16th Hilbert problem: computation of Lyapunov quantities and limit cycles in two-dimensional dynamical systems

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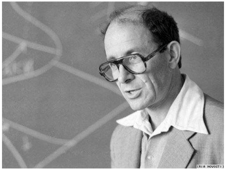
http://www.math.spbu.ru/user/nk/

http://www.math.spbu.ru/user/nk/Limit_cycles_Focus_values.htm

tutorial last version: http://www.math.spbu.ru/user/nk/PDF/Limit_cycles_Focus_values.pdf

History: existence, number & computation of limit cycles



- D.Hilbert: Limit cycles (LC) number & disposition in two-dimensional polynomial system
  $\dot{x} = P_n(x, y), \dot{y} = Q_n(x, y) = ax + by + cx^2 + dxy + ey^2 ...$
- A.Kolmogorov: Calculation of limit cycles in two-dimensional quadratic systems
- V.Arnold: Estimation of parameters domain corresponding to existence of limit cycles

V. Arnold wrote (2005): *To estimate the number of LCs of square vector fields on plane, A.N. Kolmogorov had distributed several hundreds of such fields (with randomly chosen coefficients of quadratic expressions) among a few hundreds of students of Mech.&Math. Faculty of Moscow Univ. as a mathematical practice. Each student had to find the number of LCs of his/her field. The result of this experiment was absolutely unexpected: not a single field had a LC!...The fact that this did not occur suggests that the above-mentioned domains are, apparently, small.*

16th Hilbert problem: quadratic system

$$\dot{x} = a_1x^2 + b_1xy + c_1y^2 + \alpha_1x + \beta_1y$$
$$\dot{y} = a_2x^2 + b_2xy + c_2y^2 + \alpha_2x + \beta_2y$$

- N.N. Bautin 1949-1952: 3 limit cycles [around one focus]
- I.G.Petrovskii, E.M.Landis 1955–1959: **only** 3 limit cycles
- L.Chen & M.Wang, S.Shi 1979-80: 4 limit cycles [(1,3), 2 focuses]
- R. Bamon 1985: number of LC in QS is finite

Number of limit cycles $H(n)$: $H(2) \geqslant 4$

- small-amplitude limit cycles: analytical methods
  - Lyapunov quantity (focus value, Poincare-Lyapunov constant) weak focus & Andronov-Hopf bifurcation
- normal-amplitude limit cycles: analytical & numerical methods

numerical methods: hidden oscillation — nested and semistable cycles

Lyapunov quantity (focus value, Poincare-Lyapunov constant)

$$\dot{x} = f_{10}x + f_{01}y + f(x,y) \qquad \text{eig}(A)=\text{eig}\begin{pmatrix} f_{10} & f_{01} \\ g_{10} & g_{01} \end{pmatrix}=\pm i\omega_0: \quad \widetilde{L}_1=0$$
$$\dot{y} = g_{10}x + g_{01}y + g(x,y)$$

$$f = \sum_{k+j=2}^{n} f_{kj}x^k y^j + o\big((|x|+|y|)^n\big), \quad g = \sum_{k+j=2}^{n} g_{kj}x^k y^j + o\big((|x|+|y|)^n\big)$$

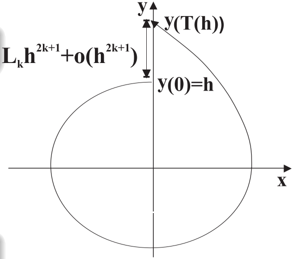
Poincare map: $\mathrm{L}(h) = y(T(h),h) - h$

Solution with suff. small h:

$x(t,h) = x(t,0,h),\ y(t,h) = y(t,0,h)$

$T(h)$ — time of first return

of $(x(t,h),y(t,h))$ to $\{x=0, y>0\}$



$$y(T(h),h)=h(1+\widetilde{L}_1)+\widetilde{L}_2h^2+\widetilde{L}_3h^3+\widetilde{L}_4h^4+..+o(h^n)$$

First nonzero $\widetilde{L}_i$ has odd index: $\mathrm{L}(h)\mathrm{L}(-h)\leqslant 0$

Lyapunov quantity $\mathrm{L}_k \stackrel{\text{def}}{=} \widetilde{L}_{2k+1}$ (first $\neq 0$) : $y(T(h),h)-h=\mathrm{L}_k h^{2k+1}+o(h^{2k+1})$

trajectory winding or unwinding & equilibrium stability or instability

A.Lyapunov: similar procedure for dynamical system higher dimension

Symbolic expression of Lyapunov quantities: in terms of coefficients of right hand side of the system

To compute general expression of kth Lyapunov quantity it is necessary to consider expansion upto $2k+1$: $\mathrm{L}_k = \mathrm{L}_k(\{g_{k,j}\}_{k+j=2}^{2k+1}, \{f_{k,j}\}_{k+j=2}^{2k+1})$

$$\dot{x} = -y + f_{20}x^2 + f_{11}xy + f_{02}y^2 + \ldots,$$
$$\dot{y} = x + g_{20}x^2 + g_{11}xy + g_{02}y^2 + \ldots.$$

- **1949**, Bautin:

$$\mathrm{L}_1 = \frac{\pi}{4}(g_{21} + f_{12} + 3f_{30} + 3g_{03} + f_{20}f_{11} + f_{02}f_{11} - g_{11}g_{20} + 2g_{02}f_{02} - 2f_{20}g_{20} - g_{02}g_{11})$$

- **1959**, Serebryakova: $\mathrm{L}_2 = \frac{\pi}{72}($ ...1 page... $)$
- **1968**, Shuko: first computer program for Lq calculation
- **2008**, Kuznetsov, Leonov: $\mathrm{L}_3 = \frac{\pi}{1728}($ ...4 pages... $)$

To simplify the expressions of Lyapunov quantities often used change of coordinate (complex, polar) & reduction to normal forms (but such reduction is not unique and often a laborious).

Direct method for computation of Lyapunov quantities in Euclidian coordinates and in the time domain

$$\dot{x}=-y+f(x,y)=-y+\sum_{k+j=2}^{n} f_{kj}x^k y^j+o\big((|x|+|y|)^n\big),\ \ x(t,h)=x(t,0,h)$$
$$\dot{y}=+x+g(x,y)=+x+\sum_{k+j=2}^{n} g_{kj}x^k y^j+o\big((|x|+|y|)^n\big),\ \ y(t,h)=y(t,0,h)$$

1. Approximation of solution $x(t,h), y(t,h)$

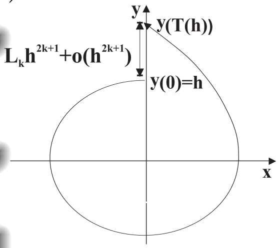
$$x(t,h)=\sum_{k=1}^{n}\widetilde{x}_{h^k}(t)h^k+o(h^n),\ y(t,h)=\sum_{k=1}^{n}\widetilde{y}_{h^k}(t)h^k+o(h^n)$$

2. Approximation of return time $T(h): x(T(h),h)=0$

$$T(h)=2\pi+\Delta T(h)=2\pi+\sum_{j=1}^{n}\widetilde{T}_j h^j+o(h^n)$$

3. Computation of Lyapunov quantities $\mathrm{L_k}$: $\{\widetilde{\mathrm{L}}_\mathrm{i}\}_{\mathrm{i}=2}^{2\mathrm{k}}=0$

$$y(T(h),h)=h+\widetilde{L}_2h^2+\widetilde{L}_3h^3+\widetilde{L}_4h^4+\ldots+o(h^n)\qquad=\mathbf{h}+\mathrm{L}_\mathbf{k}\mathbf{h}^{\mathbf{2k+1}}+\mathbf{o}(\mathbf{h}^{\mathbf{2k+1}})$$



In the study of real systems in applied problems it is more convenient to study the system in the initial "physical" space.

Kuznetsov N.V., Leonov G.A.,
Lyapunov quantities, limit cycles and strange behavior of trajectories in two-dimension
Journal of Vibroengineering, Vol. 10, Iss. 4, 2008, 460-467 [PDF]

Direct method for computation of Lq: solution approximation

$$\dot{x}=-y+f(x,y)=-y+\sum_{k+j=2}^{n} f_{kj}x^k y^j+o\big((|x|+|y|)^n\big),\ \ x(t,h)=x(t,0,h)$$
$$\dot{y}=+x+g(x,y)=+x+\sum_{k+j=2}^{n} g_{kj}x^k y^j+o\big((|x|+|y|)^n\big),\ \ y(t,h)=y(t,0,h)$$

$$x(t,h)=\sum_{k=1}^{n}\widetilde{x}_{h^k}(t)h^k+o(h^n),\quad y(t,h)=\sum_{k=1}^{n}\widetilde{y}_{h^k}(t)h^k+o(h^n)$$

$$x(t,h)=h\frac{\partial x(t,\eta)}{\partial\eta}\Big|_{\eta=0}+\frac{h^2}{2}\frac{\partial^2 x(t,\eta)}{\partial\eta^2}\Big|_{\eta=h\theta_x(t,h)}=\widetilde{x}_{h^1}(t)h+o(h),\ 0\leq\theta_x(t,h)\leq 1$$
$$y(t,h)=h\frac{\partial y(t,\eta)}{\partial\eta}\Big|_{\eta=0}+\frac{h^2}{2}\frac{\partial^2 y(t,\eta)}{\partial\eta^2}\Big|_{\eta=h\theta_y(t,h)}=\widetilde{y}_{h^1}(t)h+o(h),\ 0\leq\theta_y(t,h)\leq 1$$

$$k=1:\quad \dot{\widetilde{x}}_{h^1}(t)=-\widetilde{y}_{h^1}(t),\ \dot{\widetilde{y}}_{h^1}(t)=\widetilde{x}_{h^1}(t)\ \ f,g\big(x(t,h),y(t,h)\big)=o(h^1)$$

$$x_{h^1}(t,h)=\widetilde{x}_{h^1}(t)h=-h\sin(t),\ \ y_{h^1}(t,h)=\widetilde{y}_{h^1}(t)h=h\cos(t)$$

Let $x_{h^{k-1}}(t,h)=\sum_{i=1}^{k-1}\widetilde{x}_{h^k}(t)h^k,\ y_{h^{k-1}}(t,h)=\sum_{i=1}^{k-1}\widetilde{y}_{h^k}(t)h^k$ known f-ns(t,h)

$$x_{h^k}(t,h)=x_{h^{k-1}}(t,h)+\widetilde{x}_{h^k}(t)h^k+o(h^k),\ y_{h^k}(t,h)=y_{h^{k-1}}(t,h)+\widetilde{y}_{h^k}(t)h^k+o(h^k)$$

$$f\big(x_{h^{k-1}}(t,h)+o(h^{k-1}),y_{h^{k-1}}(t,h)+o(h^{k-1})\big)=O(h^{k-1})+u^f_{h^k}(t)h^k+o(h^k)$$
$$g\big(x_{h^{k-1}}(t,h)+o(h^{k-1}),y_{h^{k-1}}(t,h)+o(h^{k-1})\big)=O(h^{k-1})+u^g_{h^k}(t)h^k+o(h^k)$$

$\Rightarrow u^{f,g}_{h^k}(t)=u^{f,g}_{h^k}\big(\{\widetilde{x}_{h^m}(t),\widetilde{y}_{h^m}(t)\}_{m\leq k-1}\big)$ known functions (t)

$$k-1\to k:\quad \dot{\widetilde{x}}_{h^k}(t)=-\widetilde{y}_{h^k}(t)+u^f_{h^k}(t),\ \ \dot{\widetilde{y}}_{h^k}(t)=\widetilde{x}_{h^k}(t)+u^g_{h^k}(t)$$

Direct method for computation of Lq: time constants & Lq

$x_{h^n}(t,h) = \sum_{k=1}^{n} \widetilde{x}_{h^k}(t) h^k : \quad x_{h^n}\left(t,h,\{f_{ij},g_{ij}\}_{i+j=2}^{n}\right) :$ known f-n(t)

$y_{h^n}(t,h) = \sum_{k=1}^{n} \widetilde{y}_{h^k}(t) h^k : \quad y_{h^n}\left(t,h,\{f_{ij},g_{ij}\}_{i+j=2}^{n}\right) :$ known f-n(t)

$$T(h) = 2\pi + \Delta T(h) = 2\pi + \sum_{j=1}^{n} \widetilde{T}_j h^j + o(h^n) : \quad x(2\pi + \Delta T(h), h) = 0$$

$$\widetilde{x}_{h^k}(2\pi+\Delta T(h)) = \widetilde{x}_{h^k}(2\pi) + \sum_{m=1}^{n} \widetilde{x}_{h^k}^{(m)}(2\pi)\frac{(\Delta T(h))^m}{m!} + o\big((\Delta T(h))^n\big)$$

$$\widetilde{y}_{h^k}(2\pi+\Delta T(h)) = \widetilde{y}_{h^k}(2\pi) + \sum_{m=1}^{n} \widetilde{y}_{h^k}^{(m)}(2\pi)\frac{(\Delta T(h))^m}{m!} + o\big((\Delta T(h))^n\big)$$

$$x(2\pi+\Delta T(h), h) = \sum_{k=1}^{n} \widetilde{x}_{h^k}(2\pi+\Delta T(h)) h^k/k! + o(h^n) = 0$$

$$y(2\pi+\Delta T(h), h) = \sum_{k=1}^{n} \widetilde{y}_{h^k}(2\pi+\Delta T(h)) h^k/k! + o(h^n) = \sum_{k=1}^{n} \widetilde{L}_k + o(h^n)$$

h :	$0 = \widetilde{x}_{h^1}(2\pi)$	$\widetilde{L}_1 = \widetilde{y}_{h^1}(2\pi)$
h^2 :	$0 = \widetilde{x}_{h^2}(2\pi) + \widetilde{x}'_{h^1}(2\pi)\widetilde{T}_1$	$\widetilde{L}_2 = \widetilde{y}_{h^2}(2\pi) + \widetilde{y}'_{h^1}(2\pi)\widetilde{T}_1$
...	...	...
h^n :	$0 = \widetilde{x}_{h^n}(2\pi) + .. + \widetilde{x}'_{h^1}(2\pi)\widetilde{T}_{n-1}$	$\widetilde{L}_n = \widetilde{y}_{h^n}(2\pi) + .. + \widetilde{y}'_{h^1}(2\pi)\widetilde{T}_{n-1}$

$\widetilde{T}_{k-1} = \widetilde{T}_{k-1}(\{f_{ij},g_{ij}\}_{i+j=2}^{k}) \qquad \widetilde{L}_k = L_k(\{T_i\}_{i=1}^{k-1}, \{f_{ij},g_{ij}\}_{i+j=2}^{k})$

Lyapunov quantity $\mathrm{L}_k \stackrel{\text{def}}{=} \widetilde{L}_{2k+1}(\text{first} \neq 0) : y(T(h),h) - h = \mathrm{L}_k h^{2k+1} + o(h^{2k+1})$

Symbolic computation: solution appr-n, time constants & Lq

```
function [L,T,xt,yt] = fLQ_kuzleo(fxy,gxy,N)
syms x y h t 'real'
  NL=2*N+1; Nfg=NL; % CREATE SYMBOLIC REPRESENTATION
  xt_s(1:Nfg-1)=0*h; yt_s(1:Nfg-1)=0*h; xth_s=0*t; yth_s=0*t;
  for n=1:Nfg %1. Create solution as a series of h (x(0,h)=0; y(0,h)=h)
     xt_s(n)=sym(['xt_',int2str(n)],'real'); xth_s=xth_s+xt_s(n)*h^n;
     yt_s(n)=sym(['yt_',int2str(n)],'real'); yth_s=yth_s+yt_s(n)*h^n;
  end
  disp(['NL=',int2str(NL)]); % To calculate L_m , set NL= 2m+1
  sT_h_cur=0; %2. Create crossing time T (x(T,h)=0,y(T,h)>0 ) as a series of h
  for i=1:NL-1
    sT_h(i,1)=sym(['T',int2str(i)],'real');
    sT_h_cur=sT_h_cur + sT_h(i,1)*h^i;
  end;
  % CALCULATION OF LYAPUNOV QUANTITIES
  %1. Calculation x(t,h) y(t,h) as series in terms of  t
  ugt(1:Nfg)=0*t; xt(1:Nfg)=0*t; yt(1:Nfg)=0*t;
  %  solution of the first approximation system
  xt(1)=-sin(t); yt(1)=cos(t); xt_cur=xt(1)*h; yt_cur=yt(1)*h;
  for i=2:NL
    %create approx-n of right-hand sides of the system depending on t
    uft_s=subs(diff(subs(fxy, [x y], [xth_s yth_s]),h,i)/factorial(i),h,0);
    uft(i)= subs(uft_s, [xt_s yt_s], [xt yt]);
    ugt_s=subs(diff(subs(gxy, [x y], [xth_s yth_s]),h,i)/factorial(i),h,0);
    ugt(i)= subs(ugt_s, [xt_s yt_s], [xt yt]);
    uIt=diff(ugt(i),t)+uft(i); %create approximation of solution depending on t
    Iucos=int(cos(t)*uIt,t); Iucos_t0=(Iucos - subs(Iucos,t,0));
    Iusin=int(sin(t)*uIt,t); Iusin_t0=(Iusin - subs(Iusin,t,0));
    ug0=subs(ugt(i),t,0);
    xt(i)=simplify(cos(t)*ug0+Iucos_t0*cos(t)+Iusin_t0*sin(t)-ugt(i));
    yt(i)=simplify(sin(t)*ug0+Iucos_t0*sin(t)-Iusin_t0*cos(t));
    xt_cur=xt_cur+xt(i)*h^i; yt_cur=yt_cur+yt(i)*h^i;
  end;
```

Symbolic computation: solution appr-n, time constants & Lq

```
%2. Calculation coeffisients of x(t,h) in terms of T_k
xh_cur=subs(xt_cur,t,2*pi);
for k=1:NL
    xh_cur=xh_cur + subs(diff(xt_cur,k,t),t,2*pi)*sT_h_cur^k/factorial(k);
end;
for k=1:NL
    xh(k,1)=subs(diff(xh_cur,k,h)/factorial(k),h,0);
end;
%3.  Find T_k from x_k=0
xh_temp=xh; T_cur=0; T(1,1)=0*x;
for k=2:NL
    T(k-1,1)=solve(xh_temp(k,1),sT_h(k-1,1));
    T_cur=T_cur + T(k-1,1)*h^(k-1);
    xh_temp=subs(xh_temp,sT_h(k-1,1),T(k-1,1));
end;
%4.
yh_cur=subs(yt_cur,t,2*pi);
for k=1:NL
    yh_cur=yh_cur + subs(diff(yt_cur,k,t),t,2*pi)*T_cur^k /factorial(k);
end;
for k=1:NL
    yh(k,1)=subs(diff(yh_cur,k,h)/factorial(k),h,0);
end;
for k=1:N
  L(k)=factor(yh(2*k+1))
end;
```

Example: non isochronous center in Duffing $\dot{x} = -y, \dot{y} = x + x^3$

For solution $(x(t), y(t))$ with i.d. $x_0 = 0, y_0 = h$: $y(t)^2 + x(t)^2 + \frac{1}{2}x(t)^4 = h^2$
$\Rightarrow$ all trajectories are closed and periodic: $y(0) = h, x(0) = x(T(h)) = 0$

$\Rightarrow$ for $(x < 0 < y)$: $\frac{\mathrm{d}t}{\mathrm{d}y} = \frac{1}{x(1+x^2)} = \frac{1}{-\sqrt{-1+\sqrt{1+2h^2-2y^2}}\sqrt{1+2h^2-2y^2}}$

$$T(h) = 4\int_h^0 \frac{\mathrm{d}y}{-\sqrt{-1+\sqrt{1+2h^2-2y^2}}\sqrt{1+2h^2-2y^2}}$$

$y = h\cos(z) \Rightarrow z = \arccos\frac{y}{h}$,
$\mathrm{d}y = -h\sin(z)\mathrm{d}z$

$$T(h) = 4\int_0^{\pi/2} \frac{h\sin(z)\mathrm{d}z}{\sqrt{-1+\sqrt{1+2h^2\sin^2 z}}\sqrt{1+2h^2\sin^2 z}} = 2\pi + \sum_{k=1}^{n} \widetilde{T}_k h$$

$$\widetilde{T}_1 = 0,\ \widetilde{T}_2 = -\frac{3\pi}{4},\ \widetilde{T}_3 = 0,\ \widetilde{T}_4 = \frac{105\pi}{128},\ \widetilde{T}_5 = 0,\ \widetilde{T}_6 = \frac{1155\pi}{1024}, \ldots$$

$$x(t,h) = \widetilde{x}_{h^1}(t)h + \widetilde{x}_{h^2}(t)h^2 + \widetilde{x}_{h^3}(t)h^3 + \widetilde{x}_{h^4}(t)h^4 + \ldots$$

$$y(t,h) = \widetilde{y}_{h^1}(t)h + \widetilde{y}_{h^2}(t)h^2 + \widetilde{y}_{h^3}(t)h^3 + \widetilde{y}_{h^4}(t)h^4 + \ldots$$

$$\widetilde{x}_{h^1}(t) = -\sin(t), \quad \widetilde{y}_{h^1}(t) = \cos(t); \quad \widetilde{x}_{h^2}(t) = \widetilde{y}_{h^2}(t) = 0$$

$$\widetilde{x}_{h^3}(t) = \frac{1}{8}\cos(t)^2\sin(t) - \frac{3}{8}t\cos(t) + \frac{1}{4}\sin(t), \quad \widetilde{x}_{h^4}(t) = 0.$$

$$\widetilde{y}_{h^3}(t) = -\frac{3}{8}t\sin(t) + \frac{3}{8}\cos(t) - \frac{3}{8}\cos(t)^3; \quad \widetilde{y}_{h^4}(t) = 0.$$

$T(h) \neq \mathrm{const}$, $\mathsf{L}_1 = \mathsf{L}_2 = \ldots = 0$: non isochronous center

Classical Poincare-Lyapunov method: Lyapunov function

$$\dot{x} = -y + f(x,y) \qquad f(x,y) = \sum_{k+j=2}^{n} f_{kj}x^k y^j + o\big((|x|+|y|)^n\big)$$
$$\dot{y} = +x + g(x,y) \qquad g(x,y) = \sum_{k+j=2}^{n} g_{kj}x^k y^j + o\big((|x|+|y|)^n\big)$$

$$V(x,y) = \frac{x^2+y^2}{2} + V_3(x,y) + .. + V_{n+1}(x,y) \qquad V_k(x,y) = \sum_{i+j=k} V_{i,j}x^i y^j$$

$$\dot{V}(x,y) = \frac{\partial V(x,y)}{\partial x}(-y + f_n(x,y)) + \frac{\partial V(x,y)}{\partial y}(x + g_n(x,y)) + o\big((|x|+|y|)^{n+1}\big)$$

$$\dot{V}(x,y) = W_3(x,y) + ... + W_{n+1}(x,y) + o\big((|x|+|y|)^{n+1}\big)$$

$$W_k(x,y) = \left(x\frac{\partial V_k(x,y)}{\partial y} - y\frac{\partial V_k(x,y)}{\partial x}\right) + u_k(x,y,\{V_{ij}, f_{ij}, g_{ij}\}_{i+j<k})$$

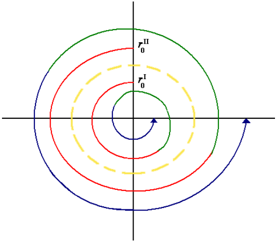
It is possible to determine $\{V_{ij}\}_{i+j=k}$ for k=3,.. step by step so that $\dot{V}(x,y) = w_1(x^2+y^2)^2 + w_2(x^2+y^2)^3 + ..,$ while $w_{1,..,k-1} = 0$: solve a system of $(k+1)$ linear equations. Uniqueness if $V_{(m+1)\,(m+1)} = 0$, for m odd, $V_{(m)\,(m+2)} + V_{(m+2)\,(m)} = 0$, for m even

Poncare-Lyapunov constant $\overset{\text{def}}{=}$ first $w_m \neq 0 \quad (2\pi w_m = \mathrm{L})$

Lyapunov quantities & small limit cycles:
Andronov-Hopf bifurcation, cyclicity and center problems

$\dot{x} = f_{10}x + f_{01}y + f(x,y), \quad \dot{y} = g_{10}x + g_{01}y + g(x,y)$

Solution $x(t,h) = x(t,0,h)$, $y(t,h) = y(t,0,h)$, return time $T(h)$



Small limit cycles: $\mathrm{L}_0 = \widetilde{L}_1 = 0, \mathrm{L}_1 = \widetilde{L}_3 > 0$

$y(T(h),h) - h = \mathrm{L}_1 h^3 + o(h^3)$

$g_{01}^{\varepsilon} = g_{01} + \varepsilon_1, \ g_{03}^{\varepsilon} = g_{03} + \varepsilon_3$

$\mathrm{L}_0^{\varepsilon} = \widetilde{L}_1^{\varepsilon} < 0 < \mathrm{L}_1^{\varepsilon} = \widetilde{L}_3^{\varepsilon}, \ |\mathrm{L}_0^{\varepsilon}| << |\mathrm{L}_1^{\varepsilon}|$

$y(T(h),h) - h = \widetilde{L}_1^{\varepsilon} h + \widetilde{L}_2^{\varepsilon} h^2 + \widetilde{L}_3^{\varepsilon} h^3 + o(h^3)$:

$\exists h_1, h_2 : y(T(h_1),h_1) - h_1 < 0 < y(T(h_2),h_2) - h_2$

Number of "independent" zeros of Lyapunov quantities expressions? Algebraic methods for analysis of polynomials:
Bautin ideal, Groebner's bases ...

- C(2)=3, Bautin 1949
- C(3)=9, Makeev 1993
- C(n)=?,

e.g., Lynch 2005, a low bound
Gasull *et al.* 2010, an upper bound

A. Lienard

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

The classical Lienard theorem:

Let $f(x)$ be even, $g(x)$ be odd, $xg(x) > 0$ $\forall x \neq 0$, $f(0) < 0$, $f \in C^1(R^1)$, $g \in C^1(R^1)$, $f'(x) > 0$, $\forall x > 0$, $f(x) \to +\infty$ as $x \to +\infty$.

Thm permits to find a unique orbital stable periodic solution.

$$\dot{x} = x^2 + xy + y$$
$$\dot{y} = ax^2 + bxy + cy^2 + \alpha x + \beta y$$

Positively invariant half plane $\Gamma = \{x > -1, r \in R^1\}$

Transformation of Quadratic system to Lienard system

$$\dot{x} = u, \ \dot{u} = -f(x)u - g(x) \quad u = \left(y + \frac{x^2}{(x+1)}\right)|x+1|^q$$

$$f(x) = [(2c_2 - b_2 - 1)x^2 - (2 + b_2 + \beta_2) - \beta_2]|x+1|^{q-2}, \ q = -c_2$$

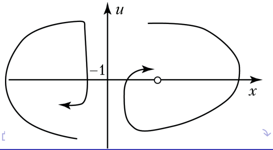
$$g(x) = [-x(x+1)^2(a_2x + \alpha_2) + x^2(x+1)(b_2x - \beta_2) - c_2x^4]\frac{|x+1|^{2q}}{(x+1)^3}$$

Large limit cycles: asymptotic integration of Lienard eq.

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$
$$f(x) = (Ax^2 + Bx + C)|x+1|^{q-2},$$
$$g(x) = (C_1x^3 + C_2x^2 + C_3x + 1)x\frac{|x+1|^{2q}}{(x+1)^3}.$$

$$f(x) = \left(A + O\left(\tfrac{1}{|x|}\right)\right)|x|^q,\ \ g(x) = \left(C + O\left(\tfrac{1}{|x|}\right)\right)x|x|^{2q}$$
$$FdF + \tfrac{\left(A+O\left(\frac{1}{|x|}\right)\right)}{(q+1)}Fd(x^{q+1}) + \tfrac{\left(C+O\left(\frac{1}{|x|}\right)\right)}{(q+1)}(x)^{q+1}d(x^{q+1}) = 0$$
$$FdF + \tfrac{A}{(q+1)}Fd(x^{q+1}) + \tfrac{C}{(q+1)}(x^{q+1})d(x^{q+1}) = 0.$$
$$z = x^{q+1}:\ \ F\tfrac{dF}{dz} + \tfrac{A}{(q+1)}F + \tfrac{C}{(q+1)}z = 0$$
$$\ddot{z} + \tfrac{A}{(q+1)}\dot{z} + \tfrac{C}{(q+1)}z = 0$$

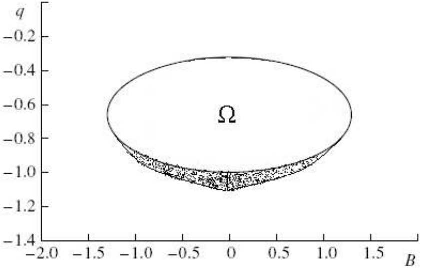
Theorem. Boundedness of $x(t), y(t)$ in $\Gamma \Leftrightarrow$ $c_2 \in (0,1), c_2 < b_2 - a_2$ and either $2c_2 > b_2 + 1$ or $2c_2 \le b_2 + 1$, $4a_2(c_2 - 1) > (b_2 - 1)^2$.



Estimation of parameters domain: LC existence (Arnold's problem)

$$\dot{x}=a_1x^2+b_1xy+c_1y^2+\alpha_1x+\beta_1y,\ \ \dot{y}=a_2x^2+b_2xy+c_2y^2+\alpha_2x+\beta_2y$$
$$\ddot{x}+f(x)\dot{x}+g(x)=0 \quad f(x)=\big(Ax^2+Bx+C\big)|x+1|^{q-2},$$
$$g(x)=\big(C_1x^4+C_2x^3+C_3x^2+C_4x+C_5\big)\frac{|x+1|^{2q}}{(x+1)^3}.$$

$$A=\tfrac{2}{5}B(q+2),$$
$$C_1=(q+3)\tfrac{B^2}{25}-\tfrac{(1+3q)}{5},$$
$$C_2=\big(15(1-2q)+3B^2\big)\tfrac{1}{25},$$
$$C_3=\tfrac{3(3-q)}{5},\ \ C_4=1,\ \ C_5=0.$$
$$L_3=-\frac{\pi B(q+2)(3q+1)[5(q+1)(2q-1)^2+B^2(q-3)]}{20000}$$
$$\Omega:\ \ B^2<-5(q+1)(3q+1),\ \ B\neq 0$$



One large LC + 3-rd weak focus:
4 LC by small perturbations

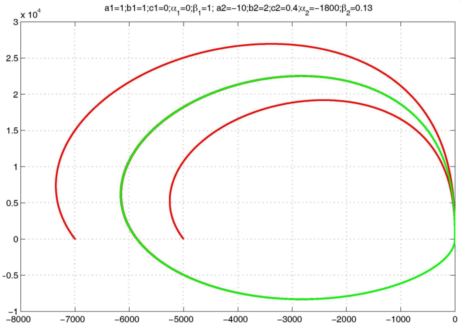
Four normal size limit cycles

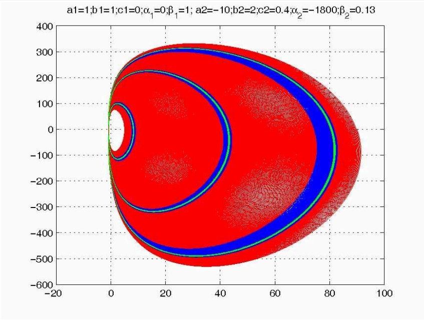
$$\begin{aligned} \dot{x} &= x^2 + xy + y \\ \dot{y} &= ax^2 + bxy + cy^2 + \alpha x + \beta y \end{aligned} \qquad (1)$$

$c \in (1/3,1), \alpha = -\varepsilon^{-1}, bc < 1, b > a + c, 2c < b+1, 4a(c-1) > (b-1)^2, \beta = 0$

Theorem. *For sufficiently small ε system (1) has three limit cycles: one to the left of line $\{x = -1\}$ and two to the right of it.*

Increase β and get four normal size limit cycles.





Main publications

✓ G.A. Leonov, N.V. Kuznetsov, and E.V. Kudryashova,
A Direct Method for Calculating Lyapunov Quantities of Two-Dimensional Dynamical Systems, *Proceedings of the Steklov Institute of Mathematics*, Volume 272, Supplement 1, **2011**, pp. S119-S127

✓ G.A. Leonov, N.V. Kuznetsov, O.A. Kuznetsova, S.M. Seledzhi, V.I.Vagaitsev,
Hidden oscillations in dynamical systems, *Transaction on Systems and Control*, Is. 2, Vol. 6, **2011**, pp. 54–67 (survey)

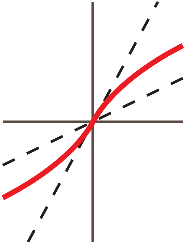
✓ G.A. Leonov, N.V. Kuznetsov.
Limit cycles in quadratic systems with perturbed weak focus of order 3 and a saddle equilibrium at infinity, *Doklady Mathematics,* 82(2), 2010, 693–696 [DOI] [PDF]

✓ Leonov G.A., Kuznetsov N.V., and Kudryashova E.V. A direct method for calculating Lyapunov quantities of two-dimensional dynamical systems. *Proceedings of the Steklov Institute of Mathematics*, 2010, Suppl. 3

✓ Kuznetsov N.V., Leonov G.A.,
Limit cycles and strange behavior of trajectories in two dimension quadratic systems, *Journal of Vibroengineering*, Vol. 10, Iss. 4, 2008, 460-467 [PDF]

✓ Kuznetsov N.V., Leonov G.A., Computation of Lyapunov quantities, Proceedings of the 6th EUROMECH Nonlinear Dynamics Conference, 2008

if $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{b}k\mathbf{c}^*\mathbf{z}$, is asympt. stable $\forall\, k \in (k_1, k_2)$: $\forall \mathbf{z}(t, \mathbf{z}_0) \to 0$
$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\varphi(\sigma)$, $\sigma = \mathbf{c}^*\mathbf{x}$, $\varphi(0) = 0$, $k_1 < \varphi(\sigma)/\sigma$, $\varphi' < k_2$, $\forall \mathbf{x}(t, \mathbf{x}_0) \to 0$?



$1949: \; k_1 < \varphi(\sigma)/\sigma < k_2$

$1957: \; k_1 < \varphi'(\sigma) < k_2$

In general, conjectures are not true (Aizerman's $n \geq 2$, Kalman's $n \geq 4$): nonlinearity can be in linear stability sector but periodic solutions exist.

✓ Bragin, Leonov, Kuznetsov, Vagaitsev (2011) Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman conjectures and Chua's circuits, *J. of Computer and Systems Sciences Int., V.50, N4, 511-544 (survey)*

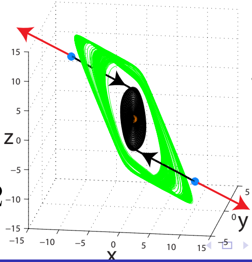
Hidden attractors localization

self-exciting oscillations and attractors - Van der Pol, Lorenz, et al.
standard computation: 1) determine equilibria 2) after transient process trajectory, starting from a point of unstable manifold in neighborhood of unstable equilibrium, reaches an oscillation and computes it.

hidden oscillations and hidden attractors — basin of attraction does not contain neighborhoods of equilibria
Leonov G.A., Kuznetsov N.V., Vagaitsev V.I, Localization of hidden Chua's attractors, *Phys. Lett. A*, 2011, 375, 2230-2233

Chua system

$$\dot{x} = \alpha(y - x - m_1 x - \psi(x))$$
$$\dot{y} = x - y + z,$$
$$\dot{z} = -(\beta y + \gamma z)$$
$$\psi(x) = (m_0 - m_1)\mathrm{sat}(x)$$
$$\alpha = 8.4562 \; \beta = 12.0732 \; \gamma = 0.0052$$
$$m_0 = -0.1768, m_1 = -1.1468$$



Stable zero eqv. and 2 symmetric saddles: trajectories "from" saddles tend to zero eqv. or to infinity: black and red
Hidden attractor(green)

Lyapunov exponents and chaos : Linearization, Perron effects

$$\begin{cases} \dot{x} = F(x),\ x \in \mathbb{R}^n, \quad F(x_0) = 0 \\ x(t) \equiv x_0, A = \frac{dF(x)}{dx}\Big|_{x=x_0} \end{cases} \quad \begin{cases} \dot{y} = Ay + o(y) \\ y(t) \equiv 0, (y = x - x_0) \end{cases} \quad \begin{cases} \dot{z} = Az \\ z(t) \equiv 0 \end{cases}$$

✓stationary: $z(t) = 0$ is exp. stable $\Rightarrow$ $y(t) = 0$ is asympt. stable

$$\begin{cases} \dot{x} = F(x), \quad \dot{x}(t) = F(x(t)) \not\equiv 0 \\ x(t) \not\equiv x_0, A(t) = \frac{dF(x)}{dx}\Big|_{x=x(t)} \end{cases} \quad \begin{cases} \dot{y} = A(t)y + o(y) \\ y(t) \equiv 0, (y = x - x(t)) \end{cases} \begin{cases} \dot{z} = A(t)z \\ z(t) \equiv 0 \end{cases}$$

? nonstationary: $z(t) = 0$ is exp. stable $\Rightarrow ?$ $y(t) = 0$ is asympt. stable

! Perron effect: $z(t)=0$ is exp. stable(unst), $y(t)=0$ is exp. unstable(st)
- positive largest Lyapunov exponent doesn't in general indicate chaos
- negative largest LE doesn't in general indicate stability [PDF slides]

[DOI] [PDF] G.A. Leonov, N.V. Kuznetsov, Time-Varying Linearization and the Perron effects, *International Journal of Bifurcation and Chaos*, Vol. 17, No. 4, 2007, pp. 1079-1107 (survey)