Nonlinear Analysis and Design of Phase-locked loops (PLL) & Costas loop

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http://www.math.spbu.ru/user/nk/

http://www.math.spbu.ru/user/nk/Nonlinear_analysis_of_PLL.htm

tutorial last version:

http://www.math.spbu.ru/user/nk/PDF/Nonlinear-analysis-Phase-locked-loop-PLL.pdf

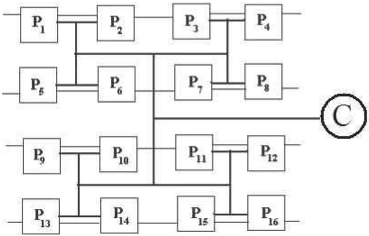
Phase-locked loops (PLL): history

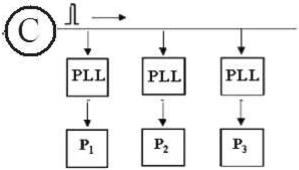
- Radio and TV
  - de Bellescize H., La réception synchrone, L'Onde Électrique, 11, 1932
  - Wendt, K. & Fredentall, G. Automatic frequency and phase control of synchronization in TV receivers, *Proceedings IRE*, 31(1), 1943
- Computer architectures (frequency multiplication)
  Ian Young, PLL in a microprocessor i486DX2-50 (1992)
  (in Turbo regime stable operation was not guaranteed)

- Theory and Technology
  ✓ F.M.Gardner, *Phase-Lock Techniques*, 1966
  ✓ A.J. Viterbi, *Principles of Coherent Comm.*, 1966
  ✓ W.C.Lindsey, *Synch. Syst. in Comm. and C.*, 1972
  ✓ W.F.Egan, *Freq. Synthesis by Phase Lock*, 2000
  ✓ B. Razavi, *Phase-Locking in High-Perf. Syst.*, 2003
  ✓ R.Best, *PLL: Design, Simulation and Appl.*, 2003
  ✓ V.Kroupa, *Phase Lock Loops and Freq. Synthesis*, 2003

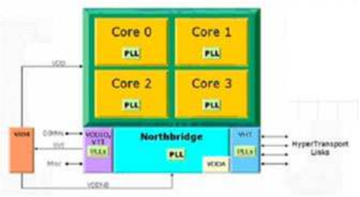
PLL in Computer architectures

Clock Skew Elimination: multiprocessor systems



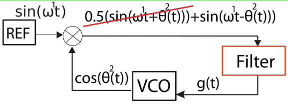


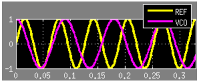
Frequencies synthesis: multicore processors, motherboards

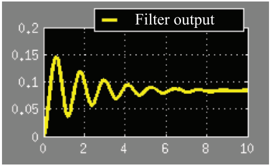


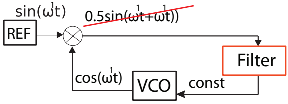
PLL analysis & design

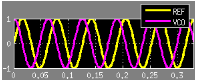
PLL operation: generation of electrical signal (voltage), phase of which is automatically tuned to phase of input reference signal (after transient process the signal, controlling the frequency of tunable osc., is constant)











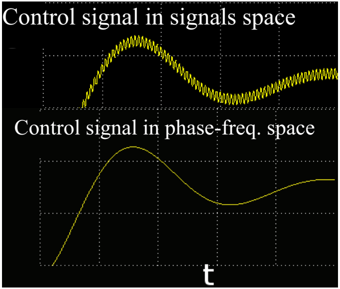
Design: Signals class (sinusoidal,impulse...), PLL type (PLL,ADPLL,DPLL...)

Analysis: Choose PLL parameters (VCO, PD, Filter etc.) to achieve stable operation for the desired range of frequencies and transient time

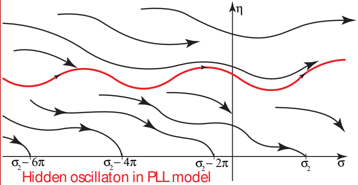
Analysis methods: simulation, linear analysis, nonlinear analysis of mathematical models in signal space and phase-frequency space

PLL analysis & design: simulation

D.Abramovitch, ACC-2008 plenary lecture:
Full simulation of PLL in space of signals is very difficult since one has to observe simultaneously very fast time scale of input signals and slow time scale of signal's phases.
How to construct model of signal's phases?

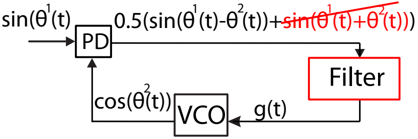


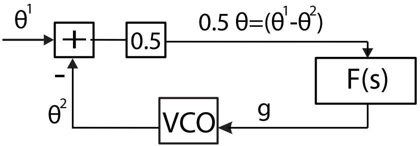
Simulation in space of phases: Could stable operation be guaranteed for all possible inputs, internal blocks states by simulation?
N.Gubar' (1961), hidden oscillation in PLL: global stability in simulation, but only bounded region of stability in reality



Survey: Leonov G.A., Kuznetsov N.V., Hidden attractors in dynamical systems. From hidden oscillations in Hilbert-Kolmogorov, Aizerman, and Kalman problems to hidden chaotic attractor in Chua circuits, International Journal of Bifurcation and Chaos, 23(1), 2013, art. no. 1330002

PLL analysis & design: analytical linear methods





Linearization: while this is useful for studying loops that are near lock, it does not help for analyzing the loop when misphasing is large.

$$\otimes - \mathrm{PD}: \ \sin(\theta^1)\cos(\theta^2) = 0.5\sin(\theta^1{+}\theta^2){+}\sin(\theta^1{+}\theta^2) \approx 0.5(\theta^1{+}\theta^2)$$

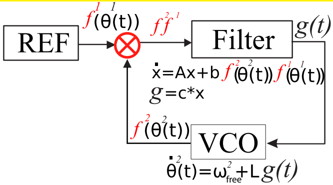
$$\mathrm{VCO}: \quad \dot{\theta}^2(t) = \omega_{free} + Lg(t)$$

Linearization justification problems: Harmonic linearization (describing function method), Aizerman & Kalman conjectures,

Time-varying linearization & Perron effects of Lyapunov exponent sign inversions

Survey: V.O. Bragin, V.I. Vagaitsev, N.V. Kuznetsov, G.A. Leonov, Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman conjectures and Chua's circuits, *J. of Computer and Systems Sciences Int., 50(4), 2011, 511-544 (doi:10.1134/S10642307110406X)*
Survey: G.A. Leonov, N.V. Kuznetsov, Time-Varying Linearization and the Perron effects, Int. J. of Bifurcation and Chaos, Vol. 17, No. 4, 2007, 1079-1107 (doi:10.1142/S0218127407017732)

?) Control signals in signals and phase-frequency spaces are equal.
?) Qualitative behaviors of signals and phase-freq. models are the same.



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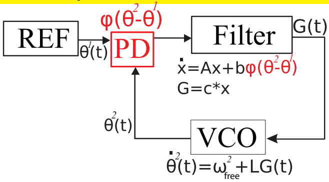


$$\dot\theta^j \geq \omega_{min},\ |\dot\theta^1 - \dot\theta^2| \leq \Delta\omega,\ |\dot\theta^j(t) - \dot\theta^j(\tau)| \leq \Delta\Omega,\ \ |\tau - t| \leq \delta,\ \forall \tau, t \in [0, T]\ \ (*)$$

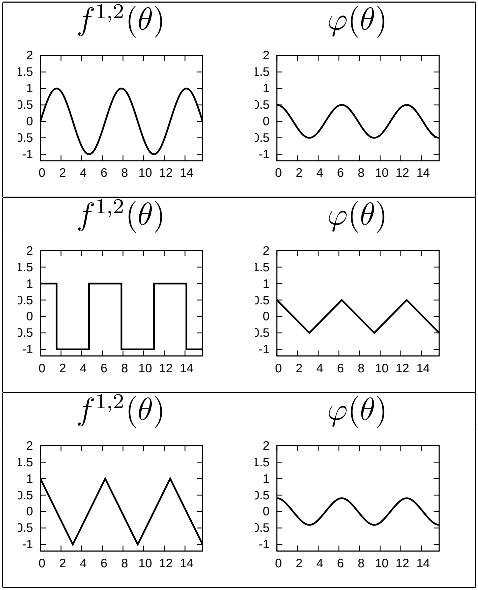
waveforms: $f^p(\theta) = \sum_{i=1}^{\infty}\big(a_i^p \sin(i\theta) + b_i^p \cos(i\theta)\big), \quad p = 1, 2$

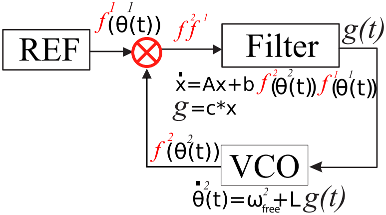
$b_i^p = \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(x)\sin(ix)dx, \quad a_i^p = \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(x)\cos(ix)dx,$

Thm. If (*), $\varphi(\theta) = \frac{1}{2}\sum_{l=1}^{\infty}\big((a_l^1 a_l^2 + b_l^1 b_l^2)\cos(l\theta) + (a_l^1 b_l^2 - b_l^1 a_l^2)\sin(l\theta)\big)$
then $|G(t) - g(t)| = O(\delta),\ \forall\, t \in [0, T]$

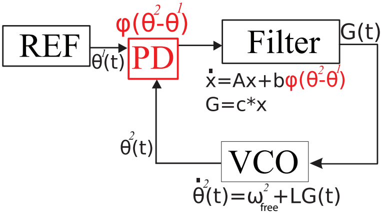
Leonov G.A., Kuznetsov N.V., Yuldahsev M.V., Yuldashev R.V., Analytical method for computation of phase-detector characteristic, IEEE Transactions on Circuits and Systems Part II, vol. 59, num. 10, 2012 (doi:10.1109/TCSII.2012.2213362)

PD characteristics examples



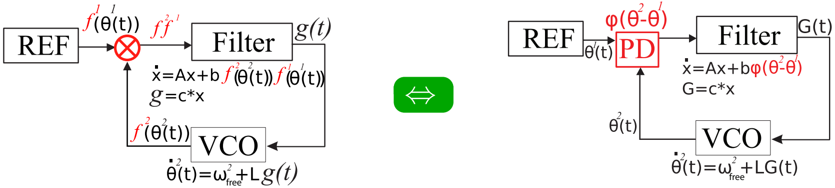


- $f^{1,2}$ — waveforms



- φ — PD characteristic

Differential equations of PLL



- master oscillator phase: $\dot{\theta}_1(t) \equiv \omega_1$
- phase & freq. difference: $\theta_d(t) = \theta_1(t) - \theta_2(t)$, $\omega_2 - \omega_1 = \Delta\omega$

Nonautonomous equations

$$\dot{x} = Ax + bf_1(\omega_1 t)f_2(\theta_d + \omega_1 t),$$
$$\dot{\theta}_d = (\omega_2 - \omega_1) + Lc^* x,$$
$$x(0) = x_0,\ \ \theta_d(0) = \theta_0$$

⇔

Autonomous equations

$$\dot{x} = Ax + b\varphi(\theta_d),$$
$$\dot{\theta}_d = \Delta\omega + Lc^* x,$$
$$x(0) = x_0,\ \ \theta_d(0) = \theta_0$$

?) Qualitative behaviors of signals and phase-freq. models are the same.

Autonomous differential equations of PLL

?) Qualitative behaviors of signals and phase-freq. models are the same.

$$\dot{z} = Az + bf^1(\omega^1 t) f^2(\eta_d + \omega^1 t), \quad \dot{\eta}_d = \omega_d + Lc^* z \quad \Leftrightarrow \quad \dot{x} = Ax + b\varphi(\theta_d), \quad \dot{\theta}_d = \omega_d + Lc^* x$$

If $\omega^1(t) \equiv \omega^1$ then averaging method [Bogolubov, Krylov] allows one to justify passing to autonomous diff. equations of PLL.

$$\tau = \omega^1 t, \qquad \omega^1 \frac{du}{d\tau} = Au + bf^1(\tau) f^2(\eta_d + \tau),$$
$$u(\tau) = z\left(\frac{\tau}{\omega^1}\right) = z(t), \quad \omega^1 \frac{d\eta_d}{d\tau} = \omega_d + Lc^* u.$$

Aver. method requires to prove $\frac{1}{T} \int\limits_0^T f^1(\tau) f^2(\eta_d + \tau) - \varphi(\eta_d) dt \underset{T\to\infty}{\rightrightarrows} 0$

(it is satisfied for phase detector characteristic φ computed above). Solutions $(z(t), \eta_d(t))$ & $(x(t), \theta_d(t))$ (with the same initial data)) of nonautonomous and autonomous models are close to each other on a certain time interval as $\omega^1 \to \infty$.

Nonlinear analysis of PLL

Continuous

$$\dot{x} = Ax + b\varphi(\theta_d),$$
$$\dot{\theta}_d = \Delta\omega + Lc^*x,$$
$$x(0) = x_0,\ \theta_d(0) = \theta_0$$

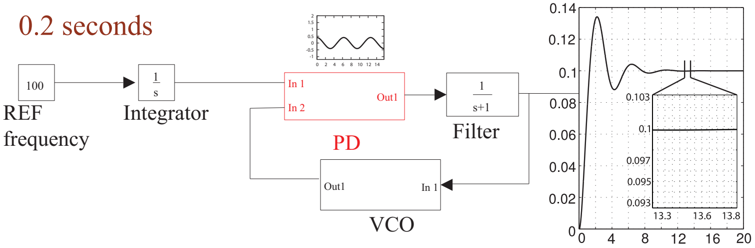
Discrete

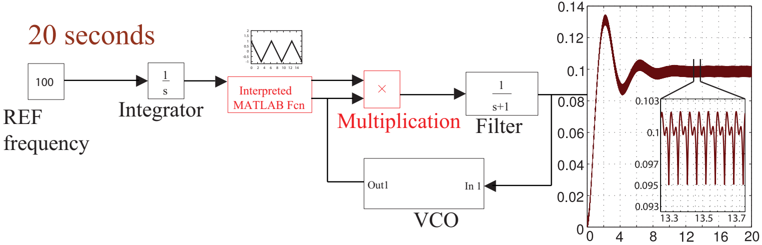
$$x(t+1) = Ax(t) + b\varphi(\theta_d(t)),$$
$$\theta_d(t+1) = \theta_d(t) + \Delta\omega + Lc^*x(t),$$
$$x(0) = x_0,\ \theta_d(0) = \theta_0$$

Here it is possible to apply various well developed methods of mathematical theory of phase synchronization

- Nonlinear Analysis and Design of Phase-Locked Loops, G.A. Leonov, N.V. Kuznetsov, S.M. Seledzhi, (chapter in "Automation control - Theory and Practice", In-Tech, **2009**), pp. 89-114
- V. Yakubovich, G. Leonov, A. Gelig, *Stability of Systems with Discontinuous Nonlinearities*, (Singapore: World Scientific), **2004**
- G. Leonov, D. Ponomarenko, V. Smirnova, *Frequency-Domain Methods for Nonlinear Analysis. Theory and Applications* (Singapore: World Scientific, **1996**).
- G. Leonov, V. Reitmann, V. Smirnova, *Nonlocal Methods for Pendulum-Like Feedback Systems* (Stuttgart; Leipzig: Teubner Verlagsgesselschaft, **1992**).

PLL simulation: Matlab Simulink



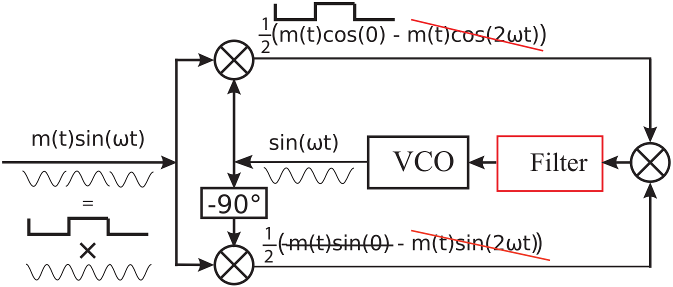


Kuznetsov, Leonov et al., Patent 2449463, 2011
Kuznetsov, Leonov et al., Patent 112555, 2011

Costas loop: digital signal demodulation

John P. Costas. General Electric, 1950s.

- signal demodulation in digital communication



$m(t) = \pm 1$ — data,

$m(t)\sin(2\omega t)$ and $m(t)\cos(2\omega t)$

— can be filtered out,

$m(t)\cos(0) = m(t)$,

$m(t)\sin(0) = 0$

- wireless receivers
- Global Positioning System (GPS)

Costas Loop:PD characteristic computation

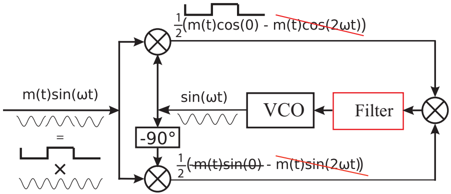
Analysis of Costas Loop can reduced to the analysis of PLL:

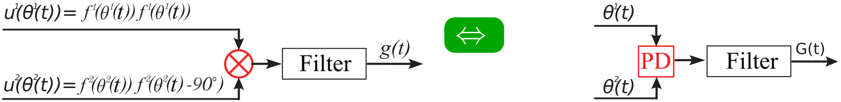
$u^1(\theta^1(t)) = f^1(\theta^1(t))\, f^1(\theta^1(t)),$

$u^2(\theta^2(t)) = f^2(\theta^2(t))\, f^2(\theta^2(t) - \frac{\pi}{2}),$

Costas Loop PD characteristic:

A_l^k, B_l^k–Fourier coefficients of $u_k(\theta)$





Theorem. If (1)–(2) and

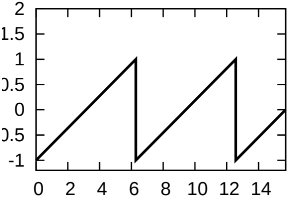
$$\varphi(\theta) = \frac{A_1^0 A_2^0}{4} + \frac{1}{2}\sum_{l=1}^{\infty}\Big((A_1^l A_2^l + B_1^l B_2^l)\cos(l\theta) + (A_1^l B_2^l - B_1^l A_2^l)\sin(l\theta)\Big)$$

then $|G(t) - g(t)| \le C\delta,\ \forall\, t \in [0, T]$

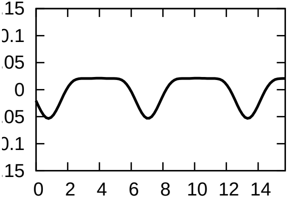
G.A. Leonov, N.V. Kuznetsov, M.V. Yuldashev, R.V. Yuldashev, Differential equations of Costas Loop, **Doklady Mathematics**, **2012**, 86(2), 149-154 (doi:10.1134/S1064562412050080)

Costas loop:PD characteristic example

$f^{1,2}(\theta)$



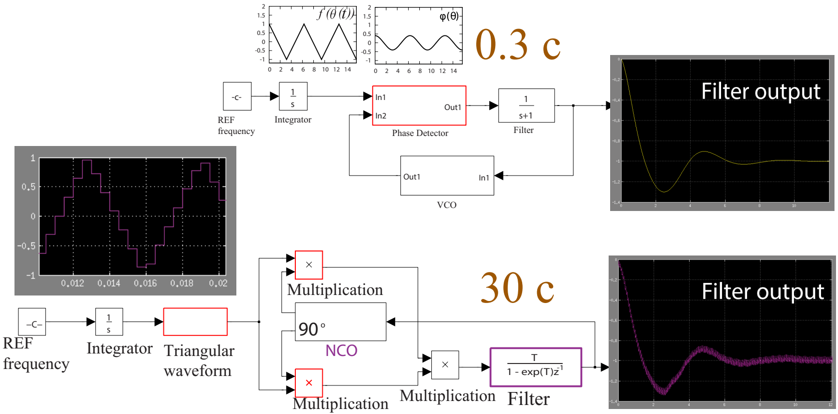
$\varphi(\theta)$



$$f^{1,2}(\theta) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(n\theta\right)$$

$$\varphi(\theta) = -\frac{1}{72} + \frac{1}{2}\sum_{l=1}^{\infty}\begin{cases} \frac{16}{\pi^4 l^4}\cos(l\theta), & l = 4p, \\ -\frac{8}{\pi^3 l^3}\sin(l\theta), & l = 4p+2 \\ -\frac{4(\pi l-2)}{\pi^4 l^4}\cos(l\theta) - \frac{4(\pi l-2)}{\pi^4 l^4}\sin(l\theta), & l = 4p+1 \\ \frac{4(\pi l+2)}{\pi^4 l^4}\cos(l\theta) - \frac{4(\pi l+2)}{\pi^4 l^4}\sin(l\theta), & l = 4p+3 \end{cases}$$

Costas loop simulation: Matlab Simulink

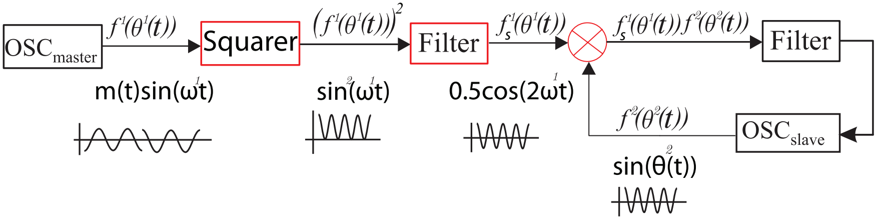


Squaring loop

PLL-based carrier recovery circuit

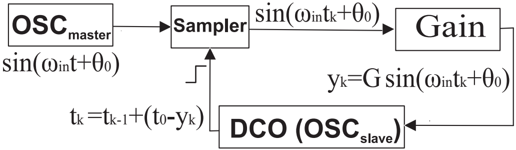
Golomb et al., 1963; Stiffler, 1964; Lindsey, 1966

- wireless recievers
- demodulation



- $m(t) = \pm 1$ — data,
- $(m(t)sin(\omega^1 t))^2 = \frac{1-\cos(2\omega t)}{2}$ — squaring allows to remove data
- $0.5\cos(2\omega t)$ — filter removes the constant

Classical DPLL: nonlinear analysis



$$\phi_{k+1} - (\phi_k + 2\pi) =$$
$$= 2\pi\left(\frac{\omega_{in}}{\omega} - 1\right) - \frac{\omega_{in}}{\omega}\omega G \sin(\phi_k)$$
$$\omega_{in} = \omega,\ \phi_k = \omega t_k + \theta_0,$$
$$\phi_k + 2\pi \to \sigma_k \in [\pi, \pi]$$
$$\sigma_{t+1} = \sigma_t - r \sin \sigma_t,\ \ r = \omega G$$

$r_1 = 2$: Global asymptotic stability disappears and appears unique and globally stable on $(-\pi, 0) \cup (0, -\pi)$ cycle with period 2

$r_2 = \pi$: Bifurcation of splitting: unique cycle loses stability and two locally stable asymmetrical cycles with period 2 appear

$r_3 = \sqrt{\pi^2 + 2}$: Two locally stable asymmetrical cycles with period 2 lose stability and two locally stable asymmetrical cycles with period 4 appear

G.A. Leonov, N.V. Kuznetsov, S.M. Seledzhi, Nonlinear Analysis and Design of Phase-Locked Loops, in '*Automation control - Theory and Practice*, In-Tech, 2009, 89-114

E.V. Kudryashova, N.V. Kuznetsov, G.A. Leonov, P. Neittaanmaki, S.M. Seledzhi, Analysis and synthesis of clock generator, in *From Physics to Control Through an Emergent View*, World Scientific, 2010, 131-136 (doi:10.1142/9789814313155_0019)

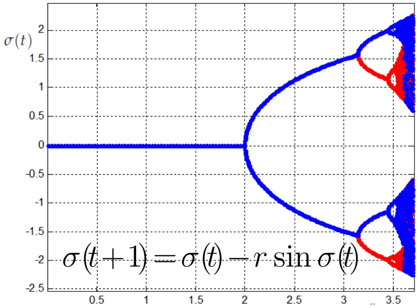
Classical DPLL: simulation of transition to chaos

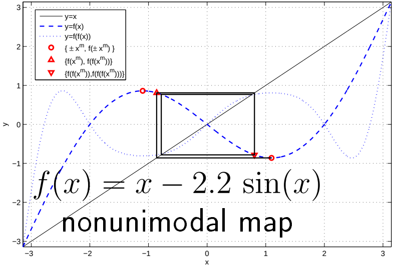
№j	Bifurcation value of parameter, r_j	Feigenbaum const $\delta_j = \frac{r_j - r_{j-1}}{r_{j+1} - r_j}$
1	2	
2	π	3.7597337326
3	3.445229223301312	4.4874675842
4	3.512892457411257	4.6240452067
5	3.527525366711579	4.6601478320
6	3.530665376391086	4.6671765089
7	3.531338162105000	4.6687679883
8	3.531482265584890	4.6690746582
9	3.531513128976555	4.6691116965
10	3.531519739097210	4.6690257365
11	3.531521154835959	

E.V. Kudryashova, N.V. Kuznetsov, G.A. Leonov, P. Neittaanmaki, S.M. Seledzhi, Analysis and synthesis of clock generator, in *From Physics to Control Through an Emergent View*, World Sci., 2010, 131-136 (doi:10.1142/9789814313155_0019)

Feigenbaum const for unimodal maps:

$$= \lim_{n \to +\infty} \delta_n = 4.669..$$





Publications: PLL and Costas loop, 2012-2011

- Leonov G.A., Kuznetsov N.V., M.V. Yuldashev, R.V. Yuldashev, Differential equations of Costas Loop, Doklady Mathematics, 86(2), 2012, 723-728 (doi: 10.1134/S1064562412050080)
- Kuznetsov N.V., Leonov G.A., Neittaanmaki P., Seledzhi S.M., Yuldashev M.V., Yuldashev R.V., Simulation of phase-locked loops in phase-frequency domain, International Congress on Ultra Modern Telecommunications and Control Systems and Workshops, 2012, IEEE art. no. 6459692, 351-356 (doi:10.1109/ICUMT.2012.6459692)
- Kuznetsov N.V., Leonov G.A., Neittaanmaki P., Seledzhi S.M., Yuldashev M.V., Yuldashev R.V., Nonlinear mathematical models of Costas Loop for general waveform of input signal, IEEE 4th International Conference on Nonlinear Science and Complexity, NSC 2012 - Proceedings, art. no. 6304729, 2012, 75-80 (doi:10.1109/NSC.2012.6304729)
- Leonov G.A., Kuznetsov N.V., Yuldashev M.V., Yuldashev R.V., Analytical method for computation of phase-detector characteristic, **IEEE Transactions on Circuits and Systems II**, 59(10), 2012, 633-637 (doi:10.1109/TCSII.2012.2213362)
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- Kuznetsov N.V., Leonov G.A., Yuldashev M.V., Yuldashev R.V., Nonlinear analysis of Costas loop circuit, 9th International Conference on Informatics in Control, Automation and Robotics, Vol. 1, 2012, 557-560, (doi:10.5220/0003976705570560)
- Kuznetsov N.V., Leonov G.A., Neittaanmaki P., Seledzhi S.M., Yuldashev M.V., Yuldashev R.V., Simulation of Phase-Locked Loops in Phase-Frequency Domain, IEEE IV International Congress on Ultra Modern Telecommunications and Control Systems, 2012, 364-368
- G.A. Leonov, N.V. Kuznetsov, M.V. Yuldashev, R.V. Yuldashev, Computation of Phase Detector Characteristics in Synchronization Systems, **Doklady Mathematics**, 84(1), 2011, 586-590, (doi:10.1134/S1064562411040223)

Publications: PLL and Costas loop, 2011-...

- N.V. Kuznetsov, G.A. Leonov, M.V. Yuldashev, R.V. Yuldashev, Analytical methods for computation of phase-detector characteristics and PLL design, ISSCS 2011-IEEE International Symposium on Signals, Circuits and Systems, Proceedings, **2011**, 7-10, (doi:10.1109/ISSCS.2011.5978639)
- G.A. Leonov, S.M. Seledzhi, N.V. Kuznetsov, P. Neittaanmaki, Nonlinear Analysis of Phase-locked loop, IFAC Proceedings Volumes (IFACPapersOnline), 4(1), **2010** (doi:10.3182/20100826-3-TR-4016.00010)
- G.A. Leonov, S.M. Seledzhi, N.V. Kuznetsov, P. Neittaanmaki, Asymptotic analysis of phase control system for clocks in multiprocessor arrays, ICINCO 2010 - Proceedings of the 7th International Conference on Informatics in Control, Automation and Robotics, 3, **2010**, 99-102 (doi:10.5220/0002938200990102)
- N.V. Kuznetsov, G.A. Leonov, S.M. Seledzhi, Nonlinear analysis of the Costas loop and phase-locked loop with squarer, IASTED Int. conference on Signal and Image Processing, SIP 2009, **2009**, 1-7
- N.V. Kuznetsov, G.A. Leonov, S.M. Seledzhi, P. Neittaanmaki, Analysis and design of computer architecture circuits with controllable delay line. ICINCO 2009 - 6th International Conference on Informatics in Control, Automation and Robotics, Proceedings, 3 SPSMC, **2009**, 221-224 (doi:10.5220/0002205002210224).
- N.V. Kuznetsov, G.A. Leonov, S.M. Seledzhi, Phase locked loops design and analysis. ICINCO 2008 - 5th International Conference on Informatics in Control, Automation and Robotics, Proceedings, SPSMC, **2008**, 114-118 (doi:10.5220/0001485401140118)
- N.V. Kuznetsov, G.A. Leonov, S.M. Seledzhi, Analysis of phase-locked systems with discontinuous characteristics of the phase detectors, IFAC Proceedings Volumes (IFACPapersOnline), 1 (PART 1), **2006**, 107-112 (doi:10.3182/20060628-3-FR-3903.00021)

Lyapunov exponent: sign inversions, Perron effects, chaos, linearization

$$\begin{cases} \dot{x} = F(x), \ x \in \mathbb{R}^n, \quad F(x_0) = 0 \\ x(t) \equiv x_0, A = \frac{dF(x)}{dx}\Big|_{x=x_0} \end{cases} \quad \begin{cases} \dot{y} = Ay + o(y) \\ y(t) \equiv 0, (y = x - x_0) \end{cases} \quad \begin{cases} \dot{z} = Az \\ z(t) \equiv 0 \end{cases}$$

✓stationary: $z(t) = 0$ is exp. stable $\Rightarrow$ $y(t) = 0$ is asympt. stable

$$\begin{cases} \dot{x} = F(x), \quad \dot{x}(t) = F(x(t)) \not\equiv 0 \\ x(t) \not\equiv x_0, A(t) = \frac{dF(x)}{dx}\Big|_{x=x(t)} \end{cases} \quad \begin{cases} \dot{y} = A(t)y + o(y) \\ y(t) \equiv 0, (y = x - x(t)) \end{cases} \begin{cases} \dot{z} = A(t)z \\ z(t) \equiv 0 \end{cases}$$

? nonstationary: $z(t) = 0$ is exp. stable $\Rightarrow$? $y(t) = 0$ is asympt. stable

! Perron effects: $z(t) = 0$ is exp. stable(unst), $y(t) = 0$ is exp. unstable(st)

Positive largest Lyapunov exponent
doesn't, in general, indicate chaos

Survey: G.A. Leonov, N.V. Kuznetsov, Time-Varying Linearization and the Perron effects, Int. Journal of Bifurcation and Chaos, 17(4), 2007, 1079-1107 (doi:10.1142/S0218127407017732)
N.V. Kuznetsov, G.A. Leonov, On stability by the first approximation for discrete systems, 2005 Int. Conf. on Physics and Control, PhysCon 2005, Proc. Vol. 2005, 2005, 596-599

Harmonic linearization without justification may lead to wrong results

Harmonic balance & Describing function method in Absolute Stability Theory

$\dot{x} = Px + q\psi(r^*x),\ \psi(0) = 0 \quad (1)$ $\qquad \dot{x} = P_0x + q\varphi(r^*x)$

$W(p) = r^*(P - pI)^{-1}q \qquad P_0 = P + kqr^*, \varphi(\sigma) = \psi(\sigma) - k\sigma$

$\mathrm{Im}W(i\omega_0) = 0, k = -(\mathrm{Re}W(i\omega_0))^{-1} \qquad P_0 : \lambda_{1,2} = \pm i\omega_0,\ \mathrm{Re}\lambda_{j>2} < 0$

DFM: exists periodic solution $\sigma(t) = r^*x(t) \approx a\cos\omega_0 t$

$$a:\ \int_0^{2\pi/\omega_0} \psi(a\cos\omega_0 t)\cos\omega_0 t\mathrm{d}t = ka\int_0^{2\pi/\omega_0} (\cos\omega_0 t)^2 \mathrm{d}t$$

Aizerman problem: *If (1) is stable for any linear* $\psi(\sigma) = \mu\sigma,\ \mu \in (\mu_1, \mu_2)$
then (1) is stable for any nonlinear $\psi(\sigma) : \mu_1\sigma < \psi(\sigma) < \mu_2\sigma,\ \forall\sigma \neq 0$

DFM: (1) is stable $\Rightarrow k : k < \mu_1, \mu_2 < k \Rightarrow k\sigma^2 < \psi(\sigma)\sigma, \psi(\sigma)\sigma < k\sigma^2$

$$\Rightarrow \forall a \neq 0 :\ \int_0^{2\pi/\omega_0} (\psi(a\cos\omega_0 t)\, a\cos\omega_0 t - k(a\cos\omega_0 t)^2)\mathrm{d}t \neq 0$$

$\Rightarrow$ no periodic solutions by harmonic linearization and DFM, but

Survey: V.O. Bragin, V.I. Vagaitsev, N.V. Kuznetsov, G.A. Leonov (2011) Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman conjectures and Chua's circuits, *J. of Computer and Systems Sciences Int., V.50, N4, 511-544 (doi:10.1134/S106423071104006X)*

Hidden attractor in classical Chua's system

In 2010 the notion of *hidden attractor* was introduced
hidden chaotic attractor was found in Chua circuits for the first time by the authors
[Leonov G.A., Kuznetsov N.V., Vagaitsev V.I., Physics Letters A, 375(23), 2011]

$$\dot{x} = \alpha(y - x - m_1 x - \psi(x))$$
$$\dot{y} = x - y + z, \dot{z} = -(\beta y + \gamma z)$$
$$\psi(x) = (m_0 - m_1)\mathrm{sat}(x)$$

$\alpha = 8.4562, \beta = 12.0732$
$\gamma = 0.0052$
$m_0 = -0.1768, m_1 = -1.1468$

equilibria: stable zero F_0 & 2 saddles $S_{1,2}$
trajectories: 'from' $S_{1,2}$ tend (black) to zero F_0 or tend (red) to infinity;
Hidden chaotic attractor (in green) with positive Lyapunov exponent