m, $0 \le n_m < 2^{j_m} = J_m$, $j_m \to \infty$, such that $|\Delta_{n_m j_m}| \to \infty$ as $m \to \infty$. It means that a sequence of the projectors

$$f \mapsto \mathcal{V}_{n_m}^{j_m}[f] \stackrel{\mathrm{def}}{=} \frac{\langle f, p_{n_m}^{j_m} \rangle}{\langle v_{n_m}^{j_m}, p_{n_m}^{j_m} \rangle} v_{n_m}^{j_m}$$

is unbounded as a sequence of operators acting from ℓ^p to ℓ^p, i.e., there exists a sequence $f_m \in \ell^p$, $\| f_m \mid \ell^p \| = 1$ such that $\| \mathcal{V}_{n_m}^{j_m}[f_m] \mid \ell^p \| \to \infty$. Obviously, we can choose f_m with all vanishing harmonics except when their numbers equal $n_m + kJ_m$, $k \in \mathbb{Z}$. Hence, a sequence of norms of the operators $\mathcal{V}^j$ is not bounded. Necessity is proved.

We note that boundedness of the set Δ_{nj} is necessary for convergence in the spaces in which a sequence of norms of operators of multiplication by the functions $\exp(inx)$ is bounded. Indeed, it follows from the fact that an operator $f(x) \mapsto \sum_{k \in \mathbb{Z}} \hat{f}(n+kJ) e^{i(n+kJ)x}$ can be represented in the form $e^{inx} \sum_{k=0}^{J-1} f(x + 2\pi k/J) e^{-in(x+2\pi k/J)}/J$.

Sufficiency. Let numbers $|\Delta_{nj}| = \| \mathcal{V}_n^j \mid \ell^p \to \ell^p \|$ are bounded by $M > 0$. We prove that $\| \mathcal{V}^j \mid \ell^p \to \ell^p \|$ are bounded by M. Indeed, introducing the notation

$$f_n^j(x) \stackrel{\mathrm{def}}{=} \sum_{k \in \mathbb{Z}} \hat{f}(n+kJ) e^{i(n+kJ)x},$$

we obtain

$$\| \mathcal{V}^j[f] \mid \ell^p \|^p = \| \sum_{n=0}^{J-1} \mathcal{V}_n^j[f] \mid \ell^p \|^p = \| \sum_{n=0}^{J-1} \mathcal{V}_n^j[f_n^j] \mid \ell^p \|^p$$
$$= \sum_{n=0}^{J-1} \| \mathcal{V}_n^j[f_n^j] \mid \ell^p \|^p \le M^p \sum_{n=0}^{J-1} \| f_n^j \mid \ell^p \|^p = M^p \| f \mid \ell^p \|^p.$$

As we wished.

Thus, in view of boundedness of norms of the approximating operators, to prove the convergence for every function of the space ℓ^p we need only to prove the convergence for the exponents. It easily follows from (2.1) and from the fact that for any MRA of $X \in \mathfrak{H}$ (in particular, for ℓ^p and ℓ^q) for any n we have $\| e^{inx} - v_n^j(x)/\hat{v}_n^j(n) \mid X \| \to 0$ as $j \to \infty$ that was shown when we proved Theorem 2. Theorem 4 is proved.

§3. Algorithms of wavelet decomposition and reconstruction

Algorithms of decomposition consists of two steps. The first of them is projection of infinite dimensional space X to a finite dimensional space V^j. This step can be realized only approximately, since it includes such operations as integration or summation of infinite series. We shall adduce only analytic formulae and shall not discuss methods of numerical realization of the projection. The second step is decomposition of $f \in V^j$ into a sum

$$f(x) = f_0(x) + \sum_{n=0}^{j-1} g_n(x) = f_0(x) + \sum_{n=0}^{j-1} \sum_{k=0}^{N-1} c_{jn} \psi^n(x - 2\pi k/N),$$

where $f_0 \in V^0$, $g_n \in W^n$, $n = 0, 1, 2, \dots, j-1$. Since here we deal with finite dimensional spaces then we can use finite numerical algorithms. We consider 3 type of algorithms. First of them is universal. It based on transfer to frequency domain, and it has complexity $O(J \log J) = O(Jj)$. The second one is applicable for wavelet projections of the space $X \in \mathfrak{H}$ when there exist locally supported bases $\{\vec{\varphi}^j\}_{j=0}^{\infty}$ of $\{V^j\}_{j=0}^{\infty}$ and $\{\vec{\xi}^j\}_{j=0}^{\infty}$ of the spaces $\{P^j\}_{j=0}^{\infty}$, determining a wavelet projection. We use the term *locally supported bases*, since for the periodic case the traditional term *compact supported basis* does not make sense. In the case of locally supported bases of translates there exists a real number $\sigma > 0$ such that support of distributions of basis $\vec{\varphi}^n$ (on a period) belongs to a segment of length $2\pi\sigma/N$, $n > 0$. In what follows we suppose that σ is positive integer and it is equal to minimal number of sequential segments of the form $[2\pi k/N, 2\pi(k+1)/N]$, covering the support of the bases on period. Complexity of such algorithm is equal to $O((\sigma_1 + \sigma_2)J)$, where σ_1 and σ_2 are sizes of supports of bases $\{\vec{\varphi}^j\}_{j=0}^{\infty}$ and $\{\vec{\xi}^j\}_{j=0}^{\infty}$ respectively. Solving systems of linear equations with sparse matrices are main components of this algorithm. The third algorithm allows for an arbitrary MRA with a finite mask of refinement equation to find an appropriate wavelet projection such that procedures of decomposition and reconstruction can be realized in the form of discrete convolutions with a size of windows of order $\mathcal{N}$ which coincides with a size of the mask. Complexity of such algorithm is equal to $O(\mathcal{N}J)$. These algorithms can be parallelizing. They are considered in §4.

Let us fix two dual MRA $\{V^j\}_{j=0}^{\infty}$ and $\{P^j\}_{j=0}^{\infty}$ of the spaces X and $(\mathring{X})^*$ respectively, $\{\vec{v}^j\}_{j=0}^{\infty}$ and $\{\vec{p}^j\}_{j=0}^{\infty}$ are their Wiener bases, $\{W^j\}_{j=0}^{\infty}$ and $\{Q^j\}_{j=0}^{\infty}$ are the corresponding wavelet spaces with the Wiener bases $\{\vec{w}^j\}_{j=0}^{\infty}$ and $\{\vec{q}^j\}_{j=0}^{\infty}$.

The coefficients of expansion of projection of an arbitrary distribution f to the space V^n in the Wiener basis are determined by formula (2.1). According to relation (1.5), coordinates of vectors of V^n in a Wiener basis and in a basis of translates are connected by direct and inverse DFT and (may be) coordinatewise multiplication in frequency domain. According to Lemma 1 and by (2.2), the functions v_k^j and w_k^j are linear combinations of the pair v_k^{j+1}, v_{k+J}^{j+1}. So if we know coefficients of decomposition in the basis $\vec{v}^{j+1}$, we need at most $4J$ operations of multiplication and $2J$ additions to calculate projections to V^j and W^j in the bases $\vec{v}^j$ and $\vec{w}^j$. The reconstruction formulae have the same complexity. Thus, the general complexity is determined by the transfer from Wiener bases to bases of translates and by the inverse one. Hence, the complexity is equal to the complexity of Fast Fourier Transform, i.e., $O(N \log N)$.

Before consideration the second algorithm we obtain some results on bases of translates. We have formula (2.1) for projecting to V^j in the Wiener basis. Let us obtain analogous formulae of projecting in a basis of translates. A basis $\vec{\xi}^j$ of translates in P^j is called *dual* one to the basis $\vec{\varphi}^j$, if

$$\langle \varphi^j(\bullet - 2\pi l/J), \xi^j(\bullet - 2\pi k/J)\rangle = \delta_{lk} \overset{\text{def}}{=} \begin{cases} 1, & l = k; \\ 0, & l \neq k \end{cases}.$$

Theorem 5. *Two distributions*

$$\varphi = \sum_{k=0}^{N-1} \alpha_k v_k^n \text{ and } \xi = \sum_{k=0}^{N-1} \beta_k p_k^n$$

are dual (i.e., their corresponding translates form dual bases of V^n and P^n) if and only if

$$\alpha_0 \bar{\beta}_0 \langle v_0^n, p_0^n\rangle = \alpha_1 \bar{\beta}_1 \langle v_1^n, p_1^n\rangle = \dots = \alpha_{N-1} \bar{\beta}_{N-1} \langle v_{N-1}^n, p_{N-1}^n\rangle = 1/N.$$

Theorem 5 for the case of the spline MRA of the space $\mathbb{L}^2$ was proved by V. A. Zheludev [13].

Proof of Theorem 5. Indeed,

$$\langle \varphi(\bullet), \xi(\bullet - 2\pi l/N)\rangle = \sum_{k=0}^{N-1} \alpha_k \bar{\beta}_k \langle v_k^n, p_k^n\rangle \exp(2\pi i lk/N) \overset{\text{def}}{=} \tau_l.$$

The right-hand expression gives values of the trigonometric polynomial $t_N(x) = \sum_{k=0}^{N-1} \gamma_k \exp(ixk)$, where $\gamma_k = \alpha_k \bar{\beta}_k \langle v_k^n, p_k^n\rangle$, at points $2\pi l/N$, $l = 0, 1, \dots, N-1$. In view of the fact that the interpolation problem has a single solution we have $\tau_l = \delta_{0l}$, if and only if $\gamma_k = 1/N$. Theorem 5 is proved.

Let us introduce the notation

$$\gamma_m = \langle \varphi^{n+1}(\bullet - 2\pi m/2N), \xi^n(\bullet)\rangle, \quad m = 0, 1, \dots, 2N-1.$$

Theorem 6. *Let*

$$\psi^n(\bullet + 2\pi/2N) = \sum_{m=0}^{2N-1} (-1)^n \gamma_m \varphi^{n+1}(\bullet + 2\pi m/2N). \tag{3.1}$$

Then the system $\{\psi^n(\bullet - 2\pi m/N)\}_{m=0}^{N-1}$ *of functions forms a basis of the space* W^n.

Relationship (3.1) is reminiscent of the well-known representation of ψ^n in terms of components of the vector $\vec{\varphi}^{n+1}$ when the components form an orthonormal basis (see, [4],[5]). In fact, (3.1) makes clear the sense of the coefficients. For the case of MRA of the space $\mathbb{L}^2(\mathbb{R})$ formula (3.1) represented in [4, p. 264] (see also formulae (5.4) from [1]).

For the case of locally supported bases $\vec{\varphi}^j$ of V^j it follows from Theorem 6 that the bases $\vec{\psi}^j$ are locally supported. Moreover, the masks of refinement equations for ψ^j has a bounded number of elements. Similar problems for MRA of $\mathbb{L}^2(\mathbb{R})$ with compactly supported bases, connected by a refinement equations with a finite mask, were considered in [3].

Proof of Theorem 6. For any k we have

$$\langle \psi^n(\bullet - 2\pi k/N), \xi^n(\bullet)\rangle = \sum_{m=0}^{2N-1} (-1)^m \gamma_m \langle \varphi^{n+1}(\bullet - 2\pi(2k+1-m)/2N)), \xi^m(\bullet)\rangle$$

$$= \sum_{m=0}^{2N-1} (-1)^m \gamma_m \gamma_{2k+1-m} = \sum_{m=0}^{N-1} \gamma_{2m}\gamma_{2k+1-2m} - \sum_{m=0}^{N-1} \gamma_{2m+1}\gamma_{2k-2m}$$

$$= \sum_{m=0}^{N-1} \gamma_{2m}\gamma_{2k+1-2m} - \sum_{m=0}^{N-1} \gamma_{2k+1-2m}\gamma_{2m} = 0.$$

We show that components of the vector $\vec{\psi}^n(x)$ are linear independent. If $A(\bullet)$ and $B(\bullet)$ are functional matrices of sizes $j_0 \times k_0$, $l_0 \times k_0$ elements of which are in X and $(\overset{\circ}{X})^*$ respectively, then the notation $\langle A, B\rangle$ is defined to be a numerical matrix C of size $j_0 \times l_0$, where $c_{j,l} = \sum_{k=0}^{k_0-1} \langle a_{j,k}, b_{l,k}\rangle$. We introduce the auxiliary matrix $\Gamma^n = \langle \vec{\varphi}^{n+1}, \vec{\xi}^n\rangle$. Its first column is $(\gamma_0, \gamma_1, \dots, \gamma_{2N-1})$. And each of the following columns is a cyclic shift of the previous one by two position. In view of relation (3.1) a dimension of a linear span of components $\vec{\psi}^n$ is equal to the rank of the matrix Γ^n. Since $\operatorname{rank}\Gamma^n = \operatorname{rank}(F^*_{2N}\Gamma^n F_N/2N^2) = \operatorname{rank}\langle \vec{v}^{n+1}, \vec{p}^n\rangle = N$, components of vector $\vec{\psi}^n$ are linear independent. Theorem 6 is proved.

Now we dwell on numerical algorithms of wavelet decomposition and reconstruction when we have MRA of $X \in \mathfrak{H}$ with locally supported bases $\vec{\varphi}^j$ and when MRA of $(\overset{\circ}{X})^*$, determining a wavelet projection, also has locally supported bases $\vec{\xi}^j$. We denote by σ_1 and σ_2 sizes of the corresponding supports. The case $X = \mathbb{L}^2$, $P^j = V^j$ is a particular one. It gives an decomposition of the space $\mathbb{L}^2$ in an orthogonal sum of wavelet spaces.

We denote by $\vec{\tilde{\xi}}^j$ and $\vec{\tilde{\varphi}}^j$ bases which are dual to $\vec{\varphi}^j$ and $\vec{\xi}^j$ respectively.

First, let us find a coefficients of projection of an arbitrary function $f \in X$ to the space V^n in the basis $\vec{\varphi}^n$. The projection f_n can be represented in the two forms

$$f_n = \vec{a}^{nT}\vec{\varphi}^n = \vec{b}^{nT}\vec{\tilde{\varphi}}^n, \tag{3.2}$$

where $\vec{a}$ and $\vec{b}$ are vectors of coefficients of decomposition in the bases $\vec{\varphi}^n$ and $\vec{\tilde{\varphi}}^n$. Obviously, $\vec{b}^n = \langle f, \vec{\xi}^n\rangle$. So, if $f\bar{\vec{\xi}}^n$ is a vector of integrable functions, to find $\vec{b}^n$ we need to calculate N integrals over segments of length $2\pi\sigma_2/N$. It follows from (3.2) that $\langle \vec{a}^{nT}\vec{\varphi}^n, \vec{\xi}^n\rangle = \langle \vec{b}^{nT}\vec{\tilde{\varphi}}^n, \vec{\xi}^n\rangle$. Hence, $\vec{a}^{nT}\Phi^n = \vec{b}^{nT}$. Thus, a vector $\vec{a}^n$ is a solution of a system of linear equations with at most $(\sigma_1+\sigma_2)$-diagonal circulant matrix. Triangle decomposition of a matrix Φ^n can be realized for $O((\sigma_1+\sigma_2)^2 N)$ arithmetic operations. However, it does not depend on input function f. The complexity of solving two systems with the triangle sparse matrices equals $O((\sigma_1+\sigma_2)N)$. So complexity of this step of decomposition is equal to $O((\sigma_1+\sigma_2)N)$.

The second and the third steps consist in projection of the function $f_{j+1} \in V^{j+1}$ to the spaces V^j and W^j. These steps repeat as many times as we need up to $j = 0$. A single difference of the second step from the first one lies in simplification of a method of obtaining the function f_j. Indeed, since $\vec{b}^j = \langle f_{j+1}, \vec{\xi}^j \rangle = \langle (\vec{a}^{j+1})^T \vec{\varphi}^{j+1}, \vec{\xi}^j \rangle = (\vec{a}^{j+1})^T \Gamma^j$, then the vector $\vec{b}^j$ is a result of convolution of $\vec{a}^{j+1}$ with a window of size of order $O(\sigma_1 + 2\sigma_2)$. In particular, for $X = \mathbb{L}^2$ we need not to calculate integrals.

Projection to the wavelet space W^j is realized in a similar way. Indeed, let $\vec{\psi}^j$ be determined by (3.1) and $\vec{\zeta}^j$ is the locally supported basis of the wavelet space Q^j, $\vec{\tilde{\zeta}}^j$ and $\vec{\tilde{\psi}}^j$ are the corresponding dual bases. Then $g_j = \vec{c}^{jT} \vec{\psi}^j = \vec{d}^{jT} \vec{\tilde{\psi}}^j$, where g_j is the wavelet projection of f_{j+1} to W^j, $\vec{d}^{jT} = \langle f_{j+1}, \vec{\zeta}^j \rangle = (\vec{a}^{j+1})^T \langle \vec{\varphi}^{j+1}, \vec{\zeta}^j \rangle = (\vec{a}^{j+1})^T \Xi^j$, $\vec{c}^j$ is a solution of the system of linear equations $\vec{c}^{jT} \Psi^j = \vec{d}^{jT}$ with the matrix $\Psi^j \overset{\text{def}}{=} \langle \vec{\psi}^j, \vec{\zeta}^j \rangle$. Obviously, complexity of this step is equal to $O((\sigma_1 + \sigma_2)N)$.

Let us construct an algorithm of reconstruction. Because

$$(\vec{a}^{j+1})^T \vec{\varphi}^{j+1} = \vec{a}^{jT} \vec{\varphi}^j + \vec{c}^{jT} \vec{\psi}^j, \tag{3.3}$$

then

$$\langle (\vec{a}^{j+1})^T \vec{\varphi}^{j+1}, \vec{\xi}^{j+1} \rangle = \langle \vec{a}^{jT} \vec{\varphi}^j, \vec{\xi}^{j+1} \rangle + \langle \vec{c}^{jT} \vec{\psi}^j, \vec{\xi}^{j+1} \rangle.$$

Hence,

$$(\vec{a}^{j+1})^T \Phi^{j+1} = \vec{a}^{jT} \Gamma + \vec{c}^{jT} \Xi. \tag{3.4}$$

The right-hand part of (3.4) consists of two convolutions with windows of a size of order $\sigma_1 + \sigma_2$, and a vector $\vec{a}^{j+1}$ is a solution of system of linear equations with the matrix Φ^{j+1}.

Thus, general computational complexity of this algorithm for MRA with locally supported bases is equal to $O((\sigma_1 + \sigma_2)N)$.

§4. Convolution algorithms of wavelet decomposition and reconstruction

Strictly speaking, for every wavelet projection algorithms of wavelet decomposition and reconstruction can be realized in the form of convolution. However, not for all cases (see, for instance, the previous section) such realization is optimal from the point of view of computational complexity. Here we consider cases when these algorithms can be realized as convolutions with short windows.

Let $\{V^k\}$ form MRA of a space X, $\{P^k\}$ determines a wavelet projection, $\{\vec{\varphi}^k\}_{k=0}^{j+1}$ and $\{\vec{\psi}^k\}_{k=0}^{j}$ are bases of the corresponding spaces V^k and W^k. Then there exist collections $\{r_m\}_{m=0}^{2J-1}$ and $\{s_m\}_{m=0}^{2J-1}$ of numbers which for every $k \le j$ determines wavelet projection

$$\sum_{n=0}^{2K-1} c_n^{k+1} \varphi^{k+1}(x - 2\pi n/2K) = \sum_{n=0}^{K-1} c_n^k \varphi^k(x - 2\pi n/K) + \sum_{n=0}^{K-1} d_n^k \psi^k(x - 2\pi n/K)$$

by formulae

$$c_m^k = \sum_{l=0}^{2J-1} c_{l+2m}^{k+1} r_l, \qquad d_m^k = \sum_{l=0}^{2J-1} c_{l+2m}^{k+1} s_l.$$

The reconstruction formula can be written in the form

$$c_m^{k+1} = \sum_{l=0}^{2J-1} c_l^k a_{m-2l} + \sum_{l=0}^{2J-1} d_l^k b_{m-2l},$$

where a_l and b_l are numbers of masks of refinement equations (1.3) and (2.3).

Conversely, if we have a space W^j which is invariant with respect to a shift of an argument by $2\pi/J$ and such that $V^{j+1} = V^j \oplus W^j$, then spaces W^k, $k < j$, can be defined by the recurrent relation $W^k = \{\, f(\bullet/2) + f(\bullet/2 + \pi) \mid f \in W^{k+1} \,\}$. It turns out, there exist MRA those settle wavelet projections with the same wavelet spaces $\{W^k\}_{k=0}^{j}$. It can be easily proved that there are many possible MRA, determining these wavelet spaces. It follows from the fact that by Theorem 2 MRA $\{P^j\}$ is defined by the function p_1^1.

At the same time any function f, which satisfies conditions $\langle v_k^m, f\rangle \neq 0$, $\langle w_k^n, f\rangle = 0$, k is odd, $m = 1, 2, \ldots$; $n = 2, 3, \ldots, j$, can be taken as above p_1^1. After a choice of bases of translates in the spaces V^{j+1} and W^j the sequences $\{r_l\}$ and $\{s_l\}$ are determined uniquely.

In this section we prove that, if for a fix $n > 0$ numbers of nonzero elements of a mask of refinement equation (1.3) are in a segment of length $\mathcal{N} \in \mathbb{Z}$ (it makes a sense to assume that $\mathcal{N} \ll N$), then there exist algorithms of decomposition and reconstruction of functions of V^n of complexity $O(\mathcal{N}N)$.

We need the following auxiliary statements.

Lemma 3. *Let P_1, P_2, R be algebraic polynomials of degree m, n, and $< m+n$ respectively. Besides, the polynomials P_1 and P_2 have not common roots. Then there exist polynomials Q_1 and Q_2 of degree at most $n-1$ and $m-1$ such that*

$$P_1(x)Q_1(x) + P_2(x)Q_2(x) = R(x). \tag{4.1}$$

The polynomials Q_1 and Q_2 are determined uniquely.

Remark. This lemma in the form, which is a little less general, is adduced by I. Daubechies [4, p. 169] and [5]. Our proof is based on other idea, and it is a little shorter.

Proof of Lemma 3. Let $\{x_k\}_{k=1}^m$ be roots of the polynomial P_1. Then Q_2 is determined by conditions $Q_2(x_k) = R(x_k)/P_2(x_k)$, where the equality is fulfilled with multiplicity of the root x_k. Since $P_2(x_k) \neq 0$, the division is determined correctly. The polynomial Q_1 is found in an analogous way by values at points $\{y_k\}_{k=1}^n$ which are roots of the polynomial P_2.

Relation (4.1) follows from its fulfillment (with regard to multiplicity) at $n+m$ points, because its the left and the right parts are polynomials of order at most $n+m-1$.

In view of uniqueness of a solution of an interpolation problem (generally speaking, with multiple points) the polynomials Q_1 and Q_2 are uniquely determined.

Let $\{V^j\}$ be some MRA of a space of $\mathfrak{H}$. We know that, in fact, the Wiener bases of the spaces are determined uniquely. However, collection of translate bases $\{\vec{\varphi}^j\}$ can be chosen in various ways. We consider the case when the basis functions are connected by the refinement equation

$$\varphi^j(x) = \sum_{n=0}^{\mathcal{N}} a_n \varphi^{j+1}(x - 2\pi n/2J),$$

where $\mathcal{N} > 0$ and $\{a_n\}$ do not depend on j, $j = 0, 1, 2, \ldots$. *The characteristic polynomial* of the bases $\{\vec{\varphi}^j\}$ is defined to be $P_\varphi(z) \stackrel{\text{def}}{=} \sum_{n=0}^{\mathcal{N}} a_n z^n$. Besides, we suppose that for any $j \geq 0$ the relation $\varphi^j(\bullet) = \frac{1}{2}\left(\varphi^{j+1}(\bullet/2) + \varphi^{j+1}(\bullet/2 + \pi)\right)$ holds. In particular, it implies the equality $P_\varphi(1) = 1$. Our immediate purpose is to find all permissible finite masks, i.e., masks for which the corresponding collection of bases $\vec{\varphi}^j$ exists.

Lemma 4. *Let $\{V^j\}$ be an arbitrary MRA. Then the mask $\{a_n\}_{n=0}^{\mathcal{N}}$ is permissible, if and only if its characteristic polynomial $P_*(z) = \sum_{n=0}^{\mathcal{N}} a_n z^n$ is equal to 0 at the point $z = -1$ and does not vanish at other points representable in the form $z = \exp(2\pi i\theta)$, where θ, $0 \leq \theta < 1$, is a dyadic rational number. The mask (with the normalizing condition $P_*(1) = 1$) together with a function φ^1 determine the bases $\{\vec{\varphi}^j\}$ uniquely.*

Proof of Lemma 4. Necessity. Let $P_*(\exp(2\pi i\theta)) = 0$, where in binary representation $\theta = 0.\theta_1\theta_2\ldots\theta_{j+1}$ (for $n = 1, 2, \ldots, j+1$, $\theta_n = 0$ or 1), $\theta_{j+1} = 1$, $j \geq 1$. The number $j+1$ we shall call *an order of* θ. We assume that $a_n = 0$ for $n > \mathcal{N}$, $\vec{a}_* = (a_0, a_1, a_2, \ldots)^T$, $a_k^j \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} a_{k+nJ}$, $\vec{a}^{jT} = (a_0^j, \ldots, a_{J-1}^j)$; $P_j(z) = \sum_{n=0}^{J-1} a_n^j z^n$. We note that for θ of order at most j we have $P_*(\exp(2\pi i\theta)) = P_j(\exp(2\pi i\theta))$.

Let us introduce a number $\varkappa = 1 - \theta = 0.\varkappa_1, \varkappa_2, \ldots, \varkappa_{j+1}$. Since we have

$$\varphi^j(x) = \left(\vec{a}^{j+1}\right)^T \vec{\varphi}^{j+1}(x) = \left(F_{2J}\vec{a}^{j+1}\right)^T F_{2J}^* \vec{\varphi}^{j+1}(x)/2J \stackrel{\text{def}}{=} \left(\vec{\alpha}^{j+1}\right)^T \vec{v}^{j+1}(x),$$

and for $k = \varkappa_1\varkappa_2\ldots\varkappa_{j+1}$ (in the binary representation) $\alpha_k^{j+1} = P_{j+1}(\exp(2\pi i(1-\varkappa)) = P_{j+1}(\exp(2\pi i\theta) = P_*(\exp(2\pi i\theta)) = 0$, then there are no harmonics with numbers of the form $n = k + 2Jm$, $m \in \mathbb{Z}$, in the function φ^j. It follows from this that the same harmonics are absent in the function v_1^1. It contradicts to Theorem 2.

If $P_*(-1) \neq 0$ or $P_*(1) = 0$, then we obviously obtain either $\varphi^0 \not\equiv \text{const}$ or $\varphi^0 \equiv 0$. Necessity is proved.

Sufficiency. We suppose that $\varphi^1 = \alpha_0^1 + \alpha_1^1 v_1^1$, $\alpha_0^1 = 1$, $\alpha_1^1 \neq 0$. Then it is easy to see that coefficients of the expansions $\varphi^{j+1} = \sum_{n=0}^{2J-1} \alpha_n^{j+1} v_n^{j+1}$ can be calculated by the recurrent formula

$$\alpha_n^{j+1} = \alpha_n^j / P_{j+1}(\exp(2\pi i\theta)),$$

where in the binary representation $n = \varkappa_1\varkappa_2 \ldots \varkappa_{j+1} \neq J$, $\varkappa = 0.\varkappa_1\varkappa_2 \ldots \varkappa_{j+1}$, $\theta = 1 - \varkappa$; $\alpha_J^{j+1} = \alpha_{J/2}^j$ $(= \alpha_1^1)$.

Lemma 5. *Let $\{V^j\}$ be an arbitrary MRA. Bases $\vec{\varphi}^j$ of V^j are taken from Theorem 1 and satisfy a refinement equation (1.3) with a finite mask and a characteristic polynomial of order $\mathcal{N} > 0$. Then for this MRA there exist bases $\vec{\phi}^j$, which for any j_1, j_2, $0 \leq j_1 < j_2$ also satisfy the conditions*

$$\phi^{j_1}(J_2/J_1 x) = \sum_{k=0}^{J_2/J_1 - 1} \phi^{j_2}(x + 2\pi k J_1/J_2),$$

and for any $j \geq 0$

$$\phi^j(\bullet) = \sum_{l=0}^{2J_0-1} \alpha_l \phi^{j+1}(\bullet - 2\pi l/2J),$$

such that for characteristic polynomials

$$P_\phi(z) \stackrel{\text{def}}{=} \sum_{l=0}^{2J_0-1} \alpha_l z^l$$

of the bases $\vec{\phi}^j$ we have $|P_\phi(z)|^2 + |P_\phi(-z)|^2 > 0$ for all z. Besides, an order of the polynomial P_ϕ does not exceed $\mathcal{N}$.

An analog of Lemma 5 for MRA of the space $\mathbb{L}^2(\mathbb{R})$ can be found, for instance, in the paper of Chui C. K. and Wang J.Z. [3].

Proof of Lemma 5. Necessity. If the characteristic polynomial

$$P_\varphi(z) \stackrel{\text{def}}{=} \sum_{l=0}^{\mathcal{N}} a_l z^l$$

has not pairs of roots of the form $\pm c$, and $P_\varphi(0) \neq 0$, then the statement of Lemma 5 holds. Let the numbers $z = \pm c_0$ be roots of polynomial P_φ, $C = c_0^2$. Then the representation

$$P_\varphi(z) = (C - z^2) \sum_{l=0}^{\mathcal{N}-2} \gamma_l z^l \tag{4.2}$$

takes place.

We consider a procedure of transfer to a new basis $\vec{\phi}_1^j$ which allows reduce an order of the characteristic polynomial P_φ. We introduce a new function

$$\phi_1^j(\bullet) = \sum_{l=0}^{\mathcal{N}-2} \gamma_l \varphi^{j+1}(\bullet - 2\pi l/2J). \tag{4.3}$$

Since, according to (4.2), $\varphi^j(\bullet) = C\phi_1^j(\bullet) - \phi_1^j(\bullet - 2\pi/J)$, then the collection $\phi_1^j(\bullet - 2\pi l/J)$, $l = 0, 1, \ldots, J-1$, of functions forms a basis of the space V^j for $j \geq 0$.

It is clear that the relationship between functions of basis for distinct j is expressed by equalities

$$\phi_1^j(\bullet) = \sum_{l=0}^{\mathcal{N}-1} \alpha_l' \phi_1^{j+1}(\bullet - 2\pi l/2J),$$

where coefficients α_l' are determined by a formula

$$\sum_{l=0}^{\mathcal{N}-1} \alpha_l' z^l = (C - z) \sum_{l=0}^{\mathcal{N}-2} \gamma_l z^l.$$

We note that the new mask is permissible, since on the one hand new characteristic polynomial has not additional roots. On the other hand $c_0 \neq -1$, because $-(-1) = 1$ is not its root. We shall repeat this procedure while the obtained characteristic polynomial has roots of the form $\pm c$. Since every time we lower a degree of the polynomial by 1, we shall come to the required bases $\vec{\varphi}_*^j$ for a finite number of steps. If $P_{\varphi_*}(0) \neq 0$, we take $\vec{\phi}^j = \vec{\varphi}_*^j$. In the opposite case we suppose $\vec{\phi}^j(\bullet) = \vec{\varphi}_*^j(\bullet + 2\pi/J)$.

We pass to constructing an algorithm of decomposition of functions. Let the collection $\{V^j\}$ forms MRA of a space of $\mathfrak{H}$, $\{\vec{\varphi}^j\}$ be collection of bases of these spaces. Moreover, these bases satisfy (1.1), and a characteristic polynomial has an order $\mathcal{N}$, and $|P_\varphi(z)|^2 + |P_\varphi(-z)|^2 > 0$ for all z.

We show that there exists a decomposition of the space V^n in a direct (generally speaking, nonorthogonal) sum of the wavelet spaces $\{W^j\}_{j=0}^{n-1}$ for which in a some basis algorithms of decomposition and reconstruction can be realized as a discrete convolution with a window of size at most $\mathcal{N}+1$. Moreover, the convoluting windows, setting projectors to the spaces W^j and V^j and reconstructing a function $g \in V^{j+1}$ by its projection to these spaces, can be chosen independent of j. It means that for an arbitrary function

$$f(\bullet) = \sum_{m=0}^{N-1} c_m^n \varphi^n(\bullet - 2\pi m/N) \in V^n$$

it is possible to realize decomposition

$$f(\bullet) = \sum_{m=0}^{J-1} c_m^j \varphi^j(\bullet - 2\pi m/J) + \sum_{l=j}^{n-1} \sum_{m=0}^{L-1} d_m^l \psi^l(\bullet - 2\pi m/L),$$

where $\vec{\psi}^l$ are bases of the spaces W^l, and coefficients $\left\{\{c_m^j\}_{m=0}^{J-1}\right\}_{j=0}^{n-1}$ and $\left\{\{d_m^j\}_{m=0}^{J-1}\right\}_{j=0}^{n-1}$ can be calculated by recurrent formulae

$$c_m^{j-1} = \sum_{l=\mathcal{K}}^{\mathcal{K}+\mathcal{N}} c_{l+2m}^j r_l, \qquad d_m^{j-1} = \sum_{l=\mathcal{L}}^{\mathcal{L}+\mathcal{N}} c_{l+2m}^j s_l, \tag{4.4}$$

where $\mathcal{K}, \mathcal{L} \in \mathbb{Z}$. In the following we shall prove that an upper limit of summation in the first formula can be reduced by 1. The sequences c_m^j and d_m^j are J-periodic by a subscript, and we suppose that the sequences r_l and s_l are determined for all integer indexes but they differ from zero only if $l = \mathcal{K}, \mathcal{K}+1, \ldots, \mathcal{K}+\mathcal{N}-1$ for the first sequence and if $l = \mathcal{L}, \mathcal{L}+1, \ldots, \mathcal{L}+\mathcal{N}$ for the second one. Keeping a size of windows we can move it, changing the parameter $\mathcal{K}$ and obtaining new solutions of our problem. A unique requirement for a location of a window r_l consists in that it should to have nonempty intersection with the segment $[0, \mathcal{N}]$ or with its shift by lN, where $l \in \mathbb{Z}$. In the opposite case the first of operators (4.4) for $j = n$ transform the function φ^{n-1} to a function of the form

$$\sum_{l=0}^{N/2-1} \alpha_l \varphi^{n-1}(\bullet - 2\pi \cdot 2l/N),$$

where $\alpha_0 = 0$, that is impossible since it is a projection operator of V^n to V^{n-1}. We take an arbitrary number $\mathcal{K}$ belonging to the segment $[1 - \mathcal{N}, \mathcal{N}]$.

We shall construct the spaces W^j starting from conditions that their bases $\vec{\psi}^j$ are determined by the relations

$$\psi^{j-1}(\bullet) = \sum_{m=0}^{\mathcal{N}} b_m \varphi^j(\bullet - 2\pi m/J),$$

where we have to find the coefficients b_m. Since relations (4.4) specify operators of projection to V^{j-1} and to W^{j-1}, for $m = 0, 1, \ldots, N/2-1$ the equalities

$$\delta_{0m} = \sum_{l=\mathcal{K}}^{\mathcal{K}+\mathcal{N}-1} a_{l-2m} r_l, \tag{4.5}$$

$$\delta_{0m} = \sum_{l=\mathcal{L}}^{\mathcal{L}+\mathcal{N}} b_{l-2m} s_l, \tag{4.6}$$

$$0 = \sum_{l=\mathcal{L}}^{\mathcal{L}+\mathcal{N}} a_{l-2m} s_l, \tag{4.7}$$

$$0 = \sum_{l=\mathcal{K}}^{\mathcal{K}+\mathcal{N}-1} b_{l-2m} r_l \tag{4.8}$$

hold (naturally, we assume that the sequences $\{a_l\}$ and $\{b_l\}$ are N-periodic, and on a period outside of its domain of definition they are completed by zero). It is well-known (see, for example, [3]) and it can be easy verified, that to solve the given system of convolution equations it suffice to solve the following system of equations

$$A(z)R(z) + A(-z)R(-z) \equiv 2, \tag{4.9}$$

$$B(z)S(z) + B(-z)S(-z) \equiv 2, \tag{4.10}$$

$$A(z)S(z) + A(-z)S(-z) \equiv 0, \tag{4.11}$$

$$B(z)R(z) + B(-z)R(-z) \equiv 0, \tag{4.12}$$

where

$$A(z) = \sum_{l=0}^{\mathcal{N}} a_l z^l, \quad B(z) = \sum_{l=0}^{\mathcal{N}} b_l z^l,$$

$$R(z) = \sum_{l=\mathcal{K}}^{\mathcal{K}+\mathcal{N}-1} r_l z^{-l} = z^{-\mathcal{K}-\mathcal{N}+1} \sum_{l=0}^{\mathcal{N}-1} r_{\mathcal{K}+\mathcal{N}-1-l} z^l \stackrel{\text{def}}{=} z^{-\mathcal{K}-\mathcal{N}+1} \check{R}(z),$$

$$S(z) = \sum_{l=\mathcal{L}}^{\mathcal{L}+\mathcal{N}} s_l z^{-l} = z^{-\mathcal{L}-\mathcal{N}} \sum_{l=0}^{\mathcal{N}} s_{\mathcal{L}+\mathcal{N}-l} z^l \stackrel{\text{def}}{=} z^{-\mathcal{L}-\mathcal{N}} \check{S}(z).$$

Equations (4.9) – (4.12) can be rewritten in the form

$$A(z)\check{R}(z) + (-1)^{\mathcal{K}+\mathcal{N}-1} A(-z)\check{R}(-z) = 2z^{\mathcal{K}+\mathcal{N}-1}, \tag{4.13}$$

$$B(z)\check{S}(z) + (-1)^{\mathcal{L}+\mathcal{N}} B(-z)\check{S}(-z) = 2z^{\mathcal{L}+\mathcal{N}}, \tag{4.14}$$

$$A(z)\check{S}(z) + (-1)^{\mathcal{L}+\mathcal{N}} A(-z)\check{S}(-z) = 0, \tag{4.15}$$

$$B(z)\check{R}(z) + (-1)^{\mathcal{K}+\mathcal{N}-1} B(-z)\check{R}(-z) = 0, \tag{4.16}$$

According to Lemma 3 and by choice of $\mathcal{K}$, there exists the unique pair of polynomials Q_1 and Q_2 of order at most $\mathcal{N}-1$, satisfying the equation

$$A(z)Q_1(z) + (-1)^{\mathcal{K}+\mathcal{N}-1} A(-z)Q_2(z) = 2z^{\mathcal{K}+\mathcal{N}-1}.$$

As this takes place, evidently, $Q_1(z) = Q_2(-z)$. Thus, the polynomial $\check{R}(z) = Q_1(z)$ satisfies equation (4.13).

If $\mathcal{K}+\mathcal{N}$ is odd, then it is not hard to verify that the functions $B(z) = z\check{R}(-z)$ and $\check{S}(z) = A(-z)$ satisfy equations (4.13) — (4.16). In the case of even $\mathcal{K}+\mathcal{N}$ the functions $B(z) = \check{R}(-z)$ and $\check{S}(z) = -A(-z)$ are solutions of these equations.

In spite of apparent arbitrariness of a choice of the polynomials B and $\check{S}$, the spaces W^j are completely determined by a choice of the polynomial $\check{R}$, since coefficients of this polynomial completely determines an operator of projection to the space V^j along the space W^j.

To show that the spaces W^j actually has a dimension J we prove that the reconstruction formulae

$$c^j_m = \sum_{l=0}^{N/2-1} c^{j-1}_l a_{m-2l} + \sum_{l=0}^{N/2-1} d^{j-1}_l b_{m-2l}, \tag{4.17}$$

which invert an operators of projection of the space V^j to the spaces W^{j-1} and V^{j-1}, hold. Hence,

$$\dim W^{j-1} = \dim V^j - \dim V^{j-1} = J - J/2 = J/2.$$

It suffices to make sure that formula (4.17) is valid for $j = n$. We denote by A a matrix of size $2^{n-1} \times 2^n$ such that its first row consists of the elements $a_0, a_1, \dots$, and the next row is obtained from the previous one by double cyclic shift to the right. In an analogous way we construct the matrices B, R, S. Then system (4.5) — (4.8) of equations can be rewritten in the matrix form

$$\begin{pmatrix} A \\ B \end{pmatrix} (R^T S^T) = E.$$

Consequently, the matrix $(R^T S^T)$ has the inverse matrix $\begin{pmatrix} A \\ B \end{pmatrix}$. Hence, it is nonsingular and an operator of reconstruction is determined by (4.17).

The presented construction of nonorthogonal wavelet decomposition of the space V^n solves a problem of minimization of computational expenditures, leaving aside a problem of "angles" between wavelet spaces. It is clear that, increasing a width of convolution window (parameter $\mathcal{N}$) we can exert a certain influence to mutual disposition (in particular, to degree of nonorthogonality for the case $V^j \in \mathbb{L}^2$) of wavelet spaces. The dependence of mutual location of the spaces $\{W^j\}$ on a size of a convolution window has need for an additional consideration.

It is self-evident that for large $\mathcal{N}$ the approach based on transfer to the Wiener bases, described in §3, is preferable.

References

1. de Boor C., DeVore R.A., and Ron A., *On the construction of multivariate (pre)wavelets*, Constr. Approx. **9** (1993), no. 2 – 3, 123 – 166.
2. Chui C.K.and Mhaskar H.N., *On trigonometric wavelets*, Constr. Approx. **9** (1993), no. 2 – 3, 167 – 190.
3. Chui C.K. and Wang J.Z., *A general framework of compactly supported splines and wavelets*, J. Approx. Th. **71** (1992), 263 – 304.
4. Daubechies I., *Ten lectures on wavelets*, SIAM, Philadelphia, 1992.
5. Daubechies I., *Orthonormal basis of compactly supported wavelets*, Comm. Pure Appl. Math. **41** (1988), 909 – 996.
6. Dahmen W. and Micchelli C.A., *Banded matrices with banded inverses II: Locally finite decomposition of spline spaces*, Constr. Approx. **9** (1993), no. 2 – 3, 263 – 281.
7. Edwards R.E., *Fourier series. A modern introduction*, Springer–Verlag, New York – Heidelberg – Berlin, 1979.
8. Kamada M., Toraici K., and Mori Riochi, *Periodic splines orthonormal bases*, J. Approx. Th. **55** (1988), 27 – 34.
9. Meyer Y., *Wavelets and operators*, Cambridge University Press, 1992.
10. Perrier V. and Basdevant C., *La décomposition en ondelettes périodiques, un outil pour l'analyse de champs inhomogènes, Théorie et algorithmes*, La Recherche Aerospatiale (1989), no. 3, 53 – 67.
11. Petukhov A.P., *On approximation of functions by operators of convolution in the Hausdorff metric*, Dokl. Rossiisk. Akad. Nauk **327** (1994), 187–190. (Russian)
12. Petukhov A.P., *On approximation of functions by convolution operators in the Hausdorff metric*, Izv. Rossiisk. Akad. Nauk (1995) (to appear). (Russian)

13. Zheludev V.A., *Periodic splines and wavelets*, Proc. of the Conference "Math. Analysis and Signal processing", Cairo, Jan. 2–9, 1994.
14. Zheludev V.A., *Operational calculus connected with periodic splines*, Dokl. Akad. Nauk SSSR **313** (1990), no. 6, 1309 – 1315. (Russian)
15. Holschneider M., *Wavelets: an analysis tool*, Clarendon Press. Oxford., 1995.
16. Plonka G., Tasche M., *On the computation of periodic spline wavelets*, Appl. and Comp. Harmonic Analysis **2** (1995), 1 – 14.
17. Koh Y.W., Lee S.L., Tan H.H., *Periodic orthogonal splines and wavelets*, Appl. Comp. Harmonic Anal. **2** (1995), 201 – 218.
18. Goh S.S., Lee S.L.,Shen Z., Tang W.S., *Construction of Schauder decomposition on Banach spaces of periodic functions* (1996), Preprint.

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