Hidden attractor localization in Chua's circuit: analytical-numerical procedure based on describing function method, small parameter and continuation methods

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http://www.math.spbu.ru/user/nk/

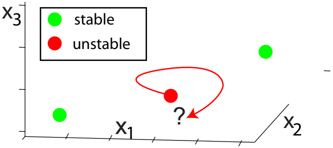
http://www.math.spbu.ru/user/nk/Hidden_oscillation_Attractor_Localization.htm

tutorial last version:

http://www.math.spbu.ru/user/nk/PDF/Hidden-attractor-localization-Chua-circuit.pdf

Classical approach to computation of oscillations

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, .., x_n) \\ f_2(x_1, x_2, .., x_n) \\ \dots \\ f_n(x_1, x_2, .., x_n) \end{pmatrix} = \mathbf{F}(\mathbf{x})$$



1. Find equilibria $\mathbf{x}^k \ : \ \mathbf{F}(\mathbf{x}^k) = 0$

local stability analysis $\dot{\mathbf{y}} = \frac{\mathrm{d}\mathbf{F}(\mathbf{x})}{\mathrm{d}\mathbf{x}}\Big|_{\mathbf{x}=\mathbf{x}^k} \mathbf{y}$

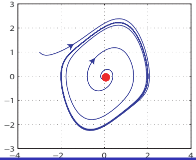
2. Numerical simulation of trajectories with initial data from neighborhoods of unstable equilibria.

self-excited attractor localization: *standard computational procedure* is 1) to find equilibria; 2) after transient process trajectory, starting from a point of unstable manifold in a neighborhood of unstable equilibrium, reaches an self-excited oscillation and localizes it:

Van der Pol

$\dot{x} = y$

$\dot{y} = -x + \varepsilon(1 - x^2)y$

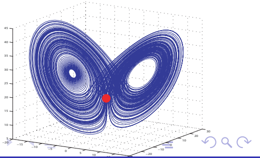


Lorenz

$x = -\sigma(x - y)$

$y = rx - y - xz$

$z = -bz + xy$

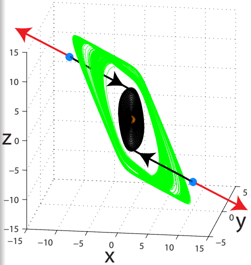


Hidden oscillations: how to compute?

hidden oscillations and hidden attractors — basin of attraction does not contain neighborhoods of equilibria

Leonov G.A., Kuznetsov N.V., Vagaitsev V.I, Localization of hidden Chua's attractors, *Phys. Lett. A*, 2011, 375, 2230-2233

✓standard computation (*trajectories from a neighborhood of unstable equilibrium reaches and identifies attractor*) does not work (all equilibria are stable or not in the basin).



The problem is: How to choose initial data in the basin of attraction of hidden oscillation? Localization is required a special analytical-numerical procedure.

Survey: Leonov G.A., Kuznetsov N.V., Hidden attractors in dynamical systems. From hidden oscillations in Hilbert-Kolmogorov, Aizerman, and Kalman problems to hidden chaotic attractor in Chua circuits, International Journal of Bifurcation and Chaos, 23(1), 2013, art. no. 1330002

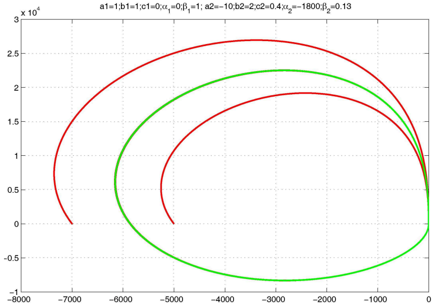
Hidden oscillations 2d: 16th Hilbert problem (second part)

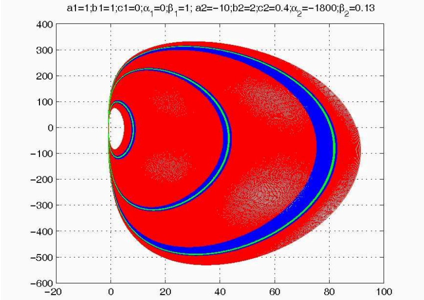
D.Hilbert 1900: number & mutual disposition of limit cycles

$$\dot{x} = P_n(x,y) = a_1x^2 + b_1xy + c_1y^2 + \alpha_1 x + \beta_1 y + ...$$
$$\dot{y} = Q_n(x,y) = a_2x^2 + b_2xy + c_2y^2 + \alpha_2 x + \beta_2 y + ...$$

N.Bautin 1949: first nontrivial results: 3 nested limit cycles in case $n = 2$ (internal limit cycles are hidden oscillations)

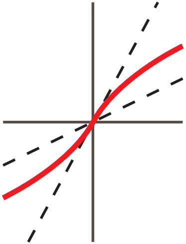
N.V. Kuznetsov, O.A. Kuznetsova, G.A. Leonov, Visualization of four normal size limit cycles in two-dimensional polynomial quadratic system, Differential equations and Dynamical systems, 21(1-2), 2013, 29-34 (doi:10.1007/s12591-012-0118-6)





Hidden oscillations 3d: Aizerman and Kalman conjectures

if $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{b}k\mathbf{c}^*\mathbf{z}$, is asympt. stable $\forall\, k \in (k_1, k_2)$: $\forall \mathbf{z}(t, \mathbf{z}_0) \to 0$, then
is $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\varphi(\sigma)$, $\sigma = \mathbf{c}^*\mathbf{x}, \varphi(0) = 0$, $k_1 < \varphi(\sigma)/\sigma < k_2$: $\forall \mathbf{x}(t, \mathbf{x}_0) \to 0$?



$1949: \; k_1 < \varphi(\sigma)/\sigma < k_2$ $\qquad$ $1957: \; k_1 < \varphi'(\sigma) < k_2$

In general, conjectures are not true(Aizerman's: $n \geq 2$, Kalman's: $n \geq 4$)
Counterexamples with hidden oscillations can be constructed:
stable periodic solution coexist with unique stable equilibrium

Survey: V.O. Bragin, V.I. Vagaitsev, N.V. Kuznetsov, G.A. Leonov (2011) Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman conjectures and Chua's circuits, *J. of Computer and Systems Sciences Int.*, 50(4), 511-544 (doi:10.1134/S106423071104006X)

Analytical-numerical algorithm: oscillation localization

$\dot{\mathbf{x}} = \mathbf{P}\mathbf{x} + \boldsymbol{\psi}(\mathbf{x}),\ \boldsymbol{\psi}(0) = 0$ $\quad \dot{\mathbf{x}} = \mathbf{P}_0\mathbf{x} + \boldsymbol{\varphi}(\mathbf{x})$ $\quad \dot{y}_1 = -\omega y_2 + \varepsilon\phi_1(\mathbf{y})$

$\mathbf{P}_0 = \mathbf{P} + \mathbf{K}$ $\quad \boldsymbol{\varphi}(\mathbf{x}) = \boldsymbol{\psi}(\mathbf{x}) - \mathbf{K}\mathbf{x}$ $\quad \dot{y}_2 = \omega y_1 + \varepsilon\phi_2(\mathbf{y})$

$\lambda_{1,2}^{\mathbf{P}_0} = \pm i\omega$, $\mathrm{Re}\lambda_{j>2}^{\mathbf{P}_0} < 0$ $\quad \mathbf{x} = \mathbf{S}\mathbf{y}, \boldsymbol{\phi} = \mathbf{S}^{-1}\boldsymbol{\varphi}$ $\quad \dot{\mathbf{y}}_3 = \mathbf{A}_3\mathbf{y}_3 + \varepsilon\boldsymbol{\phi}_3(\mathbf{y})$

Harmonic linearization, linear transformation, small parameter method:

$y_1 = \cos(\omega t)y_1(0) + O(\varepsilon), y_2 = \sin(\omega t)y_1(0) + O(\varepsilon), \mathbf{y}_3 = e^{\mathbf{A}_3 t}\mathbf{y}_3(0) + \mathbf{O}_{\mathbf{n-2}}(\varepsilon)$

Analytically: initial data for periodic solution

(1) $\dot{\mathbf{x}}^\varepsilon = \mathbf{P}_0\mathbf{x}^\varepsilon + \varepsilon\boldsymbol{\varphi}(\mathbf{x}^\varepsilon)$, $\quad \mathbf{P}_0 : \lambda_{1,2} = \pm i\omega, \mathrm{Re}\lambda_{j>2} < 0$,

for suff. small ε exists periodic solution $\mathbf{x}^\varepsilon(0, \mathbf{x}_0^\varepsilon) = \mathbf{x}^\varepsilon(T, \mathbf{x}_0^\varepsilon)$, $\mathbf{x}_0^\varepsilon = \mathbf{S}\mathbf{y}(0)$

Multistep numerical procedure: $\mathbf{x}^1(0) \approx \mathbf{x}_0^\varepsilon = \mathbf{S}\mathbf{y}(0)$ (by analytics)

(2) $\dot{\mathbf{x}}^j = \mathbf{P}_0\mathbf{x}^j + \varepsilon_j\boldsymbol{\varphi}(\mathbf{x}^j)$, $\quad$ step $j = 1, ..., m-1$

1st numerical step: ε_1 small $\approx \varepsilon$, $\varepsilon_1\boldsymbol{\varphi} \approx \varepsilon\boldsymbol{\varphi}$, $\mathbf{x}^1(\mathbf{x}_0^1, t) \approx \mathbf{x}^\varepsilon(\mathbf{x}_0^\varepsilon, t)$

next numerical steps: $\varepsilon_j = (j/m)$, $\mathbf{x}^{j+1}(0) = \mathbf{x}^j(T)$

Hidden oscillation computation in initial system: last step $j = m, \varepsilon_m = 1$

(3) $\dot{\mathbf{x}} = \dot{\mathbf{x}}^m = \mathbf{P}_0\mathbf{x}_m + \varepsilon_m\boldsymbol{\varphi}(\mathbf{x}^m) = \mathbf{P}_0\mathbf{x} + \boldsymbol{\varphi}(\mathbf{x})$,

Hidden oscillation localization: analytical-numerical procedure

(1) $\dot{\mathbf{x}} = \mathbf{P}_0\mathbf{x} + \varphi(\mathbf{x})$ (2) $\dot{\mathbf{x}} = \mathbf{P}_0\mathbf{x} + \varepsilon\boldsymbol{\varphi}(\mathbf{x})$ (3) $\dot{\mathbf{x}}^j = \mathbf{P}_0\mathbf{x}^j + \varepsilon_j\boldsymbol{\varphi}(\mathbf{x}^j)$

Small ε allows one to justify rigourously Describing Function Method (DFM) for (2) & to determine a stable nontrivial periodic solution $\mathbf{x}^0(t)$ — *oscillating attractor* $\mathcal{A}_0$.

Localization of attractor $\mathcal{A}$ in (1): numerically follow transformation of $\mathcal{A}_j$ with increasing $j=0,..,m$ ($\varepsilon_m = 1, \mathcal{A}_m = \mathcal{A}$) [special continuation method]

1. if all points of $\mathcal{A}_0$ are in the attraction domain of $\mathcal{A}_1$ (oscillating attractor of (3) with $j = 1$), then solution $\mathbf{x}^1(t)$ can be determined numerically by starting a trajectory of (3) with $j=1$ from initial point $\mathbf{x}^0(0)$. If in computational process $\mathbf{x}^1(t)$ is not fallen to equilibria and is not $\to \infty$ (on suff. large $[0, T]$), then $\mathbf{x}^1(t)$ computes attractor $\mathcal{A}_1$. Then perform similar procedure for (3) with $j=2$: by starting trajectory $\mathbf{x}^2(t)$ of (3) with $j = 2$ from init. point $\mathbf{x}^1(T)$ (last point on previous step) we compute $\mathcal{A}_2$. And so on.

2. in the change from system (2) to (3) with $j=1$, it's observed loss of stability bifurcation and vanishing of attractor $\mathcal{A}_0$(or $\mathcal{A}_{j-1}$ on j-th step).

Harmonic Balance & Describing Function Method

Describing function method (DFM) can lead to untrue results:
no periodic solution for Aizerman's or Kalman's conditions by DFM

$$\dot{\mathbf{x}} = \mathbf{P}\mathbf{x} + \mathbf{q}\psi(\mathbf{r}^*\mathbf{x}), \;\; \psi(0) = 0 \quad (1)$$

$$W(p) = \mathbf{r}^*(\mathbf{P} - p\mathbf{I})^{-1}\mathbf{q}$$

$$\mathrm{Im}W(i\omega) = 0$$

$$k = -(\mathrm{Re}W(i\omega))^{-1}$$

$$\dot{\mathbf{x}} = \mathbf{P}_0\mathbf{x} + \mathbf{q}\varphi(\mathbf{r}^*\mathbf{x})$$

$$\mathbf{P_0} = \mathbf{P} + k\mathbf{q}\mathbf{r}^*$$

$$\varphi(\sigma) = \psi(\sigma) - k\sigma$$

$$\mathbf{P}_0 : \lambda_{1,2} = \pm i\omega, \;\; \mathrm{Re}\lambda_{j>2} < 0$$

DFM: exists periodic solution $\sigma(t) = \mathbf{r}^*\mathbf{x}(t) \approx a\cos\omega t$

$$a : \;\; \int_0^{2\pi/\omega} \psi(a\cos\omega t)\cos\omega t \mathrm{d}t = ka \int_0^{2\pi/\omega} (\cos\omega t)^2 \mathrm{d}t$$

Aizerman: if $\dot{\mathbf{z}} = (\mathbf{A} + \mu\mathbf{b}\mathbf{c}^*)\mathbf{z}$, is asympt. stable $\forall\, \mu \in (\mu_1, \mu_2)$ then
$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\varphi(\sigma), \; \sigma = \mathbf{c}^*\mathbf{x}, \varphi(0) = 0, \;\; \mu_1 < \varphi(\sigma)/\sigma < \mu_2,$ all $\mathbf{x}(t, \mathbf{x}_0) \to 0$

DFM: since (1) is stable in sector $\mu_1 < \varphi(\sigma)/\sigma < \mu_2$ due to Aizerman
$\Rightarrow k$ from DFM $: k < \mu_1, \mu_2 < k \Rightarrow k\sigma^2 < \psi(\sigma)\sigma, \psi(\sigma)\sigma < k\sigma^2$
$\Rightarrow \forall a \neq 0 : \;\; \int_0^{2\pi/\omega} (\psi(a\cos\omega t)\, a\cos\omega t - k(a\,\cos\omega t)^2)\mathrm{d}t \neq 0$
$\Rightarrow$ no periodic solutions by DFM, but counterexamples are well known

Periodic solution by justified DFM (scalar case)

$\dot{\mathbf{x}} = \mathbf{P}\mathbf{x} + \mathbf{q}\psi(\mathbf{r}^*\mathbf{x})$: $W(p) = \mathbf{r}^*(\mathbf{P} - p\mathbf{I})^{-1}\mathbf{q}$, $\mathrm{Im}W(i\omega) = 0$, $k = -(\mathrm{Re}W(i\omega))^{-1}$

$\dot{\mathbf{x}} = \mathbf{P}_0\mathbf{x} + \mathbf{q}\varphi(\mathbf{r}^*\mathbf{x})$: $\mathbf{P}_0 = \mathbf{P} + \mathbf{k}\mathbf{q}\mathbf{r}^*$, $\lambda_{1,2}^{\mathbf{P}_0} = \pm i\omega$, $\mathrm{Re}\lambda_{j>2}^{\mathbf{P}_0} < 0$, $\varphi = \psi - k\mathbf{r}^*\mathbf{x}$

$\mathbf{x} = \mathbf{S}\mathbf{y}$, $\mathbf{A} = \mathbf{S}^{-1}\mathbf{P_0}\mathbf{S}$, $\mathbf{b} = (b_1, b_2, \mathbf{b}_3)^* = \mathbf{S}^{-1}\mathbf{q}$, $\mathbf{c}^* = (1, 0, \mathbf{c}_3)^* = \mathbf{r}^*\mathbf{S}$

Harmonic linearization, linear transformation, small parameter method

$$\begin{aligned} \dot{y}_1 &= -\omega y_2 + b_1\varepsilon\varphi(y_1 + \mathbf{c}_3^*\mathbf{y}_3) \\ \dot{y}_2 &= \omega y_1 + b_2\varepsilon\varphi(y_1 + \mathbf{c}_3^*\mathbf{y}_3) \\ \dot{\mathbf{y}}_3 &= \mathbf{A}_3\mathbf{y}_3 + \mathbf{b}_3\varepsilon\varphi(y_1 + \mathbf{c}_3^*\mathbf{y}_3) \\ \mathbf{y}_3^*(\mathbf{A}_3 + \mathbf{A}_3^*)\mathbf{y}_3 &\le -2d|\mathbf{y}_3|^2 \end{aligned} \qquad \begin{aligned} y_1(t) &= \cos(\omega t)y_1(0) + O(\varepsilon) \\ y_2(t) &= \sin(\omega t)y_1(0) + O(\varepsilon) \\ \mathbf{y}_3(t) &= e^{\mathbf{A}_3 t}\mathbf{y}_3(0) + \mathbf{O}_{\mathbf{n-2}}(\varepsilon) \\ t &\in (0, T] \end{aligned}$$

$$\mathbf{y}(0) \in \Omega = \{y_1 \in [a_1, a_2], y_2 = 0, |\mathbf{y}_3| \le D\varepsilon\}, \mathbf{F}\mathbf{y}(0) = \mathbf{y}(T), T = \frac{2\pi}{\omega} + O(\varepsilon)$$

Theorem. If exists $a_0 > 0$: $\Phi(a_0) = \int\limits_0^{2\pi/\omega} \varphi\big(\cos(\omega t)a_0\big)\cos(\omega t)dt = 0$

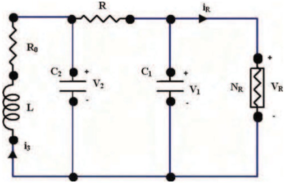
and $b_1 \frac{d\Phi(a)}{da}\Big|_{a=a_0} < 0$ then exists periodic solution $\mathbf{x}(t) = \mathbf{S}\mathbf{y}(t)$ with initial data $\mathbf{x}(0) = \mathbf{S}(a_0 + O(\varepsilon), 0, \mathbf{O}_{\mathbf{n-2}}(\varepsilon))^*$.

For all solutions with initial data suff. close to $\mathbf{x}(0)$ the modulus of their difference with $\mathbf{x}(t)$ is uniformly bounded $\forall t > 0$).

Attractors in Chua's circuits

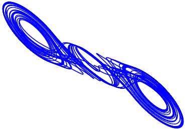
L.Chua (1983)

$$\dot{x} = \alpha\big(y - x - f(x)\big),$$
$$\dot{y} = x - y + z,$$
$$\dot{z} = -(\beta y + \gamma z),$$



$$f(x) = m_1 x + \text{sat}(x) = m_1 x + \frac{1}{2}(m_0 - m_1)(|x+1| + |x-1|)$$

Chua circuit is used in chaotic communications



1983–now: computations of Chua self-excited attractors by standard procedure: trajectory from neighborhood of unstable zero equilibrium reaches & identifies attractor. [*Bilotta&Pantano, A gallery of Chua attractors, WorldSci. 2008*]

Could an attractor exists and how lo localize it, if equilibrium is stable?

L.Chua, 1990: If zero equilibrium is stable ⇒ no chaotic attractor

Chua system: transformation to harmonic form

$\dot{\mathbf{x}}=\mathbf{P}\mathbf{x}+\mathbf{q}\psi(\mathbf{r}^*\mathbf{x})$: $W(p)=\mathbf{r}^*(\mathbf{P}-p\mathbf{I})^{-1}\mathbf{q}$, $\mathrm{Im}\,W(i\omega)=0$, $k=-(\mathrm{Re}\,W(i\omega))^{-1}$

$\dot{\mathbf{x}}=\mathbf{P}_0\mathbf{x}+\mathbf{q}\varphi(\mathbf{r}^*\mathbf{x})$: $\mathbf{P}_0=\mathbf{P}+\mathbf{k}\mathbf{q}\mathbf{r}^*$, $\lambda_{1,2}^{\mathbf{P}_0}=\pm i\omega$, $\mathrm{Re}\lambda_{j>2}^{\mathbf{P}_0}<0$, $\varphi=\psi-k\mathbf{r}^*\mathbf{x}$

$\mathbf{x}=\mathbf{S}\mathbf{y}$, $\mathbf{A}=\mathbf{S}^{-1}\mathbf{P_0}\mathbf{S}$, $\mathbf{b}=(b_1,b_2,\mathbf{b}_3)^*=\mathbf{S}^{-1}\mathbf{q}$, $\mathbf{c}^*=(1,0,\mathbf{c}_3)^*=\mathbf{r}^*\mathbf{S}$

$$\dot{x}=\alpha(y-x-m_1x-\psi(x)),\ \dot{y}=x-y+z,\ \dot{z}=-(\beta y+\gamma z),\ \psi(x)=(m_0-m_1)\mathrm{sat}(x)$$

$$\mathbf{P}_0\mathbf{q}\mathbf{r}=\begin{pmatrix}-\alpha(m_1+1+k) & \alpha & 0\\ 1 & -1 & 1\\ 0 & -\beta & -\gamma\end{pmatrix}\begin{pmatrix}-\alpha\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\rightarrow\mathbf{A}\mathbf{b}\mathbf{c}=\begin{pmatrix}0 & -\omega & 0\\ \omega & 0 & 0\\ 0 & 0 & -d\end{pmatrix}\begin{pmatrix}b_1\\b_2\\1\end{pmatrix}\begin{pmatrix}1\\0\\-h\end{pmatrix}$$

$$W_{\mathbf{P}_0}(p)=\mathbf{r}^*(\mathbf{P}_0-p\mathbf{I})^{-1}\mathbf{q}\qquad\qquad \dot{\mathbf{y}}=\mathbf{A}\mathbf{y}+\varepsilon\mathbf{b}\varphi(\mathbf{c}^*\mathbf{y})$$

$$\mathbf{S}=\begin{pmatrix}s_{11} & s_{12} & s_{13}\\ s_{21} & s_{22} & s_{23}\\ s_{31} & s_{32} & s_{33}\end{pmatrix}\qquad s_{11}=1,\quad s_{21}=1+k,\quad s_{31}=\frac{k\alpha-\omega^2}{\alpha}$$

$$W_{\mathbf{P}_0}=W_{\mathbf{A}}=\frac{-b_1p+b_2\omega}{p^2+\omega^2}+\frac{h}{p+d}$$

$$k=\frac{-\alpha\gamma+\omega^2-\gamma-\beta}{\alpha(1+\gamma)},\ d=\frac{\alpha+\omega^2-\beta+1+\gamma+\gamma^2}{1+\gamma}$$

$$h=\frac{\alpha(\gamma+\beta-(1+\gamma)d+d^2)}{\omega^2+d^2}$$

$$b_1=\frac{\alpha(\gamma+\beta-\omega^2-(1+\gamma)d)}{\omega^2+d^2}$$

$$b_2=\frac{\alpha\Big((1+\gamma-d)\omega^2+(\gamma+\beta)d\Big)}{\omega(\omega^2+d^2)}$$

Thm: $\mathbf{x}(0)=\mathbf{S}\mathbf{y}(0)=\mathbf{S}\begin{pmatrix}a_0\\0\\0\end{pmatrix}=\begin{pmatrix}a_0s_{11}\\a_0s_{21}\\a_0s_{31}\end{pmatrix}$

$a_0 : \Phi(a_0)=0$

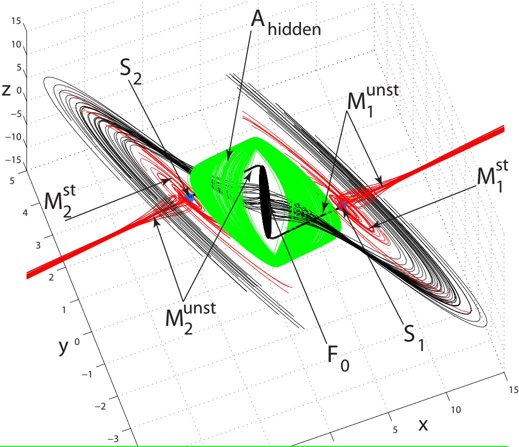
Hidden attractor in Chua's circuit

In 2009-2010 the notion of *hidden attractor* was introduced
and hidden chaotic attractor was found in Chua circuit for the first time:

$$\dot{x} = \alpha(y - x - m_1 x - \psi(x))$$
$$\dot{y} = x - y + z, \dot{z} = -(\beta y + \gamma z)$$
$$\psi(x) = (m_0 - m_1)\text{sat}(x)$$

$\alpha = 8.4562, \beta = 12.0732, \gamma = 0.0052$
$m_0 = -0.1768, m_1 = -1.1468$

Equilibria: stable zero F_0 & 2 saddles $S_{1,2}$
Trajectories: 'from' $S_{1,2}$ tend (black) to zero F_0 or tend (red) to infinity;
Hidden chaotic attractor (in green)
with positive Lyapunov exponent



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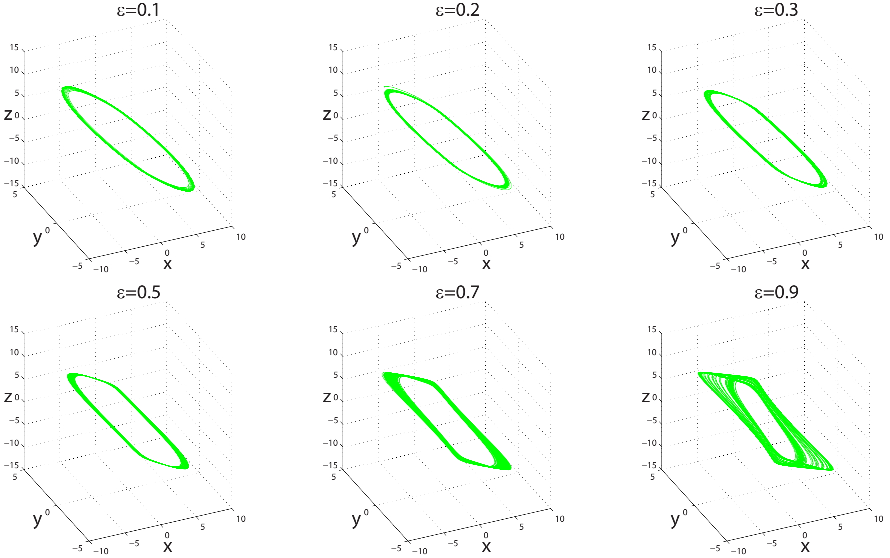
Scenario of transition to chaos: form periodic orbit to hidden chaotic attractor



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Lyapunov exponent : chaos, stability, Perron effects, linearization

$$\begin{cases} \dot{x} = F(x),\ x \in \mathbb{R}^n, \quad F(x_0) = 0 \\ x(t) \equiv x_0, A = \frac{dF(x)}{dx}\Big|_{x=x_0} \end{cases} \quad \begin{cases} \dot{y} = Ay + o(y) \\ y(t) \equiv 0, (y = x - x_0) \end{cases} \quad \begin{cases} \dot{z} = Az \\ z(t) \equiv 0 \end{cases}$$

✓stationary: $z(t) = 0$ is exp. stable $\Rightarrow y(t) = 0$ is asympt. stable

$$\begin{cases} \dot{x} = F(x), \quad \dot{x}(t) = F(x(t)) \not\equiv 0 \\ x(t) \not\equiv x_0, A(t) = \frac{dF(x)}{dx}\Big|_{x=x(t)} \end{cases} \quad \begin{cases} \dot{y} = A(t)y + o(y) \\ y(t) \equiv 0, (y = x - x(t)) \end{cases} \begin{cases} \dot{z} = A(t)z \\ z(t) \equiv 0 \end{cases}$$

? nonstationary: $z(t) = 0$ is exp. stable $\Rightarrow ?\ y(t) = 0$ is asympt. stable

! Perron effect: $z(t)=0$ is exp. stable(unst), $y(t)=0$ is exp. unstable(st)

Positive largest Lyapunov exponent doesn't, in general, indicate chaos

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