PERIODIC DISCRETE WAVELETS[1]

A. P. Petukhov

Abstract. A definition of a multiresolution analysis (MRA) on a discrete periodic net is given, and a theory of MRA is constructed. We consider two possible approaches to the definition of the wavelet spaces associated with a fixed MRA. Effective numerical algorithms of wavelet expansion and reconstruction of functions of a discrete argument are presented.

§0. Introduction

Let $N = 2^n$, where $n \in \mathbb{N}$. We consider the space $\widetilde{\mathbb{C}}^N$ of N-periodic complex-valued functions of a discrete variable. The paper is devoted to the construction of a *multiresolution analysis* (MRA) $\{V^j\}_{j=0}^n$ and to *wavelet decompositions* $V^0 \oplus W^0 \oplus \cdots \oplus W^{n-1}$ of the space $\widetilde{\mathbb{C}}^N$. Also, we find some economical algorithms of expansion and reconstruction of functions.

Wavelet decompositions have numerous applications to signal and image processing and to compressing information.

The known algorithms of wavelet expansion of functions (see [1–9]) are based on the assumption that the approximated functions have continuous argument. It is natural that the basis functions also turn out to be functions of a continuous argument. However, information for digital processing is discrete; for representing and processing such information it is appropriate to construct a theory of discrete wavelet bases.

The construction of an MRA of the space of N-periodic discrete functions involves some specific problems, which could not arise in the study of wavelets on $\mathbb{R}$. We list some of such problems.

A priori it is not clear how dilation of the argument of a function should be understood, because the function may become indefinite at the intermediate points. On the contrary, if we "compress" the argument, then some "extra" points arise. Is it true that the bases of the spaces forming an MRA are generated by translates and dilations of a single function? This question looks especially difficult in the periodic case considered here because, after dilation, we may obtain a $2N$-periodic function that we must make N-periodic. Also, it seems important that every MRA $\{V^j\}$ be uniquely recovered by any of the subspaces V^j.

All these problems find satisfactory solutions in the framework of the MRA theory proposed in §1.

At the same time, apparently there exist at least two possible approaches to the definition of wavelet spaces associated with a fixed MRA. Generally speaking, these approaches lead to different objects. The first is treated in §2, where we define a wavelet space W^j as the orthogonal complement of V^j in V^{j+1}. Such a definition looks natural, but it turns out that this collection of wavelet spaces does not possess some properties usual in the theory of wavelets in $\mathbb{L}^2(\mathbb{R})$. For example, dilation of the argument of a function belonging to W^{j+1} may lead to a function not belonging to W^j. For this and some other reasons, in §4 we consider another approach to wavelets. Among these reasons we mention the existence of economical algorithms, including those admitting parallel algorithms.

The spaces V^j forming an MRA and the spaces W^j (for both definitions) are invariant with respect to the shifts of the argument by $N/J = 2^{n-j}$. Therefore, it may be expected that the orthogonal bases consisting of eigenvectors of the operator of (N/J)-shift are of importance. V. A. Zheludev [2] applied such bases to the theory and numerical algorithms for the MRA generated by periodic splines of minimal defect. Previously, V. A. Zheludev [3] had used these bases for constructing numerical algorithms of spline approximation. These bases turned out to be useful for theoretical study of the properties of MRA as well as for constructing

1991 *Mathematics Subject Classification.* Primary 42C15, 41A58.

Key words and phrases. Multiresolution analysis, wavelets, discrete Fourier transformation, algorithms of wavelet expansion.

[1]St.Petersburg Math. J. Vol. 8(1997), No. 3, 481 – 503 (translated from Russian)

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

numerical algorithms. Actually, the operator realizing the passage to such bases plays the same role as the discrete Fourier transformation in the space $\widetilde{\mathbb{C}}^N$, and the coefficients of the corresponding expansions play the role of the spectrum. The passage to such bases underlies most of the proofs presented in §§1 and 2.

In §3, for orthogonal wavelet spaces we consider an economical expansion algorithm in the case where the spaces V^j are generated by bases of functions with *local support*. Here we use the term "local" because the term "bases of compactly supported wavelets" (traditional for MRA in $\mathbb{L}^2(\mathbb{R})$) makes no sense in the periodic case. In §4 we show that if some bases of the spaces V^j and V^{j+1} satisfy a recurrence relation of the form

$$\varphi^j(\cdot) = \sum_{k=0}^{p} a_k \varphi^{j+1}(\cdot + kN/2J),$$

then algorithms of expansion and reconstruction can be represented as discrete convolutions with window of size of order p. However, this requires that we lift the condition of the mutual orthogonality of the spaces W^j. In general, applying the procedure of sampling to orthogonal bases (of continuous function) for MRA in $\mathbb{L}^2(\mathbb{R})$, we lose the orthogonality of the elements of the bases, as well as of the spaces W^j. Therefore, in our opinion, the fact that our wavelet spaces may fail to be orthogonal is not a defect of the construction.

§1. Multiresolution analysis of the space of discrete periodic functions

The space $\widetilde{\mathbb{C}}^N$ will be equipped with the following natural inner product:

$$(f, g) = (f(\cdot), g(\cdot)) = \sum_{j=0}^{N-1} f(j)\overline{g}(j).$$

For a matrix (or a vector) C, the operation * means transposition and complex conjugation, i.e., $C^* = \overline{C}^T$. We write $C = (A(\cdot), B(\cdot))$, where A and B are functional matrices or vectors, if the matrix $A(\cdot)$ is multiplied by $B^T(\cdot)$, and then the inner product is calculated, i.e., $c_{km} = \sum_l (a_{kl}(\cdot), b_{ml}(\cdot))$. For convenience, in what follows we enumerate the entries of matrices and vectors starting with 0. Unless otherwise specified, we assume $N = 2^n$, $J = 2^j$, $L = 2^l$, and so on.

We shall use vector-matrix notation for the convolution operators and their generalizations. The matrices arising here are either circulant matrices of size $K \times K$ (their elements are of the form $a_{ij} = f(j-i)$, where $f(\cdot) \in \widetilde{\mathbb{C}}^K$), or rectangular matrices of size $K \times 2K$ that are *generalized circulant with double shift* ($a_{ij} = f(j-2i)$, $f(\cdot) \in \widetilde{\mathbb{C}}^{2K}$) and their transposes. In all the above cases we use the term *convolution with a window of size* p if the support of $f(\cdot)$ on a period can be placed on a segment of length $p-1$.

Definition 1. A sequence of linear spaces $\{V^j\}_{j=0}^n$ is called a *multiresolution analysis* of the space $\widetilde{\mathbb{C}}^N$ if the following conditions are satisfied:

1. $V^0 \subset V^1 \subset \cdots \subset V^j \subset \cdots \subset V^n = \widetilde{\mathbb{C}}^N$; $\dim V^j = 2^j$, $j = 0, 1, 2, \ldots, n$.
2. a) V^0 consists of constants.
   b) If $f(\cdot) \in V^j$, then $g(\cdot) = f(2\cdot) \in V^{j+1}$.
   c) If $f(\cdot) \in V^{j+1}$, then there exists $g(\cdot) \in V^j$ such that $g(2\cdot) = f(\cdot) + f(\cdot + N/2)$.
   d) The spaces V^j are invariant under the shift by 2^{n-j}, $j = 0, 1, 2, \ldots, n$, i.e., for any function $f \in V^j$ and any $k \in \mathbb{Z}$ we have $f(\cdot + k2^{n-j}) \in V^j$.
   e) For any $1 \le j < n$, every function $f \in V^{j+1}$ is representable in the form

$$f(\cdot) = f_1(\cdot) + f_2(\cdot + k2^{n-j-1}) + f_3(2\cdot),$$

where $k \in \mathbb{N}$, f_1, f_2, $f_3 \in V^j$.

Remark. With the exception of condition 2e), all conditions of Definition 1 are similar to those occurring in the literature on wavelets with continuous argument (see, for example, [4]). We included condition 2e) because we wished that a MRA be uniquely recovered by any space V^j; the remaining conditions do not ensure this property. For instance, it is easy to show that the periodic MRA constructed in [5] does not possess this property.

For $\varphi^l \in V^l$ we denote by $\vec{\varphi}^l$ the vector-function

$$\vec{\varphi}^l(\cdot) = \left(\varphi^l(\cdot), \varphi^l(\cdot - N/L), \ldots, \varphi^l(\cdot - (L-1)N/L)\right)^T, \quad l = 0, 1, 2, \ldots, n.$$

Theorem 1. *Let $\{V^j\}_{j=0}^n$ be an arbitrary MRA of the space $\widetilde{\mathbb{C}}^N$. Then for any fixed $0 \le l \le n$ and any function φ^l such that the components of $\vec{\varphi}^l$ form a basis of the space V^l there exists a family φ^j, $0 \le j \le n$, of functions $\varphi^j \in V^j$, such that the components of $\vec{\varphi}^j$ form bases of the spaces V^j. Moreover, for any j_1, j_2, $0 \le j_1 < j_2 \le n$, we have*

$$\varphi^{j_1}(J_2 m/J_1) = \sum_{k=0}^{J_2/J_1-1} \varphi^{j_2}(m + kNJ_1/J_2). \tag{1.1}$$

For $j < l < n$, the functions φ^j are uniquely determined by φ^l.

Before proving Theorem 1, we introduce some notation and formulate and prove some auxiliary results.

For J fixed, we put $\omega_J = \exp(2\pi i/J)$ and denote by F_J the matrix of the discrete Fourier transformation (DFT) (the elements of F_J are $F_{lk} = \omega_J^{-lk}$, $l = 0, 1, \ldots, J-1$, $k = 0, 1, \ldots, J-1$). If no confusion concerning dimensions is possible, we shall omit the subscript J. The matrix of the inverse DFT (IDFT) is F_J^*/J. We put $e_{J,k}(m) = \omega_J^{km}$, $k = 0, 1, 2, \ldots, J-1$. It is well known that the functions $e_{J,k}$ form a basis of the space $\widetilde{\mathbb{C}}^J$, and that their restrictions to the period $[0, J)$ are the columns (and rows) of the matrix F_J^*. The functions $e_{N,k}(\cdot)$ will be called *harmonics*. As usual, for an arbitrary function $f(\cdot) \in \widetilde{\mathbb{C}}^N$ we denote by $\widehat{f}(\cdot)$ the spectrum (discrete Fourier transform)

$$\widehat{f}(k) = \sum_{m=0}^{N-1} f(m)\omega_N^{-mk}, \quad k = 0, 1, \ldots, N-1.$$

Lemma 1. *For $h(\cdot) \in \widetilde{\mathbb{C}}^N$, we put $\vec{h} \stackrel{\text{def}}{=} (h(\cdot), h(\cdot - N/J), \ldots, h(\cdot - (J-1)N/J))^T$, $\vec{\eta}(\cdot) \stackrel{\text{def}}{=} F^*\vec{h}(\cdot)/J$. Then the dimension of the linear span of the components of the vector $\vec{h}(\cdot)$ is equal to the number of all nonzero components of the vector $\vec{\eta}(\cdot)$. The nonzero components of $\vec{\eta}(\cdot)$ are mutually orthogonal and can be calculated by the formulas*

$$\eta_k(\cdot) = \frac{1}{N} \sum_{m=0}^{N/J-1} \widehat{h}(k + Jm) \exp(2\pi i(k + Jm) \cdot /N), \quad k = 0, 1, \ldots, J-1. \tag{1.2}$$

Proof of Lemma 1. Let Γ be the Gram matrix of the elements of the vector $\vec{\eta}$ and G the Gram matrix of the functions forming the vector $\vec{h}$. Then

$$\Gamma = (\vec{\eta}(m), \vec{\eta}(m)) = F^* \cdot (\vec{h}(m), \vec{h}(m)) \cdot F/J^2 = F^* \cdot G \cdot F/J^2.$$

Since G is a circulant matrix, the matrix Γ is diagonal. On the other hand, since F is nonsingular, the rank of Γ coincides with that of G, and the latter rank is equal to the dimension of the linear span of the components of $\vec{h}$.

We prove (1.2):

$$\begin{aligned}
\widehat{\eta}_k(l) &= \sum_{p=0}^{N-1} \left(\frac{1}{J} \sum_{m=0}^{J-1} h(p - mN/J) e^{2\pi i k m/J} \right) e^{-2\pi i l p/N} \\
&= \frac{1}{J} \sum_{m=0}^{J-1} \left(\sum_{p=0}^{N-1} h(p - mN/J) e^{-2\pi i l p/N} \right) e^{2\pi i k m/J} \\
&= \frac{1}{J} \sum_{m=0}^{J-1} \left(\sum_{p=0}^{N-1} h(p - mN/J) e^{-2\pi i l(p - mN/J)/N} \right) e^{2\pi i (k-l)m/J} \\
&= \widehat{h}(l) \frac{1}{J} \sum_{m=0}^{J-1} e^{2\pi i (k-l)m/J} = \begin{cases} \widehat{h}(l), & l = k + qJ \\ 0, & l \ne k + qJ,\ q \in \mathbb{Z}. \end{cases}
\end{aligned}$$

□

The following statement is an obvious consequence of Lemma 1.

Corollary. *Under the conditions of Lemma* 1, *let* $g(\cdot) \in \widetilde{\mathbb{C}}^N$. *The linear span of the components of the vector* $\vec{g}\,(\cdot) = (g(\cdot), g(\cdot - N/J), \dots, g(\cdot - (J-1)N/J))^T$ *coincides with the linear span of those of* $\vec{h}$ *if and only if* $g(\cdot)$ *is a linear combination of nonzero components of* $\vec{\eta}$ *with nonzero coefficients.*

Thus, if $V \subset \widetilde{\mathbb{C}}^N$ is the linear span of the components of a vector $\vec{h}\,(\cdot)$ and $\dim V = J$, then there exists a basis of V consisting of elements of the form (1.2). These elements have mutually disjoint supports; so, they are mutually orthogonal. Besides, they fit into the pattern of the following definition.

Definition 2. A collection of functions $f_0, f_1, \dots, f_{J-1}$ belonging to a linear space $V \subset \widetilde{\mathbb{C}}^N$ is called a *Wiener basis* of V if the following discrete periodic analog of the Wiener theorem is true: a collection $\{g(\cdot - kN/J)\}_{k=0}^{J-1}$ is a basis of V if and only if g can be represented in the form $g = \sum_{k=0}^{J-1} \alpha_k f_k$, where all α_k are nonzero.

Clearly, such a basis is unique up to multiplication of its elements by nonzero factors and permutation. Formula (1.2) shows that the elements of this basis can be enumerated in a natural way; this enumeration is associated with the spectrum components involved in the corresponding elements, i. e., in the expansion of the functions $\eta_k(\cdot)$, $0 \le k < J$, with respect to the basis $\{e_{N,k}\}_{l=0}^{N-1}$ only the coefficients with the indices of the form $l = k + Jm$, $m = 0, 1, 2, \dots, N/J - 1$ are nonzero.

The Wiener basis has been obtained as the image of the components of the vector $\vec{h}$ under the IDFT. However, in accordance with (1.2), a Wiener basis can be introduced as an initial notion. The existence of a Wiener basis is necessary and sufficient for the existence of a function g such that the corresponding vector of shifts $\vec{g}$ constitutes a basis of V.

There is one more useful property of a Wiener basis, namely, the components of it are eigenfunctions of the operator of the shift by (N/J). For this reason, the coefficients in the expansion of a function with respect to a Wiener basis can be viewed as the spectrum of that operator. In particular, for a space V having a Wiener basis the following theorem on convolution holds: *The spectrum of the convolution of functions* (*with respect to the shift by* N/J) *is equal to the pointwise product of their spectra.*

As mentioned in the Introduction, in the papers [2,3] V. A. Zheludev proposed to employ such a basis for solving theoretical and numerical problems of periodic spline-approximation and periodic spline MRA.

Lemma 2. *For an arbitrary MRA of* $\widetilde{\mathbb{C}}^N$, *in each of the spaces* V^j *there exists a function* φ^j *such that the collection* $\{\varphi^j(\cdot - kN/J)\}_{k=0}^{J-1}$ *is a basis of* V^j. *The MRA can be recovered uniquely by any of the* V^j $(0 < j < n)$.

Proof of Lemma 2. By Lemma 1, the existence of a basis of translates will be proved if we check the existence of a Wiener basis $\{v_k^j\}_{k=0}^{J-1}$ for any j. We proceed by induction.

Obviously, for $j = 0$, as the above v_0^0 we can take the function identically equal to 1. Let $j = 1$, and let $h(\cdot) \equiv 1$; then $h \in V^1$. Let g be an arbitrary nonzero function in V^1 orthogonal to h. The functions h and g form a basis of V^1. We show that they form a Wiener basis. Indeed, since h and g are orthogonal, we have $g(m) = \sum_{k=1}^{N-1} \alpha_k \omega_N^{km}$. Moreover, using 2a) and 2c) we obtain

$$g(m) + g(m + N/2) = 2 \sum_{k=1}^{N/2-1} \alpha_{2k} \omega_N^{2km} \equiv \text{ const.}$$

Consequently, all α_{2k} with $k \neq 0$ are equal to zero, and there is at least one nonzero α_k with odd k. Later we shall prove that all α_k with odd k are nonzero. Thus, we put $v_0^1 = h$, $v_1^1 = g$.

Now we assume that the required Wiener bases have already been constructed for $j = 0, 1, \dots, p < n$. We show that such a construction is possible for $j = p + 1$.

Consider an arbitrary function $f_1 \in V^{p+1}$. If the dimension of the linear span $\mathcal{E}_1$ of the components of the vector-function

$$\vec{f_1}(\cdot) = (f_1(\cdot), f_1(\cdot - N/2P), \dots, f_1(\cdot - (2P-1)N/2P))^T$$

is equal to $2P$, then the components of the inverse discrete Fourier transform $F_{2P}^* \vec{f_1}/2P$ generate a subspace of dimension $2P$. Consequently, $\vec{\Phi}_1$ is a Wiener basis of the space V^{p+1}. If $\dim \mathcal{E}_1 < 2P$, then we take an arbitrary function g off $\mathcal{E}_1$ and consider the linear span $\mathcal{E}_2$ of the components of the vector-functions $\vec{f_1}$ and

$$\vec{f_2}(\cdot) = (f_2(\cdot), f_2(\cdot - N/2P), \dots, f_2(\cdot - (2P-1)N/2P))^T.$$

We supplement the system $\{f_k\}$ with new functions until the linear span obtained coincides with V^{p+1}. Now we can choose a basis of V^{p+1} consisting of the components of $\{\vec{\varphi}_k\}$, $\vec{\varphi}_k = F^*_{2P}\vec{f}_k/2P$. We show that this is a Wiener basis. This will follow from the fact that for every $l = 0, 1, \dots, 2P-1$ this basis contains precisely one element of the form

$$\sum_{m=0}^{N/2P-1} \alpha_{l,m} \exp(2\pi i(l + 2mP)\cdot/N).$$

Lemma 1 shows that all elements of the basis are of that form. Since by property 1 in the definition of a MRA we have $\dim V^{p+1} = 2^{p+1}$, it suffices to prove that for every l there exists at least one element of that form.

Suppose the contrary. This means that for some l there is no such element in the basis. If $p = n-1$, we arrive at a contradiction with property 1 mentioned above. So, we assume that $p < n-1$. We show that in this case the space V^{p+2} contains no elements of the form $f(\cdot) = \sum_{m=0}^{N/4P-1} \alpha^1_{l,m} \exp(2\pi i(2l + 4mP)\cdot/N)$. It is not hard to check that the action of the operator A of $(N/2)$-periodization,

$$A\colon h(\cdot) \mapsto h(\cdot) + h(\cdot + N/2),$$

consists in eliminating the odd harmonics and in doubling the evens ones. At the same time, deleting the odd m, on the remaining net we have $e_{N,k}(m) \equiv e_{N,k+N/2}(m)$. Moreover, the functions $\{e_{N,k}\}_{k=0}^{N/2-1}$ form a basis of the space of functions on that net. Thus, any function $g(\cdot)$ satisfying $g(2\cdot) = f(\cdot) + f(\cdot + N/2)$ involves only harmonics with indices of the form $l + 2mP$, and these harmonics are absent in V^{p+1}. Consequently,

$$2\alpha^1_{l,m} = c(\alpha_{l,m} + \alpha_{l,m+N/2}) = 0.$$

Increasing subsequently the index of the space, we see that the space V^n does not contain the function $\exp(2\pi i 2^{n-p-1} l\cdot/N)$, which contradicts property 1 in Definition 1. Thus, each space V^p has a Wiener basis.

We show by induction that for any $p \geq 1$ the Wiener basis of V^{p+1} can be expressed in terms of the elements of the Wiener basis of V^p by the formulas

$$\widehat{v}^{p+1}_k(k + 2Pl) = \widehat{v}^p_k(k + 2Pl), \qquad k = 1, 3, \dots, 2P-1, \quad l = 0, 1, \dots, N/2P - 1. \tag{1.3}$$

$$\widehat{v}^{p+1}_k(k + 2Pl) = \widehat{v}^p_{k/2}(k/2 + Pl) + \widehat{v}^p_{k/2}(k/2 + Pl + N/2), \qquad k = 0, 2, \dots, 2P-2, \quad l = 0, 1, \dots, N/2P - 1, \tag{1.4}$$

where $P = 2^p$ and $v^p_{k+P} \equiv v^p_k$ for any k. It is easy to observe that, except for $k = P$, for all even k these formulas give the same results. Next, obviously, formula (1.4) can be written in the form

$$v^{p+1}_k(\cdot) = v^p_{k/2}(2\cdot), \quad k = 0, 2, \dots, 2P-2. \tag{1.4$'$}$$

Let $j = 1$. Since $V^1 \subset V^2$, if k is odd (i. e., $k = 1, 3$) and if at least one of the numbers $\{\widehat{v}^1_k(k + 4l)\}_{l=0}^{N/4-1}$ is nonzero, we obtain (1.3) up to renormalization. We show that these numbers cannot be simultaneously zero. Let f be the kth element of the Wiener basis of the space V^2. Condition 2e) in Definition 1 implies that f can be represented in the form

$$f(\cdot) = f_1(\cdot) + f_2(\cdot + m2^{n-2}) + f_3(2\cdot), \tag{1.5}$$

where $f_1, f_2, f_3 \in V^1$. If the numbers $\widehat{v}^1_k(k + 4l)$ are all zero, the first two terms in (1.5) cannot involve the harmonics corresponding to these numbers, and the third term in (1.5) involves only even harmonics. Consequently, $f(\cdot) \equiv 0$, but this is impossible since the existence of a Wiener basis of V^p has already been proved.

The validity of (1.4) for $k = 0$ follows from condition 2b), and for $k = 2$ formula (1.4) is a direct consequence of conditions 2b) and 2e). Indeed, if the function

$$v_1^1(2\cdot) = \frac{2}{N}\sum_{k=0}^{N/4-1}\left(\widehat{v}_1^1(2k+1) + \widehat{v}_1^1(2k+1+N/2)\right)e^{2\pi i(4k+2)\cdot/N}$$

is not equal to zero identically, the validity of (1.4) follows from condition 2b). Now, suppose that the above function is equal to zero identically. By Lemma 1, the function v_2^2 contains only harmonics with indices of the form $4k+2$. We represent the function v_2^2 as a sum of three functions, as in 2e). Then the required harmonics are absent in the first two terms, because there are no such harmonics in V^1 (we recall that $v_0^1 \equiv 1$). The absence of them in the third term follows from the relation $v_1^1(2\cdot) \equiv 0$. Consequently, $v_2^2 \equiv 0$, which contradicts the existence of a Wiener basis of V^2.

Assume that (1.3) and (1.4) have been proved for the spaces V^j, $j < p$. We do the induction step for arbitrary p.

Since $V^p \subset V^{p+1}$, if k is odd and at least one of the numbers $\widehat{v}_k^p(k+2Pl)$ is nonzero, we obtain (1.3) up to renormalization. The fact that all these numbers cannot be zero simultaneously is proved as in the case $p = 1$.

For $k = 0$ or $k = P$, formula (1.4) can be checked as in the case where $p = 1$ and k is even. We consider the case of an arbitrary even $k = 2m$, $k \neq 0$, $k \neq p$. If at least one of the numbers

$$\left\{\widehat{v}_{k/2}^p(k/2+Pl) + \widehat{v}_{k/2}^p(k/2+Pl+N/2)\right\}_{l=0}^{N/2P-1}$$

is nonzero, then (1.4) follows from condition 2b). We show that these numbers cannot be zero simultaneously. Indeed, using (1.3) and (1.4) with parameters for which these formulas have already been proved, we obtain

$$\begin{aligned}\widehat{v}_k^p(k+2Pl) &= \widehat{v}_{k/2}^{p-1}(k/2+Pl) + \widehat{v}_{k/2}^{p-1}(k/2+Pl+N/2)\\ &= \widehat{v}_{k/2}^p(k/2+Pl) + \widehat{v}_{k/2}^p(k/2+Pl+N/2) = 0.\end{aligned}$$

Hence, the harmonics involved in the function v_k^{p+1} are absent in V^p, in contradiction with condition 2e) in Definition 1.

Thus, we have shown that each of the spaces V^j contains a function φ^j such that the components of the vector $\vec{\varphi}^j$ form a basis and that any space with a greater index is uniquely recovered by the spaces with smaller indices.

Now we prove that any space V^j can be recovered by the space V^{j+1}. First observe that there are no harmonics with indices $J, 2J, \ldots, N-J$ in the space V^j (and in the corresponding elements of the Wiener basis). This is a consequence of (1.3), since this formula shows that a gap in the spectrum of the space with index j arises in all spaces with greater indices, which contradicts the condition $V^n = \widetilde{\mathbb{C}}^N$.

Let $1 < j+1 < n$, $1 \le k \le J-1$. Since $V^j \subset V^{j+1}$ and since the existence of the Wiener bases has already been proved for all the spaces V^j, there exist numbers α and β such that

$$v_k^j = \alpha v_k^{j+1} + \beta v_{k+J}^{j+1}.$$

Using (1.2), we obtain $\widehat{v}_k^j(k+mJ) = \alpha\widehat{v}_k^{j+1}(k+mJ)$ for m even and $\widehat{v}_k^j(k+mJ) = \beta\widehat{v}_{k+J}^{j+1}(k+mJ)$ for m odd. As already noted, the quantities $\widehat{v}_k^j(k+mJ)$ are nonzero. Therefore, the numbers α and β are also nonzero. On the other hand, condition 2b) implies that $v_k^j(2\cdot) = \gamma v_{2k}^{j+1}(\cdot)$. Consequently, for even m we obtain

$$\alpha\left(\widehat{v}_k^{j+1}(k+mJ) + \widehat{v}_k^{j+1}(k+mJ+N/2)\right) = \widehat{v}_k^j(k+mJ) + \widehat{v}_k^j(k+mJ+N/2) = \gamma\widehat{v}_{2k}^{j+1}(2k+2mJ).$$

Thus,

$$\alpha = \gamma\widehat{v}_{2k}^{j+1}(2k+2mJ)/\left(\widehat{v}_k^{j+1}(k+mJ) + \widehat{v}_k^{j+1}(k+mJ+N/2)\right), \tag{1.6}$$

where m is an arbitrary even number. The denominator in formula (1.6) is nonzero, because otherwise, by (1.4), the spectrum of the elements of V^{j+1} would involve no harmonic with index $2k+2mJ$.

In a similar way, we prove that the coefficient β can be calculated by formula (1.6) with m odd.

Consequently, α and β are determined uniquely up to a nonzero factor γ, which does not affect the space V^j. Lemma 2 is proved. □

Let φ and ϕ be two functions such that the collections $\{\varphi(\cdot - kN/J)\}_{k=0}^{J-1}$ and $\{\phi(\cdot - kN/J)\}_{k=0}^{J-1}$ are biorthogonal bases of the space V^j (i.e., $(\vec{\varphi}, \vec{\phi})$ is the identity matrix); such functions φ and ϕ are said to be *dual*. The following statement was obtained by V. A. Zheludev [2] in the case of a spline MRA.

Lemma 3. *Let $\{V^j\}_{j=0}^{n}$ be an arbitrary MRA. Assume that a Wiener basis $\{v_k^j\}_{k=0}^{J-1}$ of each V^j is chosen so as to satisfy the normalizing condition*

$$\sum_{m=0}^{N/J-1} |\, \widehat{v}_k^j(k+mJ)\,|^2 = 1.$$

Then two function φ, $\phi \in V^j$ are dual if and only if the coefficients occurring in the representations

$$\varphi(\cdot) = \sum_{k=0}^{J-1} \alpha_k v_k^j(\cdot), \quad \phi(\cdot) = \sum_{k=0}^{J-1} \beta_k v_k^j(\cdot)$$

satisfy the relations $\alpha_0\overline{\beta}_0 = \alpha_1\overline{\beta}_1 = \dots = \alpha_{J-1}\overline{\beta}_{J-1} = 1/J$. In particular, φ gives rise to an orthogonal basis if and only if $|\,\alpha_0\,| = |\,\alpha_1\,| = \dots = |\,\alpha_{J-1}\,| \neq 0$.

Remark. Obviously, if we supplement the normalizing condition of Lemma 3 with the condition $\widehat{v}_k^j(k) > 0$, $k = 0, 1, \dots, J-1$, then the choice of a Wiener basis becomes unique.

Proof of Lemma 3. By (1.2), we obtain $v_k^j(\cdot - mN/J) = \omega_J^{-km} v_k^j(\cdot), k = 0, 1, \dots, J-1$. In particular, $(v_{k_1}^j(\cdot - m_1N/J), v_{k_2}^j(\cdot - m_2N/J)) = \delta_{k_1k_2}\omega_J^{k_1(m_2-m_1)}$, where

$$\delta_{k_1k_2} = \begin{cases} 1, & k_1 = k_2, \\ 0, & k_1 \neq k_2. \end{cases}$$

Thus, $(\varphi(\cdot - m_1N/J), \phi(\cdot - m_2N/J)) = \sum_{k=0}^{J-1} \alpha_k\overline{\beta}_k\omega_J^{k(m_2-m_1)}$. The same expression may be written in the matrix form:

$$\Phi \stackrel{\text{def}}{=} (\vec{\varphi}(\cdot), (\vec{\phi}(\cdot))^T) = F_J^* \Lambda F_J,$$

where Λ is the diagonal matrix with the elements $\lambda_i = \alpha_i\overline{\beta}_i$ on the diagonal. The matrix Φ is the identity matrix if and only if $\lambda_0 = \lambda_1 = \dots = \lambda_{J-1} = 1/J$, as claimed. This completes the proof of Lemma 3. □

Lemma 4. *Let $\{V^j\}_{j=0}^{n}$ be an arbitrary MRA. For any j, an arbitrary interpolation problem on the* (*coarser*) *net $x_k = kN/J$ is uniquely solvable in the space V^j, i.e., for any collection $\{a_k\}_{k=0}^{J-1}$ of numbers there exists a unique function $\lambda \in V^j$ such that $\lambda(x_k) = a_k$.*

Proof of Lemma 4. It is well known that any interpolation problem is solvable on the above net in the linear span of the system $\{e_{N,l}\}_{l=0}^{J-1}$. In order to find the coefficients of the expansion we need to calculate the DFT of the sequence a_k and to divide the result by J. Let $\{b_k\}$ be the resulting coefficients. We consider the lth element of the Wiener basis of V^j. On the net $\{x_k\}$, the values of its harmonics coincide with those of the function $e_{N,l}$. Thus,

$$v_l^j(x_k) = \exp(2\pi i l x_k/N) \sum_{q=0}^{N/J-1} \widehat{v}_l^j(l+qJ)/N = \exp(2\pi i l x_k/N)c_l,$$

whence we see that the coefficients of the expansion of the interpolating function with respect to the Wiener basis are equal to $\{b_l/c_l\}$. It remains to prove that the numbers c_l are all nonzero. Indeed, otherwise, applying formula (1.4) $n-j$ times, we easily show that in the space V^n there is no harmonic with index Nl/J, which contradicts condition 1 in Definition 1. Lemma 4 is proved. $\square$

Proof of Theorem 1. Let l be an arbitrary integer satisfying $0 \le l \le n$, and let $\varphi^l \in V^l$ be a function such that the elements of the vector $\vec{\varphi}^l$ form a basis of the space V^l. The existence of such a function follows from Lemma 2. Let $\{v_k^j\}_{k=0}^{J-1}$, $j = 0, 1, \dots, n$, be the Wiener bases of the spaces V^j. We assume that these bases are related by formulas (1.3) and (1.4). By the corollary to Lemma 1, we obtain

$$\varphi^l(m) = \sum_{k=0}^{L-1} \alpha_k v_k^l(m); \quad \alpha_k \neq 0, \quad k = 0, 1, \dots, L-1.$$

We show that functions φ^{l+1} and φ^{l-1} can be constructed so that the components of the vectors $\vec{\varphi}^{\,l+1}$ and $\vec{\varphi}^{\,l-1}$ form bases in the spaces V^{l+1} and V^{l-1}, respectively. As φ^{l-1} we take a function satisfying the relation

$$\varphi^{l-1}(2\cdot) = \varphi^l(\cdot) + \varphi^l(\cdot + N/2).$$

Condition 2c) shows that such a function exists; by Lemma 4, it is unique. We can write

$$\varphi^{l-1}(\cdot) = \sum_{k=0}^{L/2-1} \beta_k v_k^{l-1}(\cdot) \tag{1.7}$$

We need to prove that all coefficients β_k are nonzero. Since the operator of $(N/2)$-periodization eliminates the odd harmonics and doubles the even ones, we have

$$\varphi^l(\cdot) + \varphi^l(\cdot + N/2) = 2 \sum_{k=0}^{L/2-1} \alpha_{2k} v_{2k}^l(\cdot).$$

Combined with (1.7) and (1.4′), the above relation shows that $\beta_k = 2\alpha_{2k} \neq 0$. Applying the corollary to Lemma 1, we see that the components of the vector $\vec{\varphi}^{\,l-1}$ form a basis of the space V^{l-1}.

Now we show that the required functions φ^j can be found for $j > l$. Let $j = l+1$. We put

$$\varphi^{l+1}(m) = \sum_{k=0}^{2L-1} \gamma_k v_k^{l+1}(m)$$

and find the coefficients of the expansion. Obviously, for k odd, the coefficients can be chosen arbitrarily, subject to the sole requirement $\gamma_k \neq 0$ (which is necessary for the linear independence of the components of $\vec{\varphi}^{l+1}$). As to the even coefficients, we have $\gamma_{2k} = \alpha_k/2$ (as in the above passage from φ^l to φ^{l-1}).

In the sequel we shall need the following statement.

Lemma 5. *Let $\{\varphi^j\}_{j=1}^n$ be a sequence of functions satisfying* (1.1), *and assume that the components of the vectors $\vec{\varphi}^{\,j}$ form bases of the spaces V^j. Then there exist numbers $\{a_k\}_{k=0}^{N-1}$ independent of j such that*

$$\varphi^{j-1}(\cdot) = \sum_{k=0}^{N-1} a_k \varphi^j(\cdot - kN/J) \tag{1.8}$$

for any $0 < j \le N$.

Remark. Since φ^j is N-periodic, relation (1.8) can be written in the form

$$\varphi^{j-1}(\cdot) = \sum_{k=0}^{J-1} \Bigg(\sum_{m=0}^{N/J-1} a_{k+mJ} \Bigg) \varphi^j(\cdot - kN/J).$$

Proof of Lemma 5. We introduce the functions $\{\phi_1^j\}_{j=1}^n$, $\{\phi_2^j\}_{j=0}^{n-1}$ defined only at the points the form mN/J and $mN/2J, m \in \mathbb{Z}$, by the formulas

$$\phi_1^j(\cdot N/J) = \varphi^n(\cdot), \quad \phi_2^{j-1}(\cdot N/J) = \varphi^{n-1}(\cdot).$$

Relation (1.8) will be proved by induction.

If $j = n$, then the existence of a_k follows from condition 1 in Definition 1.

Assuming that relation (1.8) has been proved for $j = n, n-1, \dots, l+1$, we show that (1.8) is true for $j = l$.

Obviously,

$$\phi_2^{l-1}(\cdot N/L) = \sum_{k=0}^{N-1} a_k \phi_1^l(\cdot N/L - kN/L). \tag{1.9}$$

We apply a periodization procedure to both sides of the latter relation, so as to obtain N-periodic functions. By (1.1), the left-hand side will yield

$$\sum_{k=0}^{N/L-1} \phi_2^{l-1}(\cdot N/L + kN) = \sum_{k=0}^{N/L-1} \varphi^{n-1}(\cdot + kL) = \varphi^{l-1}(\cdot N/L).$$

Now we "periodize" the right-hand side of (1.9):

$$\begin{aligned} &\sum_{m=0}^{N/L-1} \sum_{k=0}^{N-1} a_k \phi_1^l(\cdot N/L - kN/L + mN) \\ &\quad = \sum_{k=0}^{N-1} a_k \sum_{m=0}^{N/L-1} \phi_1^l(\cdot N/L - kN/L + mN) \\ &\quad = \sum_{k=0}^{N-1} a_k \sum_{m=0}^{N/L-1} \varphi^n(\cdot - k + mL) = \sum_{k=0}^{N-1} a_k \varphi^l(\cdot N/L - kN/L). \end{aligned}$$

Thus, Lemma 5 is valid for the net $mN/L, m = 0, 1, \dots, L/2-1$. Since, by Lemma 4, any interpolation problem is solvable in the space V^l on that net, we obtain (1.8) at all points.

In particular, from Lemma 5 not only does it follow that any MRA of the space $\widetilde{\mathbb{C}}^N$ is determined by the function φ^n and the sequence $\{a_k\}$ describing the passage from the basis $\vec{\varphi}^n$ to the basis $\vec{\varphi}^{n-1}$, but also that the MRA in question can be recovered with the help of the simple formula (1.8), which looks uniform in j.

Thus, for the discrete periodic case a theory of MRA can be constructed; in many respects, this theory is similar to the corresponding theory for $\mathbb{L}^2(\mathbb{R})$. In particular, every MRA is generated by dilations and translates of a single function.

Now we show how an arbitrary MRA of $\widetilde{\mathbb{C}}^N$ can be constructed. By Lemma 2, an MRA can be recovered by each of the spaces $V^j, 0 < j < N$. In particular, it can be recovered by the space V^1, which, in turn, is determined by the corresponding Wiener basis. As shown in the proof of Lemma 2, the Wiener basis of V^1 is formed by the function $v_0^1(\cdot) \equiv \text{const} \neq 0$ and a function $v_1^1(\cdot)$ whose spectrum has only odd harmonics. Distinct (up to a common factor) choices of the coefficients $\widehat{v}_1^1(2k+1) \stackrel{\text{def}}{=} a_k$, $0 \le k < N/2$, of those harmonics yield distinct MRA's. It only remains to determine the admissible collections of coefficients.

As already mentioned, were at least one of the numbers $\widehat{v}_k^j(k+mJ)$ $(0 \le m < N/J, 1 \le k < j, 1 \le j \le n)$ equal to zero, the corresponding harmonic would be absent in the space V^n, which contradicts property 1 in Definition 1. Thus, it is necessary that all numbers of the form

$$\sum_{k=0}^{2^p-1} a_{m+kN/2^{p+1}}, \quad p = 0, 1, 2, \dots, n-1,$$

be nonzero. It is easy to verify that this condition is also sufficient.

Obviously, all the admissible sequences $\{a_m\}_{m=0}^{N/2-1}$ can be constructed as follows. First, we choose a sequence $\vec{b}^j \in \widetilde{\mathbb{C}}^J$, $j = 0, \dots, n-1$, by induction.

We put $\vec{b}^0 = 1$ and assume that the vectors $\{\vec{b}^k\}_{k=0}^m$ have already been chosen. As $\vec{b}^{m+1}$ we take an arbitrary vector with nonzero components satisfying the condition $b_l^{m+1} + b_{l+2^m}^{m+1} = b_l^m$ for all $l = 0, 1, \dots, 2^m - 1$.

Clearly, the vector $\vec{b}^{n-1}$ is admissible, i.e., we can put $a_k = b_k^{n-1}$, $k = 0, 1, \dots, N/2-1$.

By Theorem 1, some bases of the spaces V^j can be chosen that consist of dilations, periodizations, and translates of a single function. However, such bases are not orthogonal. By Lemma 3, orthogonalization is possible for every j, but it is not clear whether the new bases are generated by dilations, translates, and periodizations of one and the same function for all j. It turns out that not all MRA possess this property. We pass to the description of such MRA.

We say that two complex numbers are orthogonal if so are the vectors representing them on the plane. For any k, the frequencies k and $k + N/2$ will be called *symmetric points* of the spectrum.

Suppose that a function φ^n gives rise to an orthogonal basis of the space V^n; we put $v_k^n(m) = \exp(2\pi ikm/N)$. By Lemma 3, all harmonics of the function φ^n have one and the same amplitude, i.e.,

$$\varphi^n(m) = \sum_{k=0}^{N-1} a_k^n \exp(2\pi ikm/N), \quad |\, a_0 \,|=|\, a_1 \,|= \cdots =|\, a_{N-1} \,| \,.$$

Without loss of generality we may assume that the amplitude is equal to 1. Also, we assume that the Wiener bases in the spaces V^j satisfy (1.3) and (1.4).

By Lemma 4, there exists a unique function $\varphi^{n-1} \in V^{n-1}$ satisfying the relation

$$\varphi^{n-1}(2m) = \varphi^n(m) + \varphi^n(m + N/2) = \sum_{k=0}^{N/2-1} 2a_{2k}^n \exp(2\pi i2km/N). \tag{1.10}$$

On the other hand, the function φ^{n-1} is representable as a linear combination of the elements of the Wiener basis of V^{n-1}:

$$\varphi^{n-1}(m) = \sum_{k=0}^{N/2-1} a_k^{n-1} v_k^{n-1}(m), \tag{1.11}$$

where $a_k^{n-1} = 2a_{2k}^n$. Since the absolute values of the coefficients in (1.11) are equal, the components of the vector $\vec{\varphi}^{n-1}$ are orthogonal if and only if the norms of the elements

$$v_k^{n-1}(m) = b_k \exp(2\pi ikm/N) + b_{k+N/2} \exp(2\pi i(k+N/2)m/N)$$

of the Wiener basis are equal:

$$|\, b_0 \,|^2 + |\, b_{N/2} \,|^2 =|\, b_1 \,|^2 + |\, b_{1+N/2} \,|^2 = \cdots =|\, b_{N/2-1} \,|^2 + |\, b_{N-1} \,|^2 \stackrel{\text{def}}{=} c.$$

Since $b_{N/2} = 0$, formula (1.4) shows that $b_0 = 1$, whence $c = 1$. Again by (1.4), we obtain

$$b_1 + b_{1+N/2} = b_2 + b_{2+N/2} = \cdots = b_{N/2-1} + b_{N-1} = 1.$$

Obviously, the required conditions are satisfied simultaneously of and only if the complex numbers b_k and $b_{k+N/2}, k = 1, 2, \dots, N/2-1$, are orthogonal. Moreover, these numbers must be nonzero.

Any number $1 \le k \le N/2 - 1$ can be represented (in a unique way) in the form $k = (2m+1)2^p$, $0 \le m < N/4P$, $0 \le p < n-1$. In accordance with this representation, from (1.3) and (1.4) it is easy to deduce the relations

$$Nb_k = \widehat{v}_k^{n-1} = \sum_{l=0}^{P-1} \widehat{v}_1^1(1 + 2m + lN/P),$$

$$N b_{k+N/2} = \widehat{v}^{\,n-1}_{k+N/2} = \sum_{l=0}^{P-1} \widehat{v}^{\,1}_1(1 + 2m + lN/P + N/2P).$$

Clearly, a function v_1^1 satisfying these conditions is a solution of our problem. We consider an algorithm for constructing all possible functions that possess the properties indicated.

We use the same method as applied above for the construction of the Wiener bases for an arbitrary MRA. A sequence of vectors $\vec{b}^{\,j} \in \widetilde{\mathbb{C}}^J$, $j = 0, \dots, n-1$, will be constructed by induction.

We put $\vec{b}^{\,0} = 1$ and assume that the vectors $\{\vec{b}^{\,k}\}_{k=0}^m$ have already been chosen. As $\vec{b}^{\,m+1}$ we take an arbitrary vector with nonzero components satisfying $b_l^{m+1} + b_{l+2^m}^{m+1} = b_l^{m+1}$ and such that the terms of the latter sum are orthogonal; here $l = 0, 1, \dots, 2^m - 1$.

Obviously, the vector $\vec{b}^{\,n-1}$ is admissible, i.e., we can put

$$\widehat{v}^{\,1}_1(1 + 2k) = b_k^{n-1}, \quad k = 0, 1, \dots, N/2 - 1.$$

We note that if relations (1.3) and (1.4) are satisfied for $j = 1, 2, \dots, n-1$, then

$$\widehat{v}^{\,j}_{J/2}(J/2 + kJ) = b_k^{n-j}, \quad k = 0, 1, \dots, N/J - 1.$$

§2. Orthogonal spaces of discrete periodic wavelets

Let $\{V^j\}_{j=0}^n$ be an arbitrary MRA of the space $\widetilde{\mathbb{C}}^N$. Here and in §3, by the *wavelet space* W^j we mean the orthogonal complement of the space V^j to V^{j+1}. Thus, the space $\widetilde{\mathbb{C}}^N$ can be represented as the following direct sum:

$$\widetilde{\mathbb{C}}^N = V^0 \oplus W^0 \oplus W^1 \oplus \dots \oplus W^{n-1}; \tag{2.1}$$

in our case this sum is orthogonal.

In order to show that in every wavelet space there exists a function whose translates form a basis, we prove the existence of a Wiener basis of each of the spaces W^j. Since for any $k = 1, 2, \dots, 2^j - 1$ we have

$$v_k^j = \alpha v_k^{j+1} + \beta v_{k+J}^{j+1} \tag{2.2}$$

with some α and β, the functions

$$w_k^j = \overline{\beta} v_k^{j+1} \|v_{k+J}^{j+1}\|^2 - \overline{\alpha} v_{k+J}^{j+1} \|v_k^{j+1}\|^2$$

and $w_J^j = v_J^{j+1}$ can be taken as the elements of a Wiener basis. In what follows we assume that the norms of the elements of the Wiener bases in the spaces V^j are equal to 1. Then the above formula takes the form

$$w_k^j = \overline{\beta} v_k^{j+1} - \overline{\alpha} v_{k+J}^{j+1}. \tag{2.3}$$

Moreover, by (2.2), $|\alpha|^2 + |\beta|^2 = 1$, i.e., the elements of the Wiener basis of W^j are orthonormal. The mutual orthogonality of the functions w_k^j and their orthogonality to the space V^j are obvious. It is worth noting that the enumeration of these functions will start with 1 (not with zero), because if the elements of the Wiener basis are enumerated in a natural way, then the harmonic whose index is equal to the subscript turns out to be present in the spectrum of the function, whereas the zeroth harmonic is present only in the space V^0. However, this is not of great importance since in what follows it is natural to assume that $w_{k+J}^j \equiv w_k^j$.

Now we consider an algorithm of expanding an arbitrary function $f \in \widetilde{\mathbb{C}}^N$ in the basis composed of the Wiener bases of the spaces W^j. In other words, we describe an algorithm of calculation of the coefficients in the formula

$$f(\cdot) = \sum_{j=0}^{n-1} \sum_{k=1}^{J} p_k^j w_k^j(\cdot) + q_0^0.$$

As usual, we consecutively find the representations of f in the form

$$f(\cdot)=\sum_{k=0}^{2^m-1} q_k^m v_k^m(\cdot)+\sum_{j=m}^{n-1}\sum_{k=1}^{J} p_k^j w_k^j(\cdot)$$

for $m=n,n-1,\dots,1,0$.

It is clear that, for $m=n$, the coefficients q_k^n can be calculated up to the factor of $1/\sqrt{N}$ with the help of the DFT of f.

By (2.2) and (2.3), having calculated q_k^{j+1}, we can find the coefficients q_k^j and p_k^j by the formulas

$$\begin{aligned} q_0^j&=q_0^{j+1}, & q_k^j&=\alpha q_k^{j+1}+\beta q_{k+J}^{j+1},\ k=1,\dots,J-1,\\ p_J^j&=q_J^{j+1}, & p_k^j&=\overline{\beta} q_k^{j+1}-\overline{\alpha} q_{k+J}^{j+1},\ k=1,\dots,J-1, \end{aligned}\tag{2.4}$$

where α and β are as in (2.2); naturally, these numbers are different for different k and j.

Obviously, the following reconstruction formulas can also be obtained from (2.2) and (2.3):

$$\begin{aligned} q_0^{j+1}&=q_0^j, & q_k^{j+1}&=\overline{\alpha} q_k^j+\beta p_k^j,\ k=1,\dots,J-1,\\ q_J^{j+1}&=p_J^j, & q_{k+J}^{j+1}&=\overline{\beta} q_k^j-\alpha p_k^j,\ k=1,\dots,J-1. \end{aligned}\tag{2.5}$$

After applying formulas (2.5), for recovering f it remains to find the IDFT of $\vec{q}^{\,n}$ and to multiply the result by $\sqrt{N}$. Thus, the algorithm of expansion of a function with respect to the Wiener bases and the algorithm of recovering a function have complexity of order of $O(N\log N)$.

Now we consider bases of a wavelet space that consist of translates of a single function; usually, such bases arise first, and the wavelet spaces emerge only afterwards.

We already know that any basis of this type in the space W^j can be constructed as a linear combination of elements of the Wiener basis with nonzero coefficients. However, there exists another approach that does not require referring to Wiener bases.

Let φ^j and φ^{j+1} be two functions such that the collections $\{\varphi^j(\cdot-kN/J)\}_{k=0}^{J-1}$ and $\{\varphi^{j+1}(\cdot-kN/2J)\}_{k=0}^{2J-1}$ of translates form bases of the spaces V^j and V^{j+1}, respectively; we put

$$\lambda_k \overset{\text{def}}{=} (\varphi^j(\cdot),\varphi^{j+1}(\cdot-kN/2J)).$$

Theorem 2. *Let*

$$\psi^j(m+N/2J)=\sum_{n=0}^{2J-1}(-1)^n\overline{\lambda}_n\varphi^{j+1}(m+nN/2J).\tag{2.6}$$

Then the system of functions $\{\psi^j(m-kN/J)\}_{k=0}^{J-1}$ *(in what follows denoted by* $\vec{\psi}^{\,j}(\cdot)$*) is a basis of the space* W^j*.*

Remark. Formula (2.6) looks similar to the well-known formula for the expansion of ψ^j in the basis of components of the vector $\vec{\varphi}^{\,j+1}$ for the MRA of the space $\mathbb{L}^2(\mathbb{R})$ under the condition that these components form an orthonormal basis. In fact, in formula (2.6) we specify the coefficients. In some sense, formula (2.6) can be regarded as known. In any case, an analog of it can easily be obtained from formula (5.4) in [6].

Proof of Theorem 2. We show that the translates of ψ^j belong to the space W^j. Obviously, it suffices to prove that they are orthogonal to φ^j. Indeed, for any k we have

$$\begin{aligned} &(\varphi^j(\cdot),\psi(\cdot-kN/J))\\ &\quad=\sum_{n=0}^{2J-1}(-1)^n\lambda_n(\varphi^j(\cdot),\psi(\cdot-(2k+1-n)N/2J))\\ &\quad=\sum_{n=0}^{2J-1}(-1)^n\lambda_n\lambda_{2k+1-n}=\sum_{n=0}^{J-1}\lambda_{2n}\lambda_{2k+1-2n}-\sum_{n=0}^{J-1}\lambda_{2n+1}\lambda_{2k-2n}\\ &\quad=\sum_{n=0}^{J-1}\lambda_{2n}\lambda_{2k+1-2n}-\sum_{m=0}^{J-1}\lambda_{2k+1-2m}\lambda_{2m}=0. \end{aligned}$$

In order to show that the components of $\vec{\psi}^{\,j}(x)$ are linearly independent, we introduce an auxiliary matrix $\Phi^j = (\vec{\varphi}^{\,j}, \vec{\varphi}^{\,j+1})$. The first row of it is the vector $(\lambda_0, \lambda_1, \ldots, \lambda_{2J-1})$, and each subsequent row is obtained by the cyclic shift by two positions. Since the components of the vector $\vec{\varphi}^{\,j}$ form a basis of the space $V^j \subset V^{j+1}$, the rank of the matrix Φ^j equals J. Furthermore, by (2.6), the matrix $\Psi^j = \left(\vec{\psi}^{\,j}, \vec{\varphi}^{\,j+1}\right)$ is the product of two matrices. The first of them (denoted by Θ^j) is of order $(J \times 2J)$ and differs from Φ^j by complex conjugation of the elements and by the change of sign in the even columns. The second one coincides (up to nonsingular transformations) with the Gram matrix of the components of the vector $\vec{\varphi}^{\,j+1}$. Clearly, the latter matrix is nonsingular, because the components of $\vec{\varphi}^{\,j+1}$ form a basis of V^{j+1}. Consequently, the rank of the product is determined by that of the matrix Θ^j, which, obviously, coincides with the rank of Φ^j and, thus, is equal to J.

The bases of the spaces W^j generated by a single function will be called *bases of wavelets*.

We turn to algorithms of expansion of an arbitrary function $f \in \widetilde{\mathbb{C}}^N$ in the bases $\vec{\psi}^{\,j}$ and to algorithms of recovering such functions.

The passage from the coordinates corresponding to a basis of wavelets to the coordinates corresponding to a Wiener basis and conversely is realized by means of the direct and inverse DFT and, possibly, coordinatewise multiplication in the frequency domain. Indeed, for any $f \in W^j$ we have

$$f = \vec{\psi}^{\,jT}\vec{a} = \frac{1}{J}\vec{\psi}^{\,jT}F_J^* F_J \vec{a} = \vec{w}^{\,jT}\vec{b} = \vec{w}^{\,jT}F_J F_J^* \vec{b}/J,$$

where $\vec{a}$ and $\vec{b}$ are the coordinate vectors of f relative to the bases $\vec{\psi}^{\,j}$ and $\vec{w}^{\,j}$, respectively. Since the components of a Wiener basis are determined uniquely up to nonzero factors, coordinatewise multiplication in the frequency domain may occur if the Wiener basis chosen is distinct from $\vec{w}^{\,j}$. Clearly, similar relations are also valid for the spaces V^j and the bases $\vec{\varphi}^{\,j}$ and $\vec{v}^{\,j}$.

Thus, the algorithm of decomposition of functions in $\widetilde{\mathbb{C}}^N$ with respect to the basis of wavelets $\vec{\psi}^{\,j}$ and the algorithm of recovering such functions with the help of Wiener bases have complexity of order of $O(N \log N)$ independently of what was the initial representation of a function: the natural form $f(\cdot)$, the form of a spectrum, or the form of expansion with respect to some basis $\vec{\varphi}^{\,n}$.

§3. Wavelets with local support

From the point of view of economy, the algorithms for wavelets with local support can serve as an alternative to the algorithms of expansion and reconstruction that use the passage to the frequency domain. Talking of "local support" (of size p), we mean that there exists a function φ^n generating a basis of V^n such that the functions determined recursively by formulas (1.8) have supports in the segments $[0, pN/J]$, where $p \ll N$. In particular, from (2.6) it follows that the wavelet basis $\vec{\psi}^{\,j}$ in the spaces W^j can be chosen so that the functions ψ^j also have local support. One of the sources of periodic wavelets with local support is periodization of nonperiodic MRA having bases with compact support.

We recall that, by definition, a *spline* of order p and of minimal defect with knots at integral points is a $(p-1)$ times continuously differentiable function that coincides with a polynomial of order at most p on the segments of the form $(k, k+1)$, where k is an integer. A *B-spline* is a spline with support of minimal possible size. With the help of splines, it is easy to construct a discrete MRA with locally supported bases. Let $p < N$ be a nonnegative integer (the order of a spline). The role of the above φ^n will be played by a function whose values coincide with those of a B-spline at the knots. The functions φ^j are chosen so as to satisfy the recurrent relations (1.8), under the following initial conditions: for $m = 0, 1, \ldots, N-1$, the function φ^j coincides with the periodization of a B-spline of order p with knots kN/J, $k = 0, 1, \ldots, J-1$. Bases of wavelets in the spaces W^j can be found with the help of (2.6).

We turn to computational aspects of expansion and reconstruction of functions with respect to wavelet bases with local support.

We show that the procedure of expansion of any function with respect to locally supported bases of wavelets, as well as the recovering procedure, require $O(pN)$ arithmetic operations; here p is the above parameter characterizing the size of supports of the basis functions. We shall see that these algorithms consist of operations of solution of linear equations with circulant matrices having $O(p)$ nonzero diagonals, and of operations of generalized convolutions with windows of size $O(p)$.

Let $\{\varphi^j\}_{j-1}^{n}$ be a collection of locally supported functions satisfying the assumptions of Theorem 1, and let $\{\psi^j\}_{j-1}^{n-1}$ be the functions obtained as in (2.6) and determining the wavelet bases.

We need to construct an algorithm of expansion of an arbitrary function $f \in \widetilde{\mathbb{C}}^N$ in a sum of the form $f(\cdot) = f_k(\cdot) + \sum_{m=k}^{n-1} g_m(\cdot)$, where $f_k \in V^k$, $g_m \in W^m$, $k = 0, 1, \ldots, n$; more precisely, we must find the coefficients of the expansions of f_k and g_k with respect to the corresponding bases of translates. We apply inverse induction on k, beginning with $k = n$.

Since $f_n(\cdot) \stackrel{\text{def}}{=} f(\cdot) = \vec{a}^{\,nT}\vec{\varphi}^n(\cdot)$, where $\vec{a}^{\,n}$ is the vector of coefficients to be found, we see that $\vec{a}^{\,n}$ is the solution of a system of linear equations with a certain p-diagonal circulant matrix (we recall that $\vec{\varphi}^n(\cdot)$ can be viewed either as a vector-valued function or as a matrix of size $N \times N$).

Assume that the expansion of f_{k+1} $(0 \le k < n)$ with respect to the basis $\vec{\varphi}^{k+1}(\cdot)$ has already been found; we can write

$$f_{k+1}(\cdot) = \left(\vec{a}^{\,k+1}\right)^T \vec{\varphi}^{k+1}(\cdot) = f_k(\cdot) + g_k(\cdot) = \vec{a}^{\,kT}\vec{\varphi}^k(\cdot) + \vec{b}^{\,kT}\vec{\psi}^k(\cdot). \tag{3.1}$$

We take the scalar product of both sides of (3.1) first with $\vec{\varphi}^k(\cdot)$ and then with $\vec{\psi}^k(\cdot)$ to obtain

$$\left(\vec{a}^{\,k+1}\right)^T \left(\vec{\varphi}^{k+1}(\cdot), \vec{\varphi}^k(\cdot)\right) = \vec{a}^{\,kT} \left(\vec{\varphi}^k(\cdot), \vec{\varphi}^k(\cdot)\right), \tag{3.2}$$

$$\left(\vec{a}^{\,k+1}\right)^T \left(\vec{\varphi}^{k+1}(\cdot), \vec{\psi}^k(\cdot)\right) = \vec{b}^{\,kT} \left(\vec{\psi}^k(\cdot), \vec{\psi}^k(\cdot)\right). \tag{3.3}$$

Clearly, the left-hand sides of (3.2) and (3.3) can be viewed as convolutions of $\vec{a}^{\,k+1}$ with windows of size $O(p)$, and the vectors $\vec{a}^{\,k}$ and $\vec{b}^{\,k}$ are solutions of certain systems of linear equations with circulant matrices having $O(p)$ diagonals.

Some recovering algorithms can also be obtained with the help of (3.1). Indeed, if the vectors $\vec{a}^{\,k}, \vec{b}^{\,k}$ of coefficients are known for some $0 \le k < n$, then we multiply both sides of the identity

$$\left(\vec{a}^{\,k+1}\right)^T \vec{\varphi}^{k+1}(\cdot) = \vec{a}^{\,kT}\vec{\varphi}^k(\cdot) + \vec{b}^{\,kT}\vec{\psi}^k(\cdot)$$

by $\vec{\varphi}^{k+1}(\cdot)$. Now the vector $\vec{a}^{\,k+1}$ can be found from the system of linear equations

$$\left(\vec{a}^{\,k+1}\right)^T \left(\vec{\varphi}^{k+1}(\cdot), \vec{\varphi}^{k+1}(\cdot)\right) = \vec{a}^{\,kT} \left(\vec{\varphi}^k(\cdot), \vec{\varphi}^{k+1}(\cdot)\right) + \vec{b}^{\,kT} \left(\vec{\psi}^k(\cdot), \vec{\varphi}^{k+1}(\cdot)\right).$$

We note that solution of a system of L linear equations with m-diagonal matrix requires triangle decomposition of the matrix, and the complexity of this procedure is of order of $O(m^2L)$. However, these calculations do not involve the function whose expansion with respect to a wavelet basis we are looking for.

Thus, the total complexity of the expansion and recovering procedures for an arbitrary function f is at most $O(pN)$.

§4. Nonorthogonal spaces of wavelets.

Here we introduce another definition of the wavelet spaces. This definition has the following advantage: there are decomposition and recovering algorithms such that calculation of the coefficients at every level can be realized as convolution with one and the same window. Moreover, in many cases important in practice the window can be chosen to have size $p \ll N$. In this sense, these algorithms are similar to the algorithm suggested by I. Daubéchies for the orthogonal wavelets on the line. It should be noted that there is no orthogonal bases of locally supported discrete wavelets, except for the simplest case where the support of a basis of translates in the space V^j is of size less than 2^{n-j}.

In a certain sense, similar ideas were realized in [10]. In that paper, for an arbitrary spline space S on the line (including the case of nonequidistant knots) the authors proved that S can be split into the direct sum of a spline space S_1 with a coarser net of knots and a space $S_2 \subset S$ such that the corresponding canonical projections are local.

In what follows the symbol $\oplus$ means the direct (generally speaking, nonorthogonal) sum of linear spaces.

Definition 2. Assume that a sequence $\{V^j\}_{j=0}^{n}$ of linear spaces forms a MRA of the space $\widetilde{\mathbb{C}}^N$. Let φ^j, $0 < j \le n$, be the functions occurring in Theorem 1 and let a sequence $\{\psi^j\}_{j=0}^{n-1}$ be defined by the relation

$$\psi^j(\cdot) = \sum_{k=0}^{N-1} b_k \varphi^j(\cdot - kN/J).$$

We shall say that the spaces $\{W^j\}_{j=0}^{n-1}$ defined as the linear spans of the components of $\vec{\psi}^j$ give rise to a wavelet decomposition of $\widetilde{\mathbb{C}}^N$ if

$$V^0 \oplus W^0 \oplus W^1 \oplus \cdots \oplus W^{k-1} = V^k, \quad k = 1, 2, \ldots, n.$$

We show that the sole condition $\widetilde{\mathbb{C}}^N = V^{n-1} \oplus W^{n-1}$ ensures that the above spaces $\{W^j\}_{j=0}^{n-1}$ give rise to a wavelet decomposition of $\widetilde{\mathbb{C}}^N$.

We prove this by induction. Assume we have proved the relation $V^{k+1} = V^k \oplus W^k$ for some $k < n$. We shall show that $V^k = V^{k-1} \oplus W^{k-1}$. For this, we must verify that the components of the vector $\vec{\psi}^{k-1}$ are linearly independent, and that there are no elements of the space V^{k-1} among their linear combinations.

By (1.1), for any $l = 0, 1, \ldots, K-1$ we have

$$\begin{aligned}
&\psi^k(\cdot + lN/K) + \psi^k(\cdot + lN/K + N/2) \\
&\quad = \sum_{m=0}^{N-1} b_m \varphi^{k+1}(\cdot + mN/2K + lN/K) \\
&\qquad + \sum_{m=0}^{N-1} b_m \varphi^{k+1}(\cdot + mN/2K + lN/K + N/2) \\
&\quad = \sum_{m=0}^{N-1} b_m \left(\varphi^{k+1}(\cdot + mN/2K + lN/K) + \varphi^{k+1}(\cdot + mN/2K + lN/K + N/2)\right) \\
&\quad = \sum_{m=0}^{N-1} b_m \varphi^k(2(\cdot + mN/2K + lN/K)) = \psi^{k-1}(2(\cdot + lN/K)).
\end{aligned}$$

Consequently, if the components of the vector $\vec{\psi}^{k-1}$ are linearly dependent, then for some collection $\{\alpha_j\}_{j=0}^{K/2-1}$ of numbers not all of which are zero we can write

$$\begin{aligned}
0 &\equiv \sum_{j=0}^{K/2-1} \alpha_j \psi^{k-1}(2(\cdot + jN/K)) \\
&= \sum_{j=0}^{K/2-1} \alpha_j \left(\psi^k(\cdot + jN/K) + \psi^k(\cdot + jN/K + N/2)\right),
\end{aligned}$$

which contradicts the linear independence of the components of $\vec{\psi}^k$.

In a similar way we check that if

$$0 \equiv \sum_{j=0}^{K/2-1} \beta_j \varphi^{k-1}(\cdot + 2jN/K) + \sum_{j=0}^{K/2-1} \gamma_j \psi^{k-1}(\cdot + 2jN/K)$$

for some collections $\{\beta_i\}$ and $\{\gamma_i\}$ of numbers, then, in accordance with the choice of $\{\varphi^j\}$, we have

$$\begin{aligned}
0 &\equiv \sum_{j=0}^{K/2-1} \beta_j \varphi^{k-1}(2\cdot + 2jN/K) + \sum_{j=0}^{K/2-1} \gamma_j \psi^{k-1}(2\cdot + 2jN/K) \\
&= \sum_{j=0}^{K/2-1} \beta_j \left(\varphi^k(\cdot + jN/K) + \varphi^k(\cdot + jN/K + N/2)\right) \\
&\quad + \sum_{j=0}^{K/2-1} \gamma_j \left(\psi^k(\cdot + jN/K) + \psi^k(\cdot + jN/K + N/2)\right).
\end{aligned}$$

However, we have already proved that the sum of the spaces V^k and W^k is direct, a contradiction.

We shall need the following auxiliary statements.

Lemma 6. *Let P_1, P_2, R be algebraic polynomials. We assume that* $\deg P_1 = m$, $\deg P_2 = n$, $\deg P_3 < m+n$, *and that the polynomials P_1 and P_2 have no common roots. Then there exist polynomials Q_1 and Q_2 of degree at most $n-1$ and $m-1$ such that*

$$P_1(x)Q_1(x) + P_2(x)Q_2(x) = R(x). \tag{4.1}$$

The polynomials Q_1 and Q_2 are determined uniquely.

Remark. In a slightly less general form this lemma can be found in [1, p. 169] and [7]. Our proof, based on another idea, is a little shorter.

Proof of Lemma 6. Let $\{x_i\}_{i=1}^m$ be the roots of the polynomial P_1. Then Q_2 is determined by the conditions $Q_2(x_i) = R(x_i)/P_2(x_i)$ (here the multiplicities of the x_i must be taken into account). Since $P_2(x_i) \neq 0$, division makes sense. Similarly, the polynomial Q_1 is determined by its values at the roots $\{y_i\}_{i=1}^n$ of the polynomial P_2.

Now relation (4.1) follows, because it is fulfilled at $n+m$ points (with multiplicity taken into account) and both sides in (4.1) are polynomials of degree at most $n+m-1$.

The uniqueness of a solution of any interpolation problem (in general, with multiple points) implies that the polynomials Q_1 and Q_2 are uniquely determined. $\square$

Lemma 7. *Let $\{V^j\}$ be an arbitrary MRA, and let bases $\vec{\varphi}^j$ of V^j be chosen so as to satisfy* (1.1) *and* (1.8). *Then, for this MRA, there exist bases $\vec{\phi}^j$ such that they also satisfy* (1.1) *and* (1.8), *i.e., for any $0 \le j_1 < j_2 \le n$ we have*

$$\phi^{j_1}\left(2^{j_2-j_1}m\right) = \sum_{k=0}^{2^{j_2-j_1}-1} \phi^{j_2}\left(m + k2^{n-j_2+j_1}\right),$$

and for any $0 \le j < N$ we have

$$\phi^{j-1}(\cdot) = \sum_{l=0}^{N-1} b_l \phi^j(\cdot - lN/J);$$

in addition, the characteristic polynomial

$$P_\phi(z) \stackrel{\text{def}}{=} \sum_{l=0}^{N-1} b_l z^l$$

of the bases $\vec{\phi}^j$ satisfies $|P_\phi(z)|^2 + |P_\phi(-z)|^2 > 0$ for all z. Moreover, the degree of the polynomial P_ϕ does not exceed that of P_φ.

An analog of Lemma 7 for the MRA of the space $\mathbb{L}^2(\mathbb{R})$ can be found, e.g., in the paper [8] by C. K. Chui and J. Z. Wang.

Proof of Lemma 7. If the characteristic polynomial

$$P_\varphi(z) \stackrel{\text{def}}{=} \sum_{l=0}^{K} a_l z^l,$$

where $K < N$, has no pairs of roots of the form $\pm c$ and $P_\varphi(0) \neq 0$, then Lemma 7 is valid. Let some numbers $z = \pm c_0$ be roots of the polynomial P_φ; we put $C = c_0^2$. Then we can write

$$P_\varphi(z) = (C - z^2) \sum_{l=0}^{K-2} d_l z^l. \tag{4.2}$$

Now we construct a new basis $\vec{\phi}_1^j$; the passage to $\vec{\varphi}_1^j$ allows us to reduce the degree of the characteristic polynomial P_φ by 1. We introduce the following new function:

$$\phi_1^{j-1}(\cdot) = \sum_{l=0}^{K-2} d_l \varphi^j(\cdot - lN/J). \tag{4.3}$$

Since, by (4.2), we have $\varphi^{j-1}(\cdot) = C\phi_1^{j-1}(\cdot) - \phi_1^{j-1}(\cdot - 2N/J)$, the collection $\phi_1^{j-1}(\cdot - l2N/J)$, $l = 0, 1, \ldots, J/2-1$, forms a basis of the space V^{j-1} for $j > 0$. Here, for $j < n$ the functions ϕ_1^j are defined as in (4.3), and for $j = n$ the existence of the required function ϕ_1^n follows from Theorem 1 and Lemma 5.

It is clear that the functions belonging to bases with distinct j are related by the formulas

$$\phi_1^{j-1}(\cdot) = \sum_{l=0}^{K-1} a'_l \phi_1^j(\cdot - lN/J),$$

where the coefficients a'_l satisfy the relation

$$\sum_{l=0}^{K-1} a'_l z^l = (C - z) \sum_{l=0}^{K-2} d_l z^l.$$

We repeat the above procedure until we obtain a characteristic polynomial without roots of the form $\pm c$. Since at every step the degree of the polynomial reduces by 1, after finitely many steps we arrive at the required basis $\vec{\varphi}_*^j$. If $P_{\varphi_*}(0) \neq 0$, we take $\vec{\phi}^j = \vec{\varphi}_*^j$; otherwise we put $\vec{\phi}^j(\cdot) = \vec{\varphi}_*^j(\cdot + N/J)$. $\square$

We pass to the construction of an expansion algorithm. Let $\{V^j\}$ be a collection of subspaces forming an MRA of the space $\widetilde{\mathbb{C}}^N$, and let $\{\vec{\varphi}^j\}$ be a collection of bases of these subspaces. We assume that the $\vec{\varphi}^j$ satisfy (1.1), that the characteristic polynomial with coefficients occurring in (1.8) is of degree $K < N$, and that $|P_\phi(z)|^2 + |P_\phi(-z)|^2 > 0$ for all z.

We show that $\widetilde{\mathbb{C}}^N$ can be split into the direct (in general, nonorthogonal) sum of wavelet spaces $\{W^j\}_{j=0}^{n-1}$ for some basis of which the decomposition and recovering algorithms can be realized as discrete convolution with window of size at most $K+1$. Moreover, the convolution windows that determine the projections to the spaces W^j and V^j and recover a function $g \in V^{j+1}$ in terms of the projections of g to these spaces can be chosen independent of j. This means that an arbitrary function

$$f(\cdot) = \sum_{m=0}^{N-1} c_m^n \varphi^n(\cdot - m) \in \widetilde{\mathbb{C}}^N$$

can be decomposed as follows:

$$f(\cdot) = \sum_{m=0}^{J-1} c_m^j \varphi^j(\cdot - mN/J) + \sum_{l=j}^{n-1} \sum_{m=0}^{L-1} d_m^l \psi^l(\cdot - mN/L).$$

Here $\vec{\psi}^l$ are bases of the spaces W^l; the coefficients $\left\{\{c_m^j\}_{m=0}^{J-1}\right\}_{j=0}^{n-1}$ and $\left\{\{d_m^j\}_{m=0}^{J-1}\right\}_{j=0}^{n-1}$ can be calculated by the recurrence formulas

$$c_m^{j-1} = \sum_{l=\xi}^{\xi+K} c_{l+2m}^j r_l, \qquad d_m^{j-1} = \sum_{l=\zeta}^{\zeta+K} c_{l+2m}^j s_l, \tag{4.4}$$

where $\xi, \zeta \in \mathbb{Z}$. Later we shall see that the upper limit of summation in the first of the above two formulas can be reduced by 1. The sequences c_m^j and d_m^j are J-periodic relative to the subscript, and we may assume that the sequences r_l and s_l are well defined for all integral indices, but they differ from zero only if $l = \xi, \xi+1, \ldots, \xi+K-1$ (for the first sequence) or $l = \zeta, \zeta+1, \ldots, \zeta+K$ (for the second one). In what follows we put $\zeta = \xi$ if $\zeta + K$ is odd, and $\zeta = \xi - 1$ otherwise. Keeping the size of windows, we can move them by changing the parameter ξ, thus obtaining new solutions of our problem. A sole requirement imposed on the position of a window r_l is that it must have nonempty intersection with the segment $[0, K]$ or with its shift by lN, where $l \in \mathbb{Z}$. Otherwise, for $j = n$ the first of the operators (4.4) takes φ^{n-1} to a function of the form

$$\sum_{l=0}^{N/2-1} \alpha_l \varphi^{n-1}(\cdot - 2l),$$

where $\alpha_0 = 0$, which is impossible since this operator is a projection from V^n to V^{n-1}. We assume that ξ is an arbitrary number belonging to the segment $[1-K, K]$.

We construct the spaces W^j using the condition that their bases $\vec{\psi}^j$ are subject to the relations

$$\psi^{j-1}(\cdot) = \sum_{m=0}^{K} b_m \varphi^j(\cdot - mN/J),$$

where the b_m are coefficients to be determined. Since relations (4.4) give projections to V^{j-1} and to W^{j-1}, for $m = 0, 1, \dots, N/2-1$ we have

$$\delta_{0m} = \sum_{l=\xi}^{\xi+K-1} a_{l-2m} r_l, \tag{4.5}$$

$$\delta_{0m} = \sum_{l=\zeta}^{\zeta+K} b_{l-2m} s_l, \tag{4.6}$$

$$0 = \sum_{l=\zeta}^{\zeta+K} a_{l-2m} s_l, \tag{4.7}$$

$$0 = \sum_{l=\xi}^{\xi+K-1} b_{l-2m} r_l \tag{4.8}$$

(we assume that the sequences $\{a_l\}$ and $\{b_l\}$ are N-periodic and that, within a period, they are extended by zeros off their natural domain of definition). It is well known (see, for example, [8] and [9]) and can easily be checked that in order to solve the above system of convolution equations it suffices to solve the following system of equations:

$$A(z)R(z) + A(-z)R(-z) \equiv 2, \tag{4.9}$$

$$B(z)S(z) + B(-z)S(-z) \equiv 2, \tag{4.10}$$

$$A(z)S(z) + A(-z)S(-z) \equiv 0, \tag{4.11}$$

$$B(z)R(z) + B(-z)R(-z) \equiv 0, \tag{4.12}$$

where

$$A(z) = \sum_{l=0}^{K} a_l z^l, \; B(z) = \sum_{l=0}^{K} b_l z^l,$$

$$R(z) = \sum_{l=\xi}^{\xi+K-1} r_l z^{-l} = z^{-\xi-K+1} \sum_{l=0}^{K-1} r_{\xi+K-1-l} z^l \stackrel{\text{def}}{=} z^{-\xi-K+1} \check{R}(z),$$

$$S(z) = \sum_{l=\zeta}^{\zeta+K} s_l z^{-l} = z^{-\zeta-K} \sum_{l=0}^{K} s_{\zeta+K-l} z^l \stackrel{\text{def}}{=} z^{-\zeta-K} \check{S}(z).$$

Equations (4.9)–(4.12) can be rewritten in the form

$$A(z)\check{R}(z) + (-1)^{\xi+K-1}A(-z)\check{R}(-z) = 2z^{\xi+K-1}, \tag{4.13}$$

$$B(z)\check{S}(z) + (-1)^{\zeta+K}B(-z)\check{S}(-z) = 2z^{\zeta+K}, \tag{4.14}$$

$$A(z)\check{S}(z) + (-1)^{\zeta+K}A(-z)\check{S}(-z) = 0, \tag{4.15}$$

$$B(z)\check{R}(z) + (-1)^{\xi+K-1}B(-z)\check{R}(-z) = 0, \tag{4.16}$$

By Lemma 6, the choice of ξ implies that there is a unique pair of polynomials Q_1 and Q_2 of degree at most $K-1$ satisfying the equation

$$A(z)Q_1(z) + (-1)^{\xi+K-1}A(-z)Q_2(z) = 2z^{\xi+K-1}.$$

Obviously, we have $Q_1(z) = Q_2(-z)$. Thus, the polynomial $\check{R}(z) = Q_1(z)$ satisfies (4.13).

For $\xi + K$ odd, it is not hard to verify that the functions $B(z) = z\check{R}(-z)$ and $\check{S}(z) = A(-z)$ satisfy equations (4.13) — (4.16). For $\xi + K$ even, we can take $B(z) = \check{R}(-z)$ and $\check{S}(z) = -A(-z)$.

In spite of apparent freedom in the choice of B and $\check{S}$, the spaces W^j are determined completely by the choice of $\check{R}$, because the coefficients of that polynomial determine the projection to the space V^j along the space W^j.

In order to show that the space W^j obtained in this way has dimension J we shall prove the recovering formula

$$c_m^j = \sum_{l=0}^{N-1} c_l^{j-1} a_{m-2l} + \sum_{l=0}^{N-1} d_l^{j-1} b_{m-2l}, \tag{4.17}$$

which inverts the operation of projecting V^j to W^{j-1} and V^{j-1}. This formula implies that

$$\dim W^{j-1} = \dim V^j - \dim V^{j-1} = J - J/2 = J/2.$$

It suffices to check (4.17) for $j = n$. We denote by A the matrix of size $2^{n-1} \times 2^n$ such that its first row consists of the elements $a_0, a_1, \dots$, and each of the next rows is obtained from the previous one by double shift to the right. The matrices B, R, S are constructed in a similar way. Then the system of equations (4.5)–(4.8) can be rewritten in the matrix form as follows:

$$\begin{pmatrix} A \\ B \end{pmatrix} (R^T S^T) = E.$$

Consequently, the matrix $(R^T S^T)$ has the inverse matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ and, thus, is nonsingular. Therefore, (4.14) yields an operator of reconstruction.

Clearly, if K is large, then the approach based on the passage to the Wiener bases (see §2) is preferable. Since for any $j = 1, \dots, n$ and any l the linear spans of the pairs $v_l^j, v_{l+J/2}^j$ and v_l^{j-1}, w_l^{j-1} coincide (as in the orthogonal case), formulas (2.2)–(2.5) can be carried (with other coefficients) over to the case of nonorthogonal wavelet spaces.

References

1. I. Daubéchies, *Ten lectures on wavelets*, CBMS–NSF Regional Conf. Ser. in Appl. Math., **61**, SIAM, Philadelphia, 1992.
2. V. A. Zheludev, *Periodic splines and wavelets*, Mathematical Analysis, Wavelets, and Signal Processing (Cairo, 1994), Contemp. Math., **190**, Amer. Math. Soc., Providence, RI, 1995, pp. 339–354.
3. ———, *An operational calculus connected with periodic splines*, Dokl. Akad. Nauk SSSR **313** (1990), no. 6, 1309–1315; English transl., Soviet Math. Dokl. **42** (1991), no. 1, 162–167.
4. V. Perrier and C. Basdevant, *Periodical wavelet analysis, a tool for inhomogeneous field investigation. Theory and algorithms*, Rech. Aérospat. (English Edition) **1989**, no. 3, 53–67.
5. C. K. Chui and H. N. Mhaskar, *On trigonometric wavelets*, Constr. Approx. **9** (1993), no. 2–3, 167–190.
6. C. de Boor, R. A. DeVore, and A. Ron, *On the construction of multivariate (pre)wavelets*, Constr. Approx. **9** (1993), no. 2–3, 123–166.

7. I. Daubéchies, *Orthonormal bases of compactly supported wavelets*, Comm. Pure Appl. Math. **41** (1988), 909–996.
8. C. K. Chui and J. Z. Wang, *A general framework of compactly supported splines and wavelets*, J. Approx. Theory **71** (1992), 263–304.
9. C. K. Chui, *An introduction to wavelets. Wavelet analysis and its applications*, 1, Academic Press, Inc., Boston, MA, 1992.
10. W. Dahmen and C. A. Micchelli, *Banded matrices with banded inverses.* II. *Locally finite decomposition of spline spaces*, Constr. Approx. **9** (1993), no. 2–3, 263–281.

Translated by THE AUTHOR

St. Petersburg Branch, Steklov Mathematical Institute, Russian Academy of Sciences, Fontanka, 27, 191011, St. Petersburg, Russia

E-mail address: petukhov@pdmi.ras.ru