Lyapunov exponent, chaos, Perron effects: time-varying linearization, stability and instability by the first approximation

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http://www.math.spbu.ru/user/nk/Lyapunov_exponent.htm

tutorial last version: http://www.math.spbu.ru/user/nk/PDF/Lyapunov-exponent-Sign-inversion-Perron-effects-Chaos.pdf

Lyapunov exponent: sign inversions, Perron effects, chaos, stability

✓ stationary: $z(t)=0$ is exp. stable $\Rightarrow$ $y(t)=0$ is asympt. stable

$$\begin{cases} \dot{x} = F(x), \ x \in \mathbb{R}^n, \quad F(x_0)=0 \\ x(t) \equiv x_0, A = \frac{dF(x)}{dx}\Big|_{x=x_0} \end{cases} \quad \begin{cases} \dot{y} = Ay + o(y) \\ y(t) \equiv 0, (y=x-x_0) \end{cases} \quad \begin{cases} \dot{z} = Az \\ z(t) \equiv 0 \end{cases}$$

? time-varying: $z(t)=0$ is exp. stable $\Rightarrow ?$ $y(t)=0$ is asympt. stable

$$\begin{cases} \dot{x} = F(x), \quad \dot{x}(t) = F(x(t)) \not\equiv 0 \\ x(t) \not\equiv x_0, A(t) = \frac{dF(x)}{dx}\Big|_{x=x(t)} \end{cases} \quad \begin{cases} \dot{y} = A(t)y + o(y) \\ y(t) \equiv 0, (y=x-x(t)) \end{cases} \begin{cases} \dot{z} = A(t)z \\ z(t) \equiv 0 \end{cases}$$

! Perron effect: $z(t)=0$ is exp. stable(unst), $y(t)=0$ is exp. unstable(st)

Positive largest Lyapunov exponent doesn't, in general, indicate chaos
Negative largest Lyapunov exponent doesn't, in general, indicate stability

Time-varying linearization (for continious and discrete system) requires justification:
Survey: G.A. Leonov, N.V. Kuznetsov, Time-Varying Linearization and the Perron effects, Int. Journal of Bifurcation and Chaos, 17(4), 2007, 1079-1107 (doi:10.1142/S0218127407017732)

Perron effects: largest Lyapunov exponent sign inversion

O.Perron

Nonlinear system in $\mathbb{R}^2$:

$$\dot{x}(t) = -ax(t) \qquad 1 < 2a < 1 + e^{-\pi}/2$$
$$\dot{y}(t) = \big(\sin\ln(t+1) + \cos(\ln(t+1)) - 2a\big)y(t) + x^2(t)$$

$$x(t) = x(0)e^{-at},\ y(t) = e^{(t+1)\sin\ln(t+1) - 2at}\Big(y(0) + x^2(0)\int_0^t e^{-(\tau+1)\sin\ln(\tau+1)}\mathrm{d}\tau\Big)$$

exp. unstable: $\mathrm{LE}[x(t)] = -a,\ \mathrm{LE}[y(t)] = 1 - 2a + e^{-\pi}/2 > 0 \qquad (1)$

Linearized system (time-varying linearization along $(x(t), y(t)) \equiv (0,0)$):

$$\dot{x}_{lin}(t) = -ax_{lin}(t), \quad \dot{y}_{lin}(t) = \big(\sin\ln(t+1) + \cos(\ln(t+1)) - 2a\big)y_{lin}(t)$$
$$x_{lin}(t) = x(0)e^{-at}, \quad y_{lin}(t) = y(0)e^{(t+1)\sin\ln(t+1) - 2at}$$

exp. stable: $\mathrm{LE}[x_{lin}(t)] = -a, \mathrm{LE}[y_{lin}(t)] = 1 - 2a < 0 \qquad (2)$

Different signs of LEs in (1) and(2):
largest Lyapunov exponent sign inversion is called Perron effect

Survey: G.A. Leonov, N.V. Kuznetsov, Time-Varying Linearization and the Perron effects, Int. Journal of Bifurcation and Chaos, 17(4), 2007, 1079-1107 (doi:10.1142/S0218127407017732)

Lyapunov characteristic exponents (CE)

Def. *characteristic exponent* of the vector-function $f(t)$

$$\mathrm{CE}(f(t)) = \limsup_{t\to+\infty} \frac{1}{t}\ln|f(t)|,$$

$$\text{exact } \mathrm{CE}(f(t)) = \lim_{t\to+\infty} \frac{1}{t}\ln|f(t)|,$$

$$\dot{x} = A(t)x, \qquad x_{t+1} = A(t)x_t$$

$$\sup_{t\in[0,+\infty)} |A(t)| < +\infty \qquad \sup_{t\in\mathbb{N}} |A(t)|, |A(t)^{-1}| < +\infty$$

A.M. Lyapunov

$X(t)=(x_1(t),..,x_n(t))$ a solutions fundamental matrix(FM), $\lambda_j=\mathrm{CE}(x_j(t))$

Def. $X(t)$ is *normal* if $\sum_{j=1}^{n}\lambda_j$ is minimal in comparison to other FM.

Thm. For any $X(t)$ there exists a constant matrix C $(\det C \neq 0)$: matrix $X(t)C$ is a normal fundamental matrix (NFM), and all NFM have the same set of $(\lambda_1,\ldots,\lambda_n)$.

Def. The set $(\lambda_1,\ldots,\lambda_n)$ of certain NFM $X(t)$ is called the *complete spectrum* of linear system, and λ_j are called the *characteristic exponents*.

Regular system

$$(1) \quad \dot{x} = A(t)x, \quad x \in \mathbb{R}^n \quad x_{t+1} = A(t)x_t \quad (1')$$

$X(t) = (x_1(t), ., x_n(t))$ normal fundamental matrix(NFM), $\lambda_j = \mathrm{CE}(x_j(t))$

$$\text{Lyapunov inequality: } \Sigma = \sum_{j=1}^{n} \lambda_j \geq \limsup_{t \to +\infty} \frac{1}{t} \ln |\det X(t)|$$

$$\mathrm{CE}(\textstyle\prod_1^n |x_i(t)|) \geq \mathrm{CE}(\det\{x_i(t)\}_1^n)$$

$$\mathrm{CE}(\text{n-cube volume }) \geq \mathrm{CE}(\text{n-parallelotope volume})$$

Def. Linear system is *regular* if $\Sigma = \liminf_{t \to +\infty} \frac{1}{t} \ln |\det X(t)|$

$$(1) \; \Sigma = \liminf_{t \to +\infty} \frac{1}{t} \int_0^t \mathrm{Tr}\, A(\tau)\, d\tau \quad \Sigma = \liminf_{t \to +\infty} \frac{1}{t} \ln \prod_{j=0}^{t-1} |\det A(j)| \; (1')$$

Def. *Coefficient of irregularity*: $\Gamma = \Sigma - \liminf_{t \to +\infty} \frac{1}{t} \ln |\det X(t)|$

Systems with constant and periodic coefficients are regular

A. Lyapunov: There are no Perron effects in regular systems

Vinograd counterexample: characteristic exponents and regularity

Regular system ⇒ Exact Characteristic exponents

$$\Sigma = \sum_{j=1}^{n} \lambda_j = \lim_{t\to+\infty} \tfrac{1}{t} \ln|\det X(t)| \Rightarrow \lim_{t\to+\infty} \tfrac{1}{t} \ln|x(t)|,\ x(t) \neq 0$$

Exact Characteristic exponents ⇏ Regular system

$$\dot{x} = A(t)x, \quad A(t) = \begin{pmatrix} 0 & 1 \\ 0 & (\cos\ln t - \sin\ln t - 1) \end{pmatrix}, \quad t \geq 1 \quad (1)$$

FM: $X(t) = \big(x_1(t), x_2(t)\big) = \begin{pmatrix} 1 & \int_1^t e^{\gamma(\tau)}\,\mathrm{d}\tau \\ 0 & e^{\gamma(t)} \end{pmatrix}$, $\gamma(t) = t(\cos\ln t - 1)$

$$\lim_{t\to+\infty} \frac{1}{t} \ln|x_2(t)| = \lim_{t\to+\infty} \frac{1}{t} \ln|x_1(t)| = 0, \lim_{t\to+\infty} \inf \frac{1}{t} \ln|\det X(t)| = -2$$

Linear system (1) has exact CEs but it is nonregular: $\Gamma = 2$

Def. The *singular values* $\alpha_j(X(t))$ are square roots of eigenvalues of the matrix $X(t)^*X(t)$. ($\alpha_j(X(t))$ — axes of ellipsoid $X(t)$(unit ball))
Def. The *Lyapunov exponent* μ_j is the number

$$\mu_j = \limsup_{t\to+\infty} \frac{1}{t} \ln \alpha_j(X(t)).$$

Prop. *largest Characteristic exponent = largest Lyapunov exponent.*
Example (LEs $\neq$ CEs). Consider system $\dot{x} = A(t)x$ with the matrix

$$A(t) = \begin{pmatrix} 0 & \sin(\ln t) + \cos(\ln t) \\ \sin(\ln t) + \cos(\ln t) & 0 \end{pmatrix}, \quad t > 1$$

and NFM $X(t) = \begin{pmatrix} e^{\gamma(t)} & e^{-\gamma(t)} \\ e^{\gamma(t)} & -e^{-\gamma(t)} \end{pmatrix}$, where $\gamma(t) = t\sin(\ln t)$.

$\alpha_1(X(t)) = \sqrt{2}\max(e^{\gamma(t)}, e^{-\gamma(t)}), \alpha_2(X(t)) = \sqrt{2}\min(e^{\gamma(t)}, e^{-\gamma(t)})$.

LEs $\neq$ CEs: $\mu_1 = 1$, $\mu_2 = 0$ and $\lambda_1 = \lambda_2 = 1$: $\Rightarrow \lambda_2 \neq \mu_2$

Nemytskii–Vinograd counterexample

Example: Sign of Eig values $\neq$ Sign of Lyapunov exponents.

$$\dot{x} = A(t)x, \quad A(t) = \begin{pmatrix} 1 - 4(\cos 2t)^2 & 2 + 2\sin 4t \\ -2 + 2\sin 4t & 1 - 4(\sin 2t)^2 \end{pmatrix}$$

$\det(A(t) - pI_n) = p^2 + 2p + 1 \Rightarrow eig(A) = -1$

Solution $x(t) = \begin{pmatrix} e^t \sin 2t \\ e^t \cos 2t \end{pmatrix}$ $\quad \mathrm{CE}(x(t)) = 1$

Eigenvalues of matrix $A(t)$ have negative real parts, but the corresponding linear system has positive Lyapunov exponent

So, the formula $\lambda_j = \limsup\limits_{t \to +\infty} \frac{1}{t} \int_0^t \mathrm{Re}\nu_j(\tau)\, d\tau$, is not true.

Justification of time-varying linearization

Lyapunov exponents (LEs) were introduced by Lyapunov for the analysis of stability by the first approximation for regular time-varying linearizations, where negativeness of the largest Lyapunov exponent indicated stability. While there is no general methods for checking regularity of linearization & there are known effects of the largest Lyapunov exponent sign inversions, called Perron effects, for non regular time-varying linearizations, computation of LEs for linearization of nonlinear autonomous system along non stationary trajectories is widely used for investigation of chaos, where positiveness of the largest LE is often considered as indication of chaotic behavior in considered nonlinear system.

Moreover, there are a number of computer experiments, in which the various numerical methods for calculating LE of linear systems of the first approx. are used. As a rule, authors ignore the justification of the linearization procedure and use the numerical values of LE so obtained to construct various numerical characteristics of attractors of the original nonlinear systems.

Survey: G.A. Leonov, N.V. Kuznetsov, Time-Varying Linearization and the Perron effects, Int. Journal of Bifurcation and Chaos, 17(4), 2007, 1079-1107 (doi:10.1142/S0218127407017732)

Computation of Lyapunov exponents in Matlab (1)

Matlab code: N.V. Kuznetsov, Mokaev T.N., Vasilyev P.A. Numerical justification of Leonov conjecture on Lyapunov dimension of Rossler attractor, Comm. in Nonl. Sci and Num. Simul., 19(4), 2014, 1027-1034 (doi:10.1016/j.cnsns.2013.07.026)

based on Benettin's approach; for the orthogonalization of fundamental matrix it is used MATLAB function qr, where Householder transformation is used for factorization procedure

```
function [t, lces, trajectory] = lyapunov_exp(ode, x_start, t_start, ...
% For given dynamical system, represented by system of differential equations
% combined with variational equation this function returns array of
% LCEs for the point x_start.
% Parameters:
%   ode - combined system (system of ode + var. eq.);
%   x_start -  initial point;
%   t_start - initial time value;
%   t_step - time-step in Gramm-Shmidt reorthogonalization procedure;
%   k_iter - number of iterations of Gramm-Shmidt reorthogonalization procedure;
%   rel_tol - relative error in Runge-Kutta 45 method;
%   abs_tol - absolute error in Runge-Kutta 45 method;

[~,n1] = size(x_start); % n1 - size of the system of odes :
n2 = n1*(n1+1); % n2 - size of combined system :
y = zeros(n2,1); % y - variable of combined system :
norms = zeros(1,n1); % norms - array of norms of vectors in Jacobi matrix :
log_sum = zeros(1,n1); % log_sum - array of sums of logarithms of norms :
lexp = zeros(1,n1); % l_exp - array of lyapunov exponents (in current moment) :
y(1:n1) = x_start(:) % Initializing y :
for i = 1:n1
    y((n1+1)*i) = 1.0;
end

% Initializing t_curr :
t_curr = t_start;

% Preallocations for output values :
t = zeros(k_iter,1);
lces = zeros(k_iter,3);
```

Computation of Lyapunov exponents in Matlab (2)

```
% Set options for MATLAB solver :
options = odeset('RelTol', rel_tol, 'AbsTol', abs_tol);
tr_len = 1;

% Main loop:
for i = 1 : k_iter
    sol = ode45(ode, [t_curr t_curr+t_step], y, options); % Solving combined system :
    i_last = numel(sol.x); % i_last - the last moment :
    % Getting Jacobi matrix in the moment T PhiT from vector Y :
    Y = transpose(sol.y);
    PhiT = reshape( Y(i_last, n1+1 : n2 ), n1, n1);

    [V, R] = qr(PhiT); % QR factorization of PhiT :
    for j = 1 : n1
        if R(j,j) < 0
            R(j,j) = (-1) * R(j,j);
            V(:,j) = (-1) * V(:,j);
        end
    end

    t_curr = t_curr + t_step; % Updating  y and t_curr :
    y( 1 : n1 ) = Y( i_last, 1:n1 );
    y( n1+1 : n2 ) = reshape(V, 1, []);

    for k = 1 : n1 % Computing lyapunov exponents (in moment t_curr) :
        norms(k) = R(k,k);
        log_sum(k) = log_sum(k) + log( norms(k) );
        lexp(k) = log_sum(k) / (t_curr-t_start);
    end

    t(i) = t_curr; % Saving computations in corresponding vectors :
    lces(i, :) = lexp;
    for j = 1 : i_last
        trajectory(tr_len, :) = [sol.x(j) sol.y(1:n1, j)'];
        tr_len = tr_len + 1;
    end
end
end
```

Lyapunov dimension of Rossler system (1)

Since a numerical localization of attractor in considered Rossler system is used and there is no effective way to prove ergodicity rigourously, one has to consider a mesh of initial conditions for investigation of Lyapunov exponents.

```
function OUT = rossler_syst_1(t, X)

global a b % Parameters:
OUT = zeros(12,1); % Output vector, that representing combined system:

% Rosler equation:
OUT(1) = - X(2) - X(3);
OUT(2) = X(1);
OUT(3) = -b*X(3) + a*(X(2) - X(2)*X(2));

% Variational equation:
OUT(4:12) = [0 -1 -1; 1 0 0; 0 a*(1-2*X(2)) -b] * [X(4) X(7) X(10); X(5) X(8) X(11); X(6) X(9) X(12)];
```

```
function ld = lyapunov_dim(lces)
% For the given array of lyapunov characteristic exponents of some point this function
% compute so called lyapunov dimention of this point.
ld = 0; % ld - lyupunov dimention :
[~,n] = size(lces); % n - number of LCEs :
lambda = sort(lces, 'descend'); % lambda - sorted array of LCEs :
le_sum = lambda(1); % Main loop :
if ( lambda(1) > 0 )
    for i = 1 : n-1
        if lambda(i+1) ~= 0
            ld = i + le_sum / abs( lambda(i+1) );
            le_sum = le_sum + lambda(i+1);
            if le_sum < 0
                break;
            end
        end
    end
end
end
```

Lyapunov dimension of Rossler system (2)

```
function run_rossler1
% Computes local lyapunov dimention in fixed point
% and in the points on the grid for the 1st Rossler attractor and compares thems.
global a b % Parameters :
a = 0.386; b = 0.2; % Values of parameters
T = 1.0; % T - time-step in iterative procedure
K = 200; % K - number of iterations of iterative procedure
rel_tol = 1e-8; abs_tol = 1e-8; % Relative and absolute errors for Runge-Kutta 45 method
eps = 1e-1; % Epsilon -- is step on the grid :
x0 = [0 0 0]; % Fixed point :
% Attractor is located in cube : x \in [-1; 1.3]; y \in [-0.7; 1.8]; z \in [-1.1; 0];
x_begin = -1;    x_end = 1.3;   y_begin = -0.7;   y_end = 1.8;   z_begin = -1.1;   z_end = 0;
x_iterations=(x_end-x_begin)/eps+1;y_iterations=(y_end-y_begin)/eps+1;z_iterations=(z_end-z_begin)/eps+1;

% Infinity factor: if trajectory leaves cube with side 'infinity_factor',
% then we conclude, that trajectory will leave basin of attraction :
infinity_factor = 10;
grid_results = zeros(x_iterations*y_iterations*z_iterations, 7); i_res = 1; % Result array
for i = 1 : x_iterations % Looping the attractor grid :
  for j = 1 : y_iterations
    for k = 1 : z_iterations % Main logic :
      curr_point = [x_begin+(i-1)*eps y_begin+(j-1)*eps z_begin+(k-1)*eps];
      [~, lces, trajectory] = lyapunov_exp(@rossler_syst_1, curr_point, 0, T, K, rel_tol, abs_tol);
      len = size(trajectory, 1);
      if (abs(trajectory(len, 2)) < infinity_factor && abs(trajectory(len, 3)) < infinity_factor ...
                    && abs(trajectory(len, 4)) < infinity_factor)
      % Saving results for current point :
      grid_results(i_res, :) = [curr_point lyapunov_dim(lces(end, : )) lces(end, : )];
      i_res = i_res + 1;
      end
    end
  end
end
% Computing (local) lyapunov dimention for the fixed point :
[~, lces, ~] = lyapunov_exp(@rossler_syst_1, x0, 0, T, K, rel_tol, abs_tol);
LCEs = lces(end, : );
fid = fopen('hypothesis_roessler_1.txt'); % Saving results in file :
fprintf(fid, '%4s %4s %4s %10s %10s %10s %10s\r\n','x', 'y', 'z', 'dim_L', 'lce1', 'lce3', 'lce3');
fprintf(fid, '%.2f, %.2f, %.2f, %.8f, %.8f, %.8f, %.8f\r\n', grid_results);
fprintf(fid, '\r\nLyapunov dimension in fixed point:\r\n');
fprintf(fid, '%.2f, %.2f, %.2f, %.8f, %.8f, %.8f, %.8f\r\n', [x0 lyapunov_dim(LCEs) LCEs]);
fclose(fid);

end
```

References

✓ N.V. Kuznetsov, Mokaev T.N., Vasilyev P.A. Numerical justification of Leonov conjecture on Lyapunov dimension of Rossler attractor, **Comm. in Nonl. Sci and Num. Simul.**, 19(4), **2014**, pp. 1027-1034 (doi:10.1016/j.cnsns.2013.07.026)
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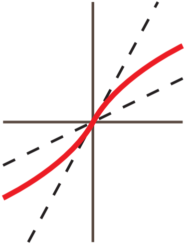
Hidden oscillations: Aizerman and Kalman conjectures

if $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{b}k\mathbf{c}^*\mathbf{z}$, is asympt. stable $\forall\, k \in (k_1, k_2)$: $\forall \mathbf{z}(t, \mathbf{z}_0) \to 0$, then
is $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\varphi(\sigma)$, $\sigma = \mathbf{c}^*\mathbf{x}, \varphi(0) = 0$, $k_1 < \varphi(\sigma)/\sigma < k_2$: $\forall \mathbf{x}(t, \mathbf{x}_0) \to 0$?



$1949: \; k_1 < \varphi(\sigma)/\sigma < k_2$ $\qquad$ $1957: \; k_1 < \varphi'(\sigma) < k_2$

In general, conjectures are not true(Aizerman's: $n \geq 2$, Kalman's: $n \geq 4$)
Periodic solution can exist for nonlinearity from linear stability sector

Aizerman's: I.G.Malkin, N.P.Erugin, N.N.Krasovsky (1952)$n=2$; V.A.Pliss (1958)$n=3$

Survey: V.O. Bragin, V.I. Vagaitsev, N.V. Kuznetsov, G.A. Leonov (2011) Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman conjectures and Chua's circuits, *J. of Computer and Systems Sciences Int., V.50, N4, 511-544 (doi:10.1134/S106423071104006X)*

Harmonic linearization and describing function method requires justification

Harmonic balance & Describing function method (DFM) in Absolute stability theory

$\dot{x} = Px + q\psi(r^*x),\ \psi(0) = 0 \quad (1)$ $\qquad \dot{x} = P_0x + q\varphi(r^*x)$

$W(p) = r^*(P - pI)^{-1}q \qquad P_0{=}P + kqr^*, \varphi(\sigma){=}\psi(\sigma) - k\sigma$

$\mathrm{Im}W(i\omega_0) = 0, k = -(\mathrm{Re}W(i\omega_0))^{-1} \qquad P_0: \lambda_{1,2}{=}\pm i\omega_0,\ \mathrm{Re}\lambda_{j>2}{<}0$

DFM: exists periodic solution $\sigma(t) = r^*x(t) \approx a\cos\omega_0 t$ if $\exists a$:

$$\int_0^{2\pi/\omega_0} \psi(a\cos\omega_0 t)\cos\omega_0 t\mathrm{d}t = ka\int_0^{2\pi/\omega_0}(\cos\omega_0 t)^2\mathrm{d}t$$

Aizerman problem: *If (1) is stable for any linear* $\psi(\sigma){=}\mu\sigma,\ \mu{\in}(\mu_1,\mu_2)$ *then (1) is stable for any nonlinear* $\psi(\sigma) : \mu_1\sigma < \psi(\sigma) < \mu_2\sigma,\ \forall\sigma\neq 0$

DFM: (1) is stable $\Rightarrow k: k{<}\mu_1, \mu_2{<}k \Rightarrow k\sigma^2{<}\psi(\sigma)\sigma, \psi(\sigma)\sigma{<}k\sigma^2$

$$\Rightarrow \forall a\neq 0:\ \int_0^{2\pi/\omega_0}(\psi(a\cos\omega_0 t)\,a\cos\omega_0 t - k(a\,\cos\omega_0 t)^2)\mathrm{d}t \neq 0$$

$\Rightarrow$ no periodic solutions by DFM, but

G.A. Leonov, N.V. Kuznetsov, Algorithms for Searching for Hidden Oscillations in the Aizerman and Kalman Problems, Doklady Mathematics, 2011, 84(1), 475-481 (doi:10.1134/S1064562411040120)
Survey: V.O. Bragin, V.I. Vagaitsev, N.V. Kuznetsov, G.A. Leonov (2011) Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman conjectures and Chua's circuits, *J. of Computer and Systems Sciences Int., V.50, N4, 511-544 (doi:10.1134/S106423071104006X)*

Computation of self-excited and hidden attractors

self-excited attractor localization: *standard computational procedure* is: 1) to find equilibria; 2) after transient process trajectory, starting from a point of unstable manifold in a neighborhood of unstable equilibrium, reaches a state of oscillation therefore one can easily identify it.

hidden oscillations and hidden attractors — basin of attraction does not intersect with small neighborhoods of equilibria

Leonov G.A., Kuznetsov N.V., Vagaitsev V.I, Localization of hidden Chua's attractors, *Phys. Lett. A*, 2011, 375, 2230-2233

Classical Chua's circuit

$\dot{x}=\alpha(y-x-m_1x-\psi(x))$

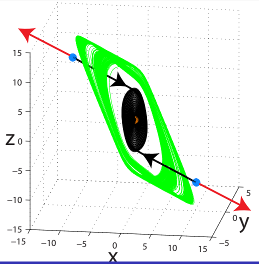
$\dot{y}=x-y+z,\ \dot{z}=-(\beta y+\gamma z)$

$\psi(x)=(m_0-m_1)\mathrm{sat}(x)$

$\alpha=8.4562, \beta=12.0732$

$\gamma=0.0052$

$m_0=-0.1768, m_1=-1.1468$



Equilibria: 2 saddles (blue) & stable zero (orange)

Trajectories: 'from' saddles tend (black) to stable zero or tend (red) to infinity;

Hidden attractor (in green)