Hidden oscillations: Aizerman conjecture & Kalman conjecture on absolute stability of nonlinear control systems, describing function method and harmonic balance

Nikolay V. Kuznetsov, Gennady A. Leonov

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http://www.math.spbu.ru/user/nk/Hidden_oscillation_Attractor_Localization.htm

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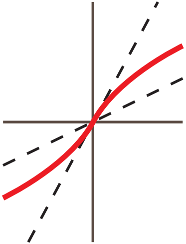
Aizerman conjecture and Kalman conjectures

if $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{b}k\mathbf{c}^*\mathbf{z}$, is asympt. stable $\forall\, k \in (k_1, k_2)$: $\forall \mathbf{z}(t, \mathbf{z}_0) \to 0$, then
is $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\varphi(\sigma)$, $\sigma = \mathbf{c}^*\mathbf{x}, \varphi(0) = 0$, $k_1 < \varphi(\sigma)/\sigma < k_2$: $\forall \mathbf{x}(t, \mathbf{x}_0) \to 0$?



$1949: \;\; k_1 < \varphi(\sigma)/\sigma < k_2$ $\qquad$ $1957: \;\; k_1 < \varphi'(\sigma) < k_2$

In general, conjectures are not true(Aizerman's: $n \geq 2$, Kalman's: $n \geq 4$)
Hidden oscillations: periodic solution and with unique stable equilibrium

Aizerman's: I.G.Malkin, N.P.Erugin, N.N.Krasovsky (1952)$n = 2$; V.A.Pliss (1958)$n = 3$

Survey: V.O. Bragin, V.I. Vagaitsev, N.V. Kuznetsov, G.A. Leonov (2011) Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman conjectures and Chua's circuits, *J. of Computer and Systems Sciences Int., V.50, N4, 511-544 (doi:10.1134/S106423071104006X)*

Kalman conjecture (Kalman problem) 1957

$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\varphi(\sigma),\ \sigma = \mathbf{c}^*\mathbf{x}, \varphi(0) = 0,\ \ 0 < \varphi'(\sigma) < k:\ \ \forall \mathbf{x}(t, \mathbf{x}_0) \to 0?$

- **Fitts R. 1966**: series of counterexamples in $\mathbb{R}^4$, nonlinearity $\varphi(\sigma) = \sigma^3$
- **Barabanov N. 1979-1988**: some of Fitts counterexamples are false; analytical 'counterex.' construction in $\mathbb{R}^4$, $\varphi(\sigma)$ 'close' to $\mathrm{sign}(\sigma)$ $(0 \leq d/d\sigma)$
  later 'gaps' were reported by Glutsyuk, Meisters, Bernat & Llibre
- **Barabanov N., Leonov G.**: Kalman conj. is true in $\mathbb{R}^3$ (Yakubovich thm)
- **Bernat J. & Llibre J. 1996**: analytical-numerical 'counterexamples' construction in $\mathbb{R}^4$, $\varphi(\sigma)$ 'close' to $\mathrm{sat}(\sigma)$ $(0 \leq d/d\sigma)$
- **Leonov G., Kuznetsov N., Bragin V. 2010**:
  some of Fitts counterexamples are true;
  smooth counterexample in $\mathbb{R}^4$ with $\varphi(\sigma) = \tanh(\sigma)$: $0 < \tanh'(\sigma) \leqslant 1$;
  analytical-numerical counterexamples construction for any type of $\varphi(\sigma)$;

G.A. Leonov, N.V. Kuznetsov, Algorithms for searching for hidden oscillations in the Aizerman and Kalman problems, Doklady Mathematics, 84(1), 2011, 475-481 (doi:10.1134/S1064562411040120)
Survey: V.O. Bragin, V.I. Vagaitsev, N.V. Kuznetsov, G.A. Leonov (2011) Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman conjectures and Chua's circuits, *J. of Computer and Systems Sciences Int., V.50, N4, 511-544 (doi:10.1134/S106423071104006X)*

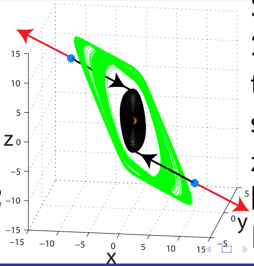
Hidden attractors localization

self-exciting oscillations and attractors - Van der Pol, Lorenz, et al.
standard computation: 1) determine equilibria 2) after transient process trajectory, starting from a point of unstable manifold in neighborhood of unstable equilibrium, reaches an oscillation and computes it.

hidden oscillations and hidden attractors — basin of attraction does not contain neighborhoods of equilibria
Leonov G.A., Kuznetsov N.V., Vagaitsev V.I, Localization of hidden Chua's attractors, *Phys. Lett. A*, 2011, 375, 2230-2233

Chua system

$$\dot{x} = \alpha(y - x - m_1 x - \psi(x))$$
$$\dot{y} = x - y + z,$$
$$\dot{z} = -(\beta y + \gamma z)$$
$$\psi(x) = (m_0 - m_1)\mathrm{sat}(x)$$
$$\alpha = 8.4562\ \beta = 12.0732\ \gamma = 0.0052$$
$$m_0 = -0.1768, m_1 = -1.1468$$



Stable zero eqv. and 2 symmetric saddles: trajectories "from" saddles tend to zero eqv. or to infinity: black and red
Hidden attractor(green)

Harmonic Balance & Describing Function Method

Describing function method (DFM) can lead to untrue results: no periodic solution for Aizerman's or Kalman's conditions by DFM

$$\dot{\mathbf{x}} = \mathbf{P}\mathbf{x} + \mathbf{q}\psi(\mathbf{r}^*\mathbf{x}), \ \ \psi(0) = 0 \quad (1)$$

$$W(p) = \mathbf{r}^*(\mathbf{P} - p\mathbf{I})^{-1}\mathbf{q}$$

$$\mathrm{Im}W(i\omega) = 0$$

$$k = -(\mathrm{Re}W(i\omega))^{-1}$$

$$\dot{\mathbf{x}} = \mathbf{P}_0\mathbf{x} + \mathbf{q}\varphi(\mathbf{r}^*\mathbf{x})$$

$$\mathbf{P_0} = \mathbf{P} + k\mathbf{q}\mathbf{r}^*$$

$$\varphi(\sigma) = \psi(\sigma) - k\sigma$$

$$\mathbf{P}_0 : \lambda_{1,2} = \pm i\omega, \ \ \mathrm{Re}\lambda_{j>2} < 0$$

DFM: exists periodic solution $\sigma(t) = \mathbf{r}^*\mathbf{x}(t) \approx a\cos\omega t$

$$a: \ \int_0^{2\pi/\omega} \psi(a\cos\omega t)\cos\omega t \mathrm{d}t = ka\int_0^{2\pi/\omega}(\cos\omega t)^2 \mathrm{d}t$$

Aizerman: if $\dot{\mathbf{z}} = (\mathbf{A} + \mu\mathbf{b}\mathbf{c}^*)\mathbf{z}$, is asympt. stable $\forall\, \mu \in (\mu_1, \mu_2)$ then $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\varphi(\sigma)$, $\sigma = \mathbf{c}^*\mathbf{x}$, $\varphi(0) = 0$, $\mu_1 < \varphi(\sigma)/\sigma < \mu_2$, all $\mathbf{x}(t, \mathbf{x}_0) \to 0$

DFM: since (1) is stable in sector $\mu_1 < \varphi(\sigma)/\sigma < \mu_2$ due to Aizerman

$\Rightarrow$ k from DFM $: k < \mu_1, \mu_2 < k \Rightarrow k\sigma^2 < \psi(\sigma)\sigma, \psi(\sigma)\sigma < k\sigma^2$

$\Rightarrow \forall a \neq 0 : \ \int_0^{2\pi/\omega}(\psi(a\cos\omega t)\, a\cos\omega t - k(a\,\cos\omega t)^2)\mathrm{d}t \neq 0$

$\Rightarrow$ no periodic solutions by DFM, but counterexamples are well known

Justification of harmonic balance and DFM

$\dot{x} = Px + q\varphi_\varepsilon(r^*x), \ x \in \mathbb{R}^n$

Eigs(P): $\lambda_{1,2} \pm i\omega_0, \ \lambda_{j>2}(\mathrm{Re}\,\lambda_{j>2} < 0)$

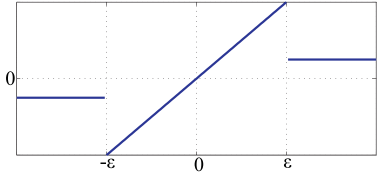
$\Rightarrow \exists x = Sy$:

$$\dot{y}_1 = -\omega_0 y_1 + b_1\varphi_\varepsilon(y_1 + c^* y_3)$$
$$\dot{y}_2 = +\omega_0 y_1 + b_2\varphi_\varepsilon(y_1 + c^* y_3)$$
$$\dot{y}_3 = Ay_3 + b\varphi_\varepsilon(y_1 + c^* y_3)$$

A–stable $(n-2)\times(n-2)$-matrix

b, c — $(n-2)$-vectors.

$$\varphi_\varepsilon(\sigma) = \begin{cases} \mu\sigma, & \forall|\sigma| \le \varepsilon \\ \mathrm{sign}(\sigma) M\varepsilon^3, & \forall|\sigma| > \varepsilon \end{cases}$$



Stab. sector: $\mu\sigma \geqslant \varphi_\varepsilon \geqslant 0$

Theorem. If $b_1 < 0$ and $0 < (\mu b_2\omega_0(c^*b + b_1) + b_1\omega_0^2)$ then for suff. small ε the system has a stable periodic solution with the initial data

$$y_1(t) = -\sin(\omega_0 t) y_2(0) + O(\varepsilon), \ y_2(t) = \cos(\omega_0 t) y_2(0) + O(\varepsilon), \ y_3(t) = O(\varepsilon)$$

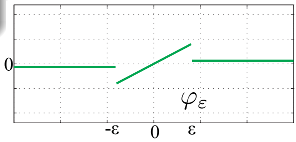
$$y_1(0) = O(\varepsilon^2), \ y_2(0) = -\sqrt{\frac{\mu(\mu b_2\omega_0(c^*b + b_1) + b_1\omega_0^2)}{-3\omega_0^2 M b_1}} + O(\varepsilon), \ y_3(0) = O(\varepsilon^2)$$

Analytical-numerical algorithm: periodic solution localization

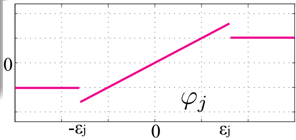
Thm. $P_0 = P - kqr^* : \lambda_{1,2} = \pm i\omega_0, \operatorname{Re}\lambda_{j>2} < 0$, for small ε $\exists$ periodic sol: $x_0 = Sy(0), x^\varepsilon(0, x_0) = x^\varepsilon(T, x_0)$

$$(1)\ \dot{x}^\varepsilon = P_0 x^\varepsilon + q\varphi_\varepsilon(r^* x^\varepsilon)$$

$$\varphi_\varepsilon(\sigma) = \begin{cases} \mu\sigma, & \forall|\sigma| \le \varepsilon \\ \operatorname{sign}(\sigma) M\varepsilon^3, & \forall|\sigma| > \varepsilon \end{cases}$$

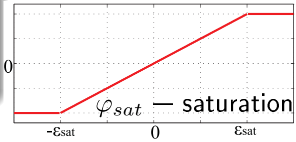


Multistep numerical proc.: $x^1(0) = Sy(0)$ by Thm
ε_1 small $\approx \varepsilon, \varphi_1(\sigma) \approx \varphi_\varepsilon(\sigma), x^1(x_0, t) \approx x^\varepsilon(x_0, t)$
$\varepsilon_j = (j/m)\sqrt{\mu/M},\ x^{j+1}(0) = x^j(T)$



$$(2)\quad \dot{x}^j = P_0 x_j + q\varphi_j(r^* x^j),\ \text{step } j = 1, \dots, m-1$$

Counterexample to Aizerman's conj: $\exists$ periodic solution in system (3) with continuous $\varphi(\sigma)$ from stable sector from the sector of stability



$$(3)\ \dot{x} = P_0 x + q\varphi_{sat}(r^* x), \quad j = m$$

$$\varepsilon_m = \varepsilon_{sat} = \sqrt{\mu/M}, \varphi_m(\sigma) = \varphi_{sat}(\sigma)$$

Analytical-numerical algorithm: stability of procedure

Harmonic linearization $P_0 = P - kqr^* : \lambda_{1,2} = \pm i\omega_0, \mathrm{Re}\,\lambda_{j>2} < 0$

(1) $\dot{x}^\varepsilon = P_0 x^\varepsilon + q\varphi_\varepsilon(r^* x^\varepsilon)$

$! \ \exists\, x^\varepsilon(0, x_0) = x^\varepsilon(T, x_0), x_0 = Sy(0)$ by Thm

(2) $\dot{x}^j = P_0 x^j + q\varphi_j(r^* x^j)$, $j = 1, ..., m$

$\varepsilon_1 \approx \varepsilon, \ \varphi_1 \approx \varphi_\varepsilon, \ x^1(x_0, t) \approx x^\varepsilon(x_0, t)$

$\varepsilon_j = (j/m)\sqrt{\mu/M}, \ x^{j+1}(0) = x^j(T)$

$\varepsilon_m = \varepsilon_{sat} = \sqrt{\mu/M}, \varphi_m(\sigma) = \varphi_{sat}(\sigma)$

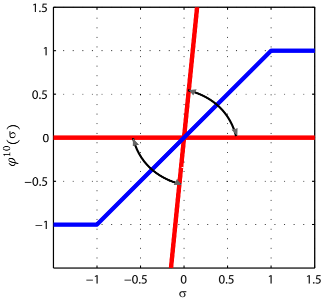
(3) $\dot{x} = P_0 x + q\varphi_{sat}(r^* x)$

$? \ \ \exists\, x(0) = x(T)$

1) If periodic solution $x^\varepsilon(t)$ is in the domain of attraction of the periodic solution $x^1(t)$ of the system with $\varphi_1(\sigma)$, then a solution of system with φ_1 can be started from $x^\varepsilon(0)$ and after transient process the computational procedure "reaches" the periodic solution $x^1(t)$. (Integration interval $[0, T]$ must be large). Otherwise, the instability bifurcation destroying periodic solution occurs and the algorithm stops.

2) Further compute the periodic solution $x^2(t)$, making use of the solution of system with $\varphi_2(\sigma)$ with the initial data $x^2(0) = x^1(T)$. And so on up to $x^m(t) = x(t)$ of system with $\varphi_m(\sigma) = \varphi_{sat}(\sigma)$. At a certain step the instability bifurcation destroying periodic solution can occur and the algorithm stops.

$$\dot{x}_1 = -x_2 - 10\,\varphi(\sigma)$$
$$\dot{x}_2 = x_1 - 10.1\,\varphi(\sigma)$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -x_3 - x_4 + \varphi(\sigma)$$
$$\sigma = x_1 - 10.1\,x_3 - 0.1\,x_4$$



Thm: $\varphi(\sigma)=\varphi^0(\sigma)\ \exists$ periodic solution with $x_1(0)=x_3(0)=x_4(0)=0, x_2(0)=-1.7513$

Aizerman's conjecture: $0 \leqslant \varphi^j(\sigma) \leqslant 1$,

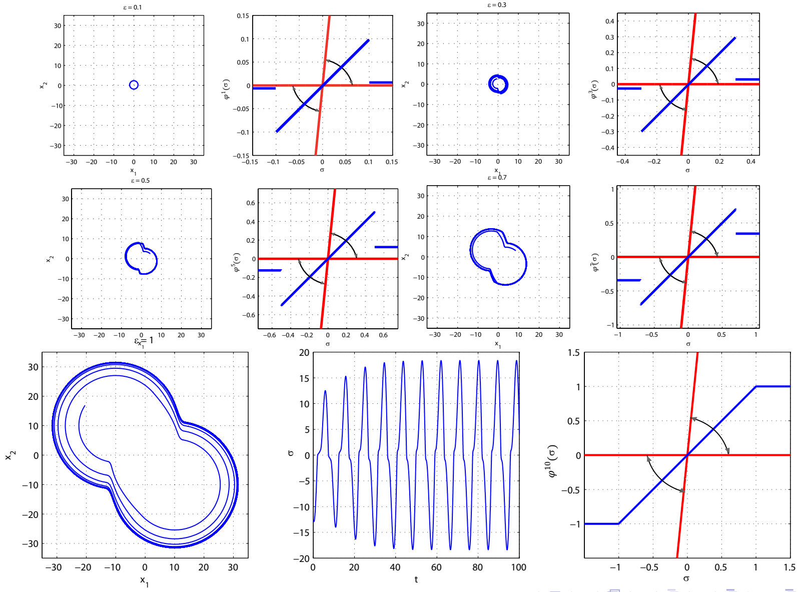
$$\varphi^j(\sigma)=\begin{cases}\sigma, & |\sigma| \le \varepsilon_j;\\ \operatorname{sign}(\sigma)\varepsilon_j^3, & |\sigma| > \varepsilon_j\end{cases} \quad \varepsilon_j = 0.1, ..., 1, \quad \varphi^{10}(\sigma) = \operatorname{sat}(\sigma)$$

Kalman's conjecture: $iN \leqslant \psi^{i\prime}(\sigma) \leqslant 1 \qquad 0 < \frac{d}{d\sigma}\tanh(\sigma) \le 1$

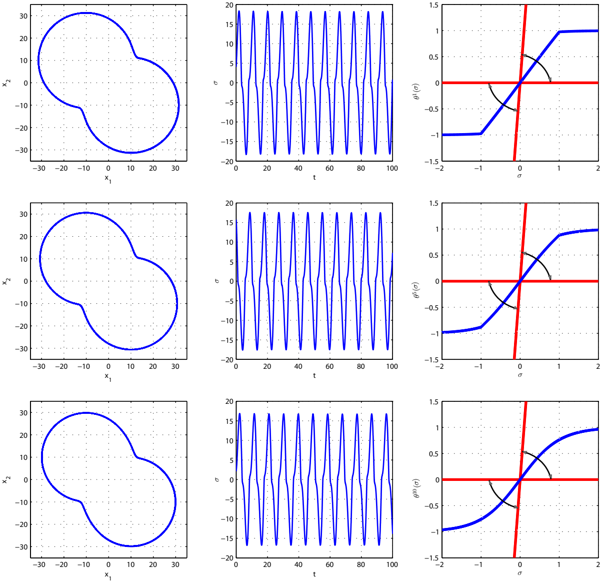
$$\psi^i(\sigma)=\begin{cases}\sigma, & |\sigma| \le 1;\\ \operatorname{sign}(\sigma) + i(\sigma - \operatorname{sign}(\sigma))N, & |\sigma| > 1\end{cases} \quad N = 0.01, i = 1, .., 5$$

$$\theta^i(\sigma)=\operatorname{sat}(\sigma)+i(\tanh(\sigma)-\operatorname{sat}(\sigma))/10 \quad i = 1, .., 10 \quad \theta^{10}(\sigma) = \tanh(\sigma)$$

Counterexample to Aizerman conjecture



Smooth counterexample to Kalman conjecture



$$\dot{x}_1 = -x_2 - 10\,\varphi(\sigma)$$
$$\dot{x}_2 = x_1 - 10.1\,\varphi(\sigma)$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -x_3 - x_4 + \varphi(\sigma)$$
$$\sigma = x_1 - 10.1\,x_3 - 0.1\,x_4$$

$$\varphi(\sigma) = \theta^i(\sigma) = \mathrm{sat}(\sigma) + i(\tanh(\sigma) - \mathrm{sat}(\sigma))/10$$

$i = 1, .., 10$

$$\tanh(\sigma) = \frac{e^{\sigma} - e^{-\sigma}}{e^{\sigma} + e^{-\sigma}}$$

Smooth counterexample to Kalman conj-re (i=10): $\left(0 < \frac{d}{d\sigma}\tanh(\sigma) \le 1\right)$, periodic solution exists, linear system is stable.

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Lyapunov exponent: sign inversions, Perron effects, chaos, linearization

$$\begin{cases} \dot{x} = F(x), \ x \in \mathbb{R}^n, \quad F(x_0) = 0 \\ x(t) \equiv x_0, A = \frac{dF(x)}{dx}\Big|_{x=x_0} \end{cases} \quad \begin{cases} \dot{y} = Ay + o(y) \\ y(t) \equiv 0, (y=x-x_0) \end{cases} \quad \begin{cases} \dot{z} = Az \\ z(t) \equiv 0 \end{cases}$$

✓stationary: $z(t) = 0$ is exp. stable $\Rightarrow$ $y(t) = 0$ is asympt. stable

$$\begin{cases} \dot{x} = F(x), \quad \dot{x}(t) = F(x(t)) \not\equiv 0 \\ x(t) \not\equiv x_0, A(t) = \frac{dF(x)}{dx}\Big|_{x=x(t)} \end{cases} \quad \begin{cases} \dot{y} = A(t)y + o(y) \\ y(t) \equiv 0, (y=x-x(t)) \end{cases} \begin{cases} \dot{z} = A(t)z \\ z(t) \equiv 0 \end{cases}$$

? nonstationary: $z(t) = 0$ is exp. stable $\Rightarrow ?$ $y(t) = 0$ is asympt. stable

! Perron effects: $z(t)=0$ is exp. stable(unst), $y(t)=0$ is exp. unstable(st)

Positive largest Lyapunov exponent
doesn't, in general, indicate chaos

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