Nonlinear Analysis and Design of Phase-Locked Loops (PLL)

Nikolay V. Kuznetsov Gennady A. Leonov
Svetlana M. Seledzhi

Department of Applied Cybernetics
Faculty of Mathematics and Mechanics
Saint-Petersburg State University
http://www.math.spbu.ru/user/nk/

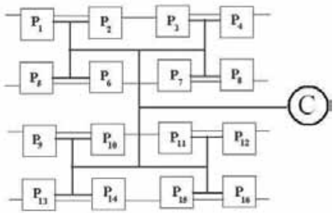
http://www.math.spbu.ru/user/nk/Nonlinear_analysis_of_PLL.htm

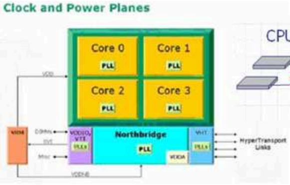
last version: http://www.math.spbu.ru/user/nk/PDF/Nonlinear_analysis_of_PLL.pdf

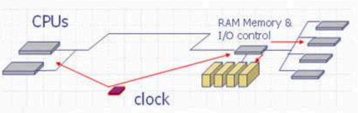
Phase-locked loops (PLL): history

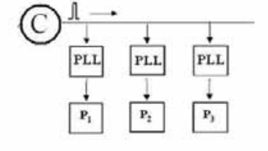
- Radio and TV
  - Henri de Bellescize, "La réception synchrone," L'Onde Électrique, vol. 11, (June 1932)
  - Wendt, K. & Fredentall, G. Automatic frequency and phase control of synchronization in TV receivers, *Proc. IRE*, 31(1), 1943
- Computer architectures (frequency multiplication)
  - Ian A. Young, PLL based clocking circuit in a microprocessor i486DX2-50 (1992) (If Turbo, stable operation is not guaranteed)
- Theory and Technology
  - ✓ F.M. Gardner, *Phase-Lock Techniques*, 1966
    ✓ A.J. Viterbi, *Principles of Coherent Communications*, 1966
    ✓ W.C. Lindsey, *Synch. Systems in Comm. and Control*, 1972
    ✓ W.F. Egan, *Frequency Synthesis by Phase Lock*, 2000
    ✓ B. Razavi, *Phase-Locking in High-Performance Systems*, 2003
    ✓ E. Best Ronald, *PLL: Design, Simulation and Appl.*, 2003
    ✓ V. Kroupa, *Phase Lock Loops and Frequency Synthesis*, 2003
    ✓ and many others

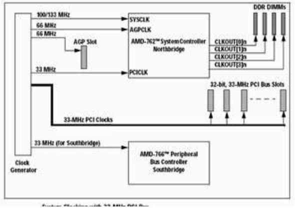
PLL in Computer architectures: Synchronization, Elimination of Clock Skew, Frequencies synthesis





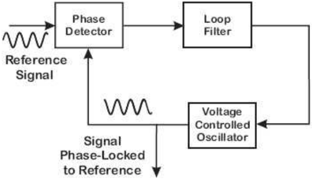






Basic Analysis and Design of PLL

D. Abramovitch (plenary lecture)
Phase-Locked Loops:
A control Centric Tutorial
American Control Conference, 2002



"First of all, PLL correct operation depends on the fact that it is nonlinear". PLL includes nonlinear devices — phase-detector and VCO, which translate the problem from signal response to phase response and back again

To fill this gap it is necessary to develop and apply
Nonlinear analysis and design of PLL

"Stability analysis and design of the loops tends to be done by a combination of linear analysis, rule of thumb, and simulation. The experts in PLLs tend to be electrical engineers with hardware design backgrounds"

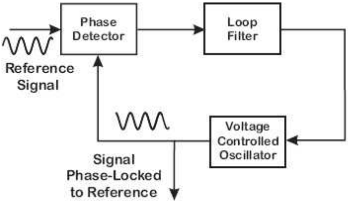
PLL Design & Analysis: Simulation

Design: Signals class (sinusoidal,impulse...), PLL type (PLL,ADPLL,DPLL...)

Analysis: Choose PLL parameters (VCO, PD, Filter etc.) to achieve stable operation for the desired range of frequencies and transient time

- Electronic realizations (signal space)
- Model in phase and frequency space [VCO freq. $= \omega_f + LG(t)$]

"Explicit simulation of the entire PLL is relatively rare. It is more typical to simulate the response of the components in signal space and then simulate the entire loop only in phase space"

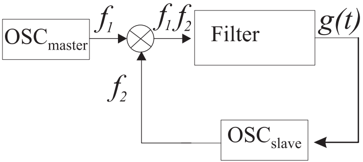


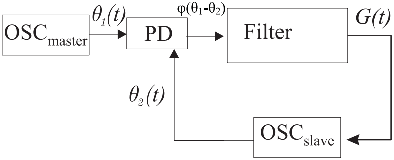
Could stable operation be guaranteed for all possible input signals, noises and internal blocks states only by simulation?

PLL Design & Analysis: Math. model analysis

PLL operation consists in the generation of an electrical signal (voltage), a phase of which is automatically tuned to the phase of input "reference" signal: phase error tends to be constant

- Math. equations in phase space: difference, differential and integro-differential
  Math. analysis: linear, nonlinear
  (e.g., Patent Appl. US 2004/0208274 A1, Patent US 7,613,268 B2)





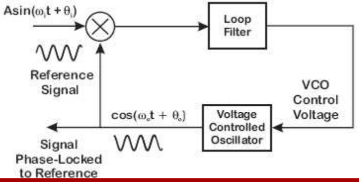
$\otimes \to$ PD, phase err. $\theta_d(t) = \theta_1(t) - \theta_2(t)$:

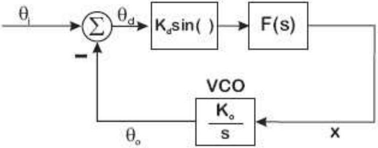
$$\dot{z} = Az + b\varphi(\theta_d),\ \dot{\theta}_d = c^* z + \rho\varphi(\theta_d)$$

$\varphi(\theta_d) = \varphi(\theta_1 - \theta_2)$ is periodic, $\theta_d \to \text{const}$?

? How to calculate PD characteristics: $\varphi(\sigma)$ depends on class of f_i

? Stability in signal & phase spaces: error signal $g(t) \to 0 \Leftrightarrow G(t) \to 0$





Linearization: while this is useful for studying loops that are near lock, it does not help for analyzing the loop when misphasing is large.

$$\bigotimes -\mathrm{PD}: 2\sin(\omega_i t+\theta_i)\cos(\omega_o t+\theta_o) = \sin(\omega_- t+\theta_-)+\sout{\sin}((\omega_+ t+\theta_+)$$
$$\omega_\pm = \omega_i \pm \omega_o, \theta_\pm = \theta_i \pm \theta_o, \omega_i \approx \omega_o, \sin(\theta_-) \approx \theta_-, cos(\theta_-) \approx 1, \theta_-^2 \approx 0$$

Linearization errors: Aizerman & Kalman problems (absolute stability, harmonic balance), Perron effects:(Lyapunov exponents sign inversion), etc.

Time-Varying Linearization and the Perron effects, G.A. Leonov, N.V. Kuznetsov, Int. J. of Biffurcation and Chaos, Vol.17, No.4, 2007, 1079-1107 (tutorial) [PDF]
[PDF] G.A. Leonov, V.O. Bragin, N.V. Kuznetsov, Algorithm for Constructing Counterexamples to the Kalman Problem, Doklady Mathematics, 82(1), 2010,

Control theory: mathematical methods of phase synch.

The theory of phase synchronization was developed in the second half of the last century on the basis of three applied theories: synchronous & induction electrical motors, auto-synchronization of the unbalanced rotors, PLL.

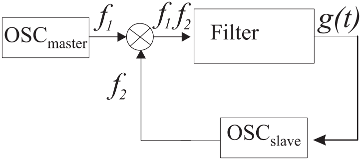
Its main principle is consideration of phase synch. systems at 3 levels:

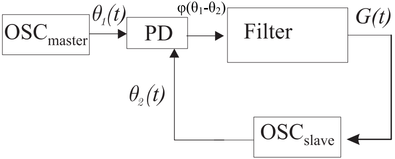
- at the level of mechanical, electromechanical, or electronic model
- at the level of phase relations
- at the level of mathematical equations

In this case the difference of oscillation phases is transformed in the control action, realizing synch. These general principles gave impetus to creation of universal methods for studying the phase synch. systems.

Application of rigorous math. methods allow to investigate nonlinear models, justify linearization and avoid errors & obtain nontrivial engineering solution

Nonlinear analysis and design of PLL Classical PLL (sin, multiplier as PD): signals $\to$ phases





OSC: $f_j(t) = A_j \sin(\omega_j(t)t + \psi_j)$ $\quad \theta_j(t) = \omega_j(t)t + \Psi_j$ — phases

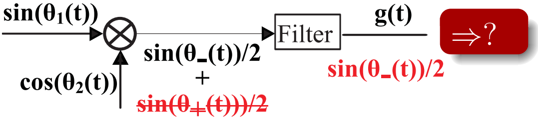
Filter: $\sigma(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau)\xi(\tau)\, d\tau$, $\gamma(t)$ – impulse response, $\alpha_0(t)$ – exp. damped function (depend on initial date of filter at $t = 0$)

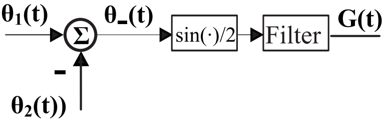
Voltage-Controlled Oscillator OCS$_{\text{slave}}$: $\omega_2 = \omega_{slave} + LG(t)$

- $\theta_j(t)$ – PD inputs, $\varphi(\theta_1(t) - \theta_2(t))$ — PD output
- signals $f_1(t)f_2(t)$ and $\varphi(\theta_1(t) - \theta_2(t))$ enter the same filters
- filter outputs are the functions $g(t)$ and $G(t)$ respectively.

1. Calculate PD characteristics: $\varphi(\sigma)$ depends on class of f_i
2. Prof. convergence in signal & phase spaces: $g(t) \to 0 \Leftrightarrow G(t) \to 0$

$$\theta_k(t)=\omega_k t+\theta_k(0),\ \theta_\pm(t)=\theta_1(t)\pm\theta_2(0)$$





Instead of ~~sin~~ trick consider math. assumption of high-frequency osc: Large time interval $[0,T]$ can be partitioned into small intervals of the form $[\tau,\tau+\delta], (\tau\in[0,T])$ such that

$$|\gamma(t)-\gamma(\tau)|\le C\delta,\quad |\omega_j(t)-\omega_j(\tau|\le C\delta\quad \forall t\in[\tau,\tau+\delta],\ \forall\tau\in[0,T]\quad (1)$$
$$|\omega_1(\tau)-\omega_2(\tau)|\le C_1,\quad \omega_j(\tau)\ge R,\qquad \forall\,\tau\in[0,T]\quad (2)$$

δ–suff. small with respect to T, C, C_1; R–suff. large with respect to δ
$\gamma(t)$ and $\omega_j(t)$ are "almost constants" on $[\tau,\tau+\delta]$
$f_k(t)=A_k\sin(\omega_k(t)t+\theta_k(0))$ rapidly oscillate as harmonic functions.

Classical PLL synthesis (continuous case)

Theorem. *[Viterbi, 1966]*
If (1)–(2) and

$$\varphi(\theta) = \frac{1}{2} A_1 A_2 \cos\theta,$$

then

$$|G(t) - g(t)| \le C_2\delta, \ \forall t \in [0, T]$$

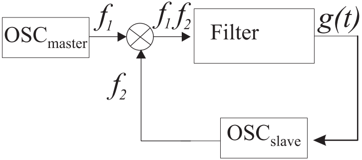
for the same initial data of the Filter

$$f_j(t) = A_j \sin(\omega_j(t)t + \psi_j)$$

$$\begin{array}{l} |\gamma(t) - \gamma(\tau)| \le C\delta \\ |\omega_j(t) - \omega_j(\tau| \le C\delta \\ \forall t \in [\tau, \tau+\delta], \forall \tau \in [0, T], \end{array} \quad (1)$$

$$\begin{array}{l} |\omega_1(\tau) - \omega_2(\tau)| \le C_1 \\ \omega_j(\tau) \ge R, \ \forall t \in [0, T] \end{array} \quad (2)$$

δ–suff. small with respect to T, C, C_1; $R \approx \delta^{-1}$.



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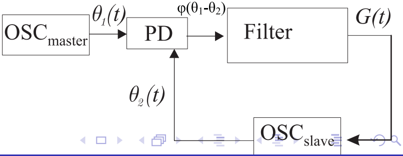

Classical PLL synthesis (discrete and mixed case)

$$f_j(t) = A_j \text{sign} \sin(\omega_j(t)t + \psi_j)$$

Theorem. If (1)–(2) and

$$\varphi(\theta) = A_1 A_2 \left(1 - \frac{2|\theta|}{\pi}\right)$$
$$\theta \in [-\pi, \pi],$$

$$f_1(t) = A_1 \sin(\omega_1(t)t + \psi_1)$$
$$f_2(t) = A_2 \text{sign} \sin(\omega_2(t)t + \psi_2)$$

Theorem. If (1)–(2) and

$$\varphi(\theta) = \frac{2A_1A_2}{\pi}\cos(\theta)$$

then

$$|G(t) - g(t)| \le C_3\delta, \ \forall\, t \in [0, T]$$

for the same initial data of Filter.

These theorems allow pass from signal space to phase space:
PD char. is calculated, convergence in signal & phase spaces is proved.

Now it is possible to pass to the level of PLL equations ⇒

Integro-differential equations of classical PLL

global stability conditions for $\dot{\omega}_j(t)$: $\dot{\theta}_j(t) = \omega_j(t) + \dot{\omega}_j(t)t \Rightarrow \dot{\theta}_j(t) = \omega_j(t)$
assumption for the control low of slave osc: $\omega_2(t) = \omega_2(0) + LG(t)$
$\omega_2(0)$ — initial frequency of slave osc, $G(t)$ is filter output

Standard PLL equation for $\omega_1(t) \equiv \omega_1(0)$, $\quad \omega_1(0) - \omega_2(0) =$

$$= (\theta_1(t) - \theta_2(t))^\bullet + L\big(\alpha_0(t) + \int_0^t \gamma(t-\tau)\varphi(\theta_1(\tau) - \theta_2(\tau))d\tau\big)$$

Filter with $W(p) = (p+\alpha)^{-1}$, $\phi(\theta) = \cos(\theta)$, $\tilde{\theta} = \theta_1 - \theta_2 + \pi/2$:

$$\ddot{\tilde{\theta}} + \alpha\dot{\tilde{\theta}} + L\sin\tilde{\theta} = \alpha\big(\omega_1(0) - \omega_2(0)\big)\text{(pendulum-like system)}$$

if take $K(p) = a + W(p)$ then in place of standard PLL equation:

$$(\theta_1(t) - \theta_2(t))^\bullet + L\big(a\varphi(\theta_1(t) - \theta_2(t)) +$$
$$+\alpha_0(t) + \int_0^t \gamma(t-\tau)\varphi(\theta_1(\tau) - \theta_2(\tau))d\tau\big) = \omega_1(0) - \omega_2(0)$$

Nonlinear analysis of PLL

$K(p) = a + W(p)$–non degenerate, $\sigma = \theta_1 - \theta_2, \rho = -aL$

$W(p) = L^{-1}c^*(A - pI)^{-1}b,\ \psi(\sigma) = \varphi(\sigma) - \frac{\omega_1(0)-\omega_2(0)}{L(a+W(0))}$ is a 2π-periodic

Continuous

$$\dot{z} = Az + b\psi(\sigma)$$
$$\dot{\sigma} = c^*z + \rho\psi(\sigma)$$

Discrete

$$z(t+1) = Az(t) + b\psi(\sigma(t))$$
$$\sigma(t+1) = \sigma(t) + c^*z(t) + \rho\psi(\sigma(t))$$

- Modification of the direct Lyapunov method with the construction of periodic Lyapunov-like functions
- Method of positively invariant cone grids
- Method of nonlocal reduction

Development of mathematical theory of phase synchronization in St. Petersburg State Univ.

G.A. Leonov (Dean, Director, Member (corr.) of RAS)
In 1975-2000 the investigations of PLLs, used in radio communication, were fulfilled:
- over 20 PhD were defended
- 1986 G.A. Leonov was awarded State Prize of USSR

- Gelig, A.Kh., Leonov, G.A., and Yakubovich, V.A. (1978). Stability of Nonlinear Systems with Nonunique Equilibrium State, Nauka
- Leonov, G., Reitmann, V., Smirnova, V. (1992). Nonlocal Methods for Pendulum-LikeFeedback Systems, Teubner Verlagsgesselschaft, Stuttgart-Leipzig
- Leonov, G., Ponomarenko, D., Smirnova, V. (1996a). Frequency-Domain Methods for Nonlinear Analysis. Theory and Applications, World Scientific [Part II Asymptotic behavior of systems with multiple equilibria, pp. 111-306]
- Leonov, G.A., Burkin, I.M., Shepeljavy, A.I. (1996b). Frequency Methods in Oscillation Theory, Kluver, Dordrecht
- Leonov, G.A. (2001). Mathematical Problems of Control Theory, World Scientific

Applied Cybernetics Dpt., Math. & Mech. Faculty

In the last ten years the group deals with the application of phase synch. systems of computer architectures and telecommunications. For the study of such systems a sophistication of previous explorations of as the group of Prof. G.A. Leonov as American scholars A.J. Viterbi and W.C. Lindsey was used.

- Nonlinear Analysis and Design of Phase-Locked Loops, G.A. Leonov, N.V. Kuznetsov, S.M. Seledzhi, (chapter in "Automation control - Theory and Practice", A.D. Rodic (Ed.), In-Tech, 2009), pp. 89–114 (ISBN 9789533070391)
- G.A. Leonov, Phase-Locked Loops. Theory and Application, *Automation and remote control*, 10, 2006, 47–55
- G.A. Leonov, S.M. Seledghi, Stability and bifurcations of PLL for digital signal processors, *Int. J. of Bifurcation and chaos*, 15(4), 2005, 1347-1360.
- G.A. Leonov, S.M. Seledghi, Design of PLL for digital signal processors, *Int. J. of Innovative Computing, Information Control*, 1(4), 2005, 779-789.

The researches, which are carried out in the group, received positive feedback from companies "Intel" and "Hewlett-Packard".

Joint Russian-Finnish research group

Prof. G.A. Leonov

2007 – now
Joint research group
within the framework of
agreement between
St. Petersburg State Univ.
and
University of Jyväskylä

Prof. P. Neittaanmäki

A main aim of this research group is to gather together the accumulated research work experience in the field of analytical methods of the theory of phase synchronization, numerical procedures and industrial applications, and to use these experiences for rigorous mathematical analysis and synthesis of real applied systems.

Main publications

• Nonlinear Analysis and Design of Phase-Locked Loops, G.A. Leonov, N.V. Kuznetsov, S.M. Seledzhi, (chapter in "Automation control - Theory and Practice", A.D. Rodic (Ed.), In-Tech, 2009), pp. 89–114 (ISBN 9789533070391)

• G.A. Leonov, Phase-Locked Loops. Theory and Application, *Automation and remote control*, 10, 2006

• V. Yakubovich, G. Leonov, A. Gelig, *Stability of Systems with Discontinuous Nonlinearities*, Singapore: World Scientific, 2004

Lyapunov exponents, chaos, stability & Perron effects:

Linearization without justification may lead to errors

$$\begin{cases} \dot{x} = F(x), \ x \in \mathbb{R}^n, \quad F(x_0) = 0 \\ x(t) \equiv x_0, A = \frac{dF(x)}{dx}\Big|_{x=x_0} \end{cases} \quad \begin{cases} \dot{y} = Ay + o(y) \\ y(t) \equiv 0, (y = x - x_0) \end{cases} \quad \begin{cases} \dot{z} = Az \\ z(t) \equiv 0 \end{cases}$$

✓stationary: $z(t) = 0$ is exp. stable $\Rightarrow$ $y(t) = 0$ is asympt. stable

$$\begin{cases} \dot{x} = F(x), \quad \dot{x}(t) = F(x(t)) \not\equiv 0 \\ x(t) \not\equiv x_0, A(t) = \frac{dF(x)}{dx}\Big|_{x=x(t)} \end{cases} \quad \begin{cases} \dot{y} = A(t)y + o(y) \\ y(t) \equiv 0, (y = x - x(t)) \end{cases} \begin{cases} \dot{z} = A(t)z \\ z(t) \equiv 0 \end{cases}$$

? nonstationary: $z(t) = 0$ is exp. stable $\Rightarrow ?$ $y(t) = 0$ is asympt. stable

! Perron effect: $z(t) = 0$ is exp. stable(unst), $y(t) = 0$ is exp. unstable(st)
- positive largest Lyapunov exponents doesn't in general indicate chaos
- negative largest LE doesn't in general indicate stability [PDF slides]

[DOI] [PDF] G.A. Leonov, N.V. Kuznetsov, Time-Varying Linearization and the Perron effects, *International Journal of Bifurcation and Chaos*, Vol. 17, No. 4, 2007, pp. 1079-1107 (survey)

Hidden oscillations in control: Aizerman & Kalman problem:

Linearization without justification may lead to errors

Harmonic balance & Describing function method in Absolute Stability Theory

$\dot{x} = Px + q\psi(r^*x), \ \psi(0) = 0 \quad (1)$

$W(p) = r^*(P - pI)^{-1}q$

$\mathrm{Im}W(i\omega_0) = 0, k = -(\mathrm{Re}W(i\omega_0))^{-1}$

$\dot{x} = P_0x + q\varphi(r^*x)$

$P_0 = P + kqr^*, \varphi(\sigma) = \psi(\sigma) - k\sigma$

$P_0 : \lambda_{1,2} = \pm i\omega_0, \ \mathrm{Re}\lambda_{j>2} < 0$

DFM: exists periodic solution $\sigma(t) = r^*x(t) \approx a\cos\omega_0 t$

$$a: \ \int_0^{2\pi/\omega_0} \psi(a\cos\omega_0 t)\cos\omega_0 t\mathrm{d}t = ka\int_0^{2\pi/\omega_0}(\cos\omega_0 t)^2\mathrm{d}t$$

Aizerman problem: *If (1) is stable for any linear* $\psi(\sigma) = \mu\sigma, \ \mu \in (\mu_1, \mu_2)$ *then (1) is stable for any nonlinear* $\psi(\sigma) : \mu_1\sigma < \psi(\sigma) < \mu_2\sigma, \ \forall\sigma \neq 0$

DFM: (1) is stable $\Rightarrow k : k < \mu_1, \mu_2 < k \Rightarrow k\sigma^2 < \psi(\sigma)\sigma, \psi(\sigma)\sigma < k\sigma^2$

$$\Rightarrow \forall a \neq 0 : \ \int_0^{2\pi/\omega_0}(\psi(a\cos\omega_0 t)\, a\cos\omega_0 t - k(a\cos\omega_0 t)^2)\mathrm{d}t \neq 0$$

$\Rightarrow$ no periodic solutions by harmonic linearization and DFM, but

Counterexamples to Aizerman's conjecture & Kalman's conjecture: [PDF slides].

[PDF] G.A. Leonov, V.O. Bragin, N.V. Kuznetsov, Algorithm for constructing counterexamples to the Kalman problem, Doklady Mathematics, 82(1), 2010,