Lyapunov exponent, chaos, Perron effects: linearization, stability and instability by the first approximation

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http://www.math.spbu.ru/user/nk/Lyapunov_exponent.htm

tutorial last version: http://www.math.spbu.ru/user/nk/PDF/Lyapunov_exponent.pdf

Lyapunov exponent : chaos, stability, Perron effects, linearization

$$\begin{cases} \dot{x} = F(x), \ x \in \mathbb{R}^n, \quad F(x_0) = 0 \\ x(t) \equiv x_0, A = \frac{dF(x)}{dx}\Big|_{x=x_0} \end{cases} \quad \begin{cases} \dot{y} = Ay + o(y) \\ y(t) \equiv 0, (y = x - x_0) \end{cases} \quad \begin{cases} \dot{z} = Az \\ z(t) \equiv 0 \end{cases}$$

✓stationary: $z(t) = 0$ is exp. stable $\Rightarrow y(t) = 0$ is asympt. stable

$$\begin{cases} \dot{x} = F(x), \quad \dot{x}(t) = F(x(t)) \not\equiv 0 \\ x(t) \not\equiv x_0, A(t) = \frac{dF(x)}{dx}\Big|_{x=x(t)} \end{cases} \quad \begin{cases} \dot{y} = A(t)y + o(y) \\ y(t) \equiv 0, (y = x - x(t)) \end{cases} \begin{cases} \dot{z} = A(t)z \\ z(t) \equiv 0 \end{cases}$$

? nonstationary: $z(t) = 0$ is exp. stable $\Rightarrow ?\ y(t) = 0$ is asympt. stable

! Perron effect: $z(t)=0$ is exp. stable(unst), $y(t)=0$ is exp. unstable(st)

Positive largest Lyapunov exponent doesn't, in general, indicate chaos

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Perron effects: largest Lyapunov exponent sign inversion

O.Perron

Nonlinear system in $\mathbb{R}^2$:

$$\dot{x}(t) = -ax(t) \qquad 1 < 2a < 1 + e^{-\pi}/2$$
$$\dot{y}(t) = \big(\sin \ln(t+1) + \cos(\ln(t+1)) - 2a\big)y(t) + x^2(t)$$

$$x(t) = x(0)e^{-at},\ y(t) = e^{(t+1)\sin\ln(t+1)-2at}\Big(y(0) + x^2(0)\int_0^t e^{-(\tau+1)\sin\ln(\tau+1)}\mathrm{d}\tau\Big)$$

exp. unstable: $\mathrm{LE}[x(t)] = -a,\ \mathrm{LE}[y(t)] = 1 - 2a + e^{-\pi}/2 > 0$ (1)

Linearized system (linearization along solution $\big(x(t), y(t)\big) \equiv (0,0)$):

$$\dot{x}_{lin}(t) = -ax_{lin}(t), \quad \dot{y}_{lin}(t) = \big(\sin\ln(t+1) + \cos(\ln(t+1)) - 2a\big)y_{lin}(t)$$

$$x_{lin}(t) = x(0)e^{-at}, \quad y_{lin}(t) = y(0)e^{(t+1)\sin\ln(t+1)-2at}$$

exp. stable: $\mathrm{LE}[x_{lin}(t)] = -a, \mathrm{LE}[y_{lin}(t)] = 1 - 2a < 0$ (2)

Different signs of LEs in (1) and (2):
largest Lyapunov exponent sign inversion shows Perron effect

Lyapunov characteristic exponents (CE)

Def. *characteristic exponent* of the vector-function $f(t)$

$$\mathrm{CE}(f(t)) = \limsup_{t\to+\infty} \frac{1}{t} \ln|f(t)|,$$

$$\text{exact } \mathrm{CE}(f(t)) = \lim_{t\to+\infty} \frac{1}{t} \ln|f(t)|,$$

$$\dot{x} = A(t)x, \qquad x_{t+1} = A(t)x_t$$

$$\sup_{t\in[0,+\infty)} |A(t)| < +\infty \qquad \sup_{t\in\mathbb{N}} |A(t)|, |A(t)^{-1}| < +\infty$$

A.M. Lyapunov

$X(t)=(x_1(t), .., x_n(t))$ a solutions fundamental matrix(FM), $\lambda_j = \mathrm{CE}(x_j(t))$

Def. $X(t)$ is *normal* if $\sum_{j=1}^{n} \lambda_j$ is minimal in comparison to other FM.

Thm. For any $X(t)$ there exists a constant matrix C $(\det C \neq 0)$: matrix $X(t)C$ is a normal fundamental matrix (NFM), and all NFM have the same set of $(\lambda_1, \ldots, \lambda_n)$.

Def. The set $(\lambda_1, \ldots, \lambda_n)$ of certain NFM $X(t)$ is called the *complete spectrum* of linear system, and λ_j are called the *characteristic exponents*.

Regular system

$$(1)\quad \dot{x} = A(t)x, \quad x \in \mathbb{R}^n \quad x_{t+1} = A(t)x_t \quad (1')$$

$X(t)=(x_1(t),..,x_n(t))$ normal fundamental matrix(NFM), $\lambda_j = \mathrm{CE}(x_j(t))$

$$\text{Lyapunov inequality: } \Sigma = \sum_{j=1}^{n} \lambda_j \geq \limsup_{t\to+\infty} \frac{1}{t} \ln|\det X(t)|$$

$$\mathrm{CE}(\textstyle\prod_1^n |x_i(t)|) \geq \mathrm{CE}(\det\{x_i(t)\}_1^n)$$

$$\mathrm{CE}(\text{n-cube volume}) \geq \mathrm{CE}(\text{n-parallelotope volume})$$

Def. Linear system is *regular* if $\Sigma = \liminf\limits_{t\to+\infty} \frac{1}{t} \ln|\det X(t)|$

$$(1)\ \Sigma = \liminf_{t\to+\infty} \frac{1}{t} \int_0^t \mathrm{Tr}\, A(\tau)\, d\tau \quad \Sigma = \liminf_{t\to+\infty} \frac{1}{t} \ln \prod_{j=0}^{t-1} |\det A(j)|\ (1')$$

Def. *Coefficient of irregularity*: $\Gamma = \Sigma - \liminf\limits_{t\to+\infty} \frac{1}{t} \ln|\det X(t)|$

Systems with constant and periodic coefficients are regular

Lyapunov: There are no Perron effects in regular systems

Regular system $\Rightarrow$ Exact Characteristic exponents

$$\Sigma = \sum_{j=1}^{n} \lambda_j = \lim_{t \to +\infty} \tfrac{1}{t} \ln |\det X(t)| \Rightarrow \lim_{t \to +\infty} \tfrac{1}{t} \ln |x(t)|, \; x(t) \neq 0$$

Exact Characteristic exponents $\not\Rightarrow$ Regular system

$$\dot{x} = A(t)x, \quad A(t) = \begin{pmatrix} 0 & 1 \\ 0 & (\cos\ln t - \sin\ln t - 1) \end{pmatrix}, \quad t \geq 1$$

FM: $X(t) = \big(x_1(t), x_2(t)\big) = \begin{pmatrix} 1 & \int\limits_1^t e^{\gamma(\tau)}\, \mathrm{d}\tau \\ 0 & e^{\gamma(t)} \end{pmatrix}, \; \gamma(t) = t(\cos\ln t - 1)$

$$\lim_{t \to +\infty} \frac{1}{t} \ln |x_2(t)| = \lim_{t \to +\infty} \frac{1}{t} \ln |x_1(t)| = 0, \lim_{t \to +\infty} \inf \frac{1}{t} \ln |\det X(t)| = -2$$

Here linear system has exact CEs but it is nonregular: $\Gamma = 2$

Lyapunov exponents (LE) and Characteristic exponents (CE)

Def. The *singular values* $\alpha_j(X(t))$ are square roots of eigenvalues of the matrix $X(t)^*X(t)$. ($\alpha_j(X(t))$ — axes of ellipsoid $X(t)$(unit ball))

Def. The *Lyapunov exponent* μ_j is the number

$$\mu_j = \limsup_{t\to+\infty} \frac{1}{t} \ln \alpha_j(X(t)).$$

Prop. *largest Characteristic exponent = largest Lyapunov exponent.*

Example (LEs $\neq$ CEs). Consider system $\dot{x} = A(t)x$ with the matrix

$$A(t) = \begin{pmatrix} 0 & \sin(\ln t) + \cos(\ln t) \\ \sin(\ln t) + \cos(\ln t) & 0 \end{pmatrix}, \quad t > 1$$

and NFM $X(t) = \begin{pmatrix} e^{\gamma(t)} & e^{-\gamma(t)} \\ e^{\gamma(t)} & -e^{-\gamma(t)} \end{pmatrix}$, where $\gamma(t) = t\sin(\ln t)$.

$\alpha_1(X(t)) = \sqrt{2}\max(e^{\gamma(t)}, e^{-\gamma(t)}), \alpha_2(X(t)) = \sqrt{2}\min(e^{\gamma(t)}, e^{-\gamma(t)})$.

LEs $\neq$ CEs: $\mu_1 = 1$, $\mu_2 = 0$ and $\lambda_1 = \lambda_2 = 1$: $\Rightarrow \lambda_2 \neq \mu_2$

Nemytskii–Vinograd counterexample

Example: Sign of Eig values $\neq$ Sign of Lyapunov exponents.

$$\dot{x} = A(t)x, \quad A(t) = \begin{pmatrix} 1 - 4(\cos 2t)^2 & 2 + 2\sin 4t \\ -2 + 2\sin 4t & 1 - 4(\sin 2t)^2 \end{pmatrix}$$

$\det(A(t) - pI_n) = p^2 + 2p + 1 \Rightarrow eig(A) = -1$

Solution $x(t) = \begin{pmatrix} e^t \sin 2t \\ e^t \cos 2t \end{pmatrix}$ $\quad \mathrm{CE}(x(t)) = 1$

All eigenvalues of the matrix $A(t)$ can have negative real parts even if the corresponding linear system has positive Lyapunov exponents

So, the formula is not true:

$\lambda_j = \limsup\limits_{t \to +\infty} \frac{1}{t} \int_0^t \mathrm{Re}\nu_j(\tau)\, d\tau,$

Justification of time-varying linearization

Nowadays the problem of the justification of the nonstationary linearizations for complicated nonperiodic motions on strange attractors bears a striking resemblance to the situation that occurred 120 years ago. The founders of the automatic control theory Maxwell[1868], and Vyschegradskii[1877] courageously used a linearization in a neighborhood of stationary motions, leaving the justification of such linearization to Poincare[1886] and Lyapunov[1892].

At present, many specialists in chaotic dynamics believe that the positiveness of the largest Lyapunov exponent of a linear system of the first approximation implies the instability of solutions of the original system. Moreover, there are a number of computer experiments, in which the various numerical methods for calculating LE of linear systems of the first approximation are used. As a rule, authors ignore the justification of the linearization procedure and use the numerical values of LE so obtained to construct various numerical characteristics of attractors of the original nonlinear systems.

G.A. Leonov, N.V. Kuznetsov, Time-Varying Linearization and the Perron effects, International Journal of Bifurcation and Chaos, Vol.17, No.4, 2007, pp.1079-1107

Main publications

✓ G.A. Leonov, N.V. Kuznetsov,
Time-Varying Linearization and the Perron effects, International Journal of Bifurcation and Chaos, Vol. 17, No. 4, 2007, pp. 1079-1107 (survey) [DOI] [PDF]

✓ Kuznetsov N.V., Leonov G.A.,
On stability by the first approximation for discrete systems, Second International conference Physics and Control, 2005, Proceedings, pp. 596-599

✓ Kuznetsov N.V., Leonov G.A.,
Criterion of stability to first approximation of nonlinear discrete systems, Vestnik St.Petersburg University. Mathematics. Vol. 38, No. 2, 2005, pp. 52-60, (Allerton Press, USA)

✓ Kuznetsov N.V., Leonov G.A.,
Criteria of stability by the first approximation for discrete nonlinear systems, Vestnik St.Petersburg University. Mathematics. Vol. 38, No. 3, 2005, pp. 21-30, (Allerton Press, USA)

✓ Kuznetsov N.V., Leonov G.A.,
Stability by the first approximation for discrete systems, Vestnik St.Petersburg University. Mathematics. Vol. 36, No. 1, 2003, pp. 21-27, (Allerton Press, USA)

✓ Leonov G. A., Strange attractors and the classical theory of motion stability, Advances of mechanics, N1, 2002, pp. 3–42.

Harmonic balance & Describing function method (DFM) in Absolute stability theory

$\dot{x} = Px + q\psi(r^*x),\ \psi(0) = 0 \quad (1)$ $\qquad \dot{x} = P_0x + q\varphi(r^*x)$

$W(p) = r^*(P - pI)^{-1}q \qquad P_0 = P + kqr^*, \varphi(\sigma) = \psi(\sigma) - k\sigma$

$\mathrm{Im}W(i\omega_0) = 0, k = -(\mathrm{Re}W(i\omega_0))^{-1} \qquad P_0: \lambda_{1,2} = \pm i\omega_0,\ \mathrm{Re}\lambda_{j>2} < 0$

DFM: exists periodic solution $\sigma(t) = r^*x(t) \approx a\cos\omega_0 t$ if $\exists a:$

$$\int_0^{2\pi/\omega_0} \psi(a\cos\omega_0 t)\cos\omega_0 t \mathrm{d}t = ka\int_0^{2\pi/\omega_0}(\cos\omega_0 t)^2\mathrm{d}t$$

Aizerman problem: *If (1) is stable for any linear $\psi(\sigma) = \mu\sigma,\ \mu \in (\mu_1, \mu_2)$ then (1) is stable for any nonlinear $\psi(\sigma): \mu_1\sigma < \psi(\sigma) < \mu_2\sigma,\ \forall\sigma \neq 0$*

DFM: (1) is stable $\Rightarrow k: k < \mu_1, \mu_2 < k \Rightarrow k\sigma^2 < \psi(\sigma)\sigma, \psi(\sigma)\sigma < k\sigma^2$

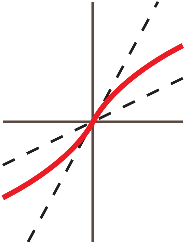
$$\Rightarrow \forall a \neq 0: \int_0^{2\pi/\omega_0}(\psi(a\cos\omega_0 t)\, a\cos\omega_0 t - k(a\cos\omega_0 t)^2)\mathrm{d}t \neq 0$$

$\Rightarrow$ no periodic solutions by DFM, but

G.A. Leonov, V.O. Bragin, N.V. Kuznetsov,
Algorithm for constructing counterexamples to the Kalman problem,
Doklady Mathematics, 82(1), 2010, 540-542 [DOI] [PDF]

Counterexamples to Aizerman's conjecture & Kalman's conjecture: [PDF slides]

if $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{b}k\mathbf{c}^*\mathbf{z}$, is asympt. stable $\forall\, k \in (k_1, k_2) : \forall \mathbf{z}(t, \mathbf{z}_0) \to 0$
$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\varphi(\sigma),\ \sigma = \mathbf{c}^*\mathbf{x},\ \varphi(0) = 0,\ k_1 < \varphi(\sigma)/\sigma,\ \varphi' < k_2,\ \forall \mathbf{x}(t, \mathbf{x}_0) \to 0?$



$1949:\ k_1 < \varphi(\sigma)/\sigma < k_2$ $\qquad$ $1957:\ k_1 < \varphi'(\sigma) < k_2$

In general, conjectures are not true (Aizerman's $n \geq 2$, Kalman's $n \geq 4$): nonlinearity can be in linear stability sector but periodic solutions exist.

✓ Bragin, Leonov, Kuznetsov, Vagaitsev (2011) Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman conjectures and Chua's circuits, *J. of Computer and Systems Sciences Int., V.50, N4, 511-544 (survey)*

Hidden attractors localization

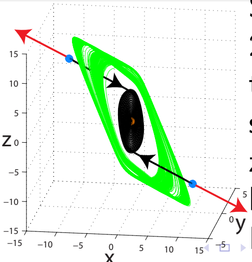
self-exciting oscillations and attractors - Van der Pol, Lorenz, et al.
standard computation: 1) determine equilibria 2) after transient process trajectory, starting from a point of unstable manifold in neighborhood of unstable equilibrium, reaches an oscillation and computes it.

hidden oscillations and hidden attractors — basin of attraction does not contain neighborhoods of equilibria

Leonov G.A., Kuznetsov N.V., Vagaitsev V.I, Localization of hidden Chua's attractors, *Phys. Lett. A*, 2011, 375, 2230-2233

Chua system

$$\dot{x} = \alpha(y - x - m_1 x - \psi(x))$$
$$\dot{y} = x - y + z,$$
$$\dot{z} = -(\beta y + \gamma z)$$
$$\psi(x) = (m_0 - m_1)\mathrm{sat}(x)$$
$$\alpha = 8.4562 \ \beta = 12.0732 \ \gamma = 0.0052$$
$$m_0 = -0.1768, m_1 = -1.1468$$



Stable zero eqv. and 2 symmetric saddles: trajectories "from" saddles tend to zero eqv. or to infinity: black and red

Hidden attractor(green)