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Chapter 3
Analytical-Numerical Methods for Hidden Attractors' Localization: The 16th Hilbert Problem, Aizerman and Kalman Conjectures, and Chua Circuits

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Abstract This survey is devoted to analytical-numerical methods for hidden attractors' localization and their application to well-known problems and systems. From the computation point of view, in nonlinear dynamical systems the attractors can be regarded as *self-exciting* and *hidden attractors*. Self-exciting attractors can be localized numerically by the following *standard computational procedure*: after a transient process a trajectory, started from a point of an unstable manifold in a small neighborhood of unstable equilibrium, reaches an attractor and computes it. In contrast, a hidden attractor is an attractor whose basin of attraction does not contain neighborhoods of equilibria. In well-known Van der Pol, Belousov-Zhabotinsky, Lorenz, Chua, and many other dynamical systems classical attractors are self-exciting attractors and can be obtained numerically by the standard computational procedure. However, for localization of hidden attractors it is necessary to develop special analytical-numerical methods, in which at the first step the initial data are chosen in a basin of attraction and then the numerical localization (visualization) of the attractor is performed. The simplest examples of hidden attractors are internal nested limit cycles (hidden oscillations) in two-dimensional systems (see, e.g., the results concerning the second part of the *16th Hilbert's problem*). Other examples of hidden oscillations are counterexamples to *Aizerman's conjecture* and *Kalman's conjecture* on absolute stability in the automatic control theory (a unique stable equilibrium coexists with a stable periodic solution in these counterexamples). In 2010, for the first time, a *chaotic hidden attractor* was computed first by the authors in a *generalized Chua's circuit* and then one chaotic hidden attractor was discovered in a *classical Chua's circuit*.

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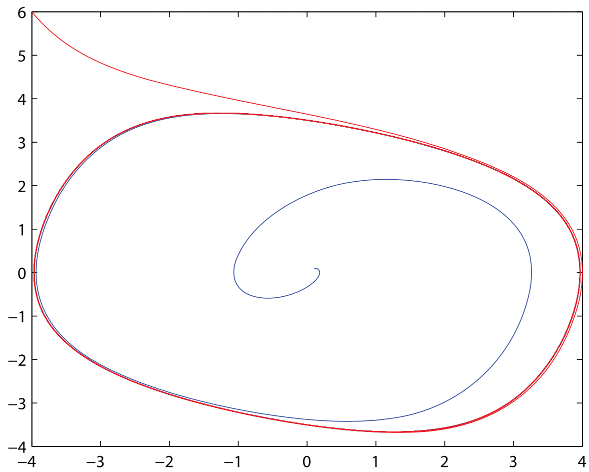

Fig. 3.1 Numerical localization of the limit cycle in the Rayleigh system

3.1 Introduction

In the first half of last century, during the initial period of development of the theory of nonlinear oscillations [2, 11, 32, 33], main attention has been given to analysis and synthesis of oscillating systems, for which the existence problem of oscillations can be solved relatively easily. The structure of many mechanical, electromechanical, and electronic systems is such that the existence of oscillations in them is almost obvious, namely the oscillations are excited from unstable equilibria. From the computational point of view it means that one can use a *standard numerical method*, in which after a transient process a trajectory, started from a point of an unstable manifold in a small neighborhood of equilibrium, reaches an attractor and identifies it.

Consider the following classical examples.

Example 3.1 (The Rayleigh string oscillator) In studying string oscillations [31] Rayleigh discovered first that in the two-dimensional nonlinear dynamical system

$$\ddot{x} - \big(a - b\dot{x}^2\big)\dot{x} + x = 0, \tag{3.1}$$

undamped vibrations (namely limit cycles—this term was introduced later by Poincare) can arise. A well-known generalization of this system is the Van der Pol equation [34] that describes the nonlinear electrical circuits used in radio engineering. The result of the simulation of this system (3.1) for $a = 1$, $b = 0.1$ is presented in Fig. 3.1.

Example 3.2 (The Belousov-Zhabotinsky (BZ) reaction) In 1951 B.P. Belousov discovered the first oscillations in the chemical reactions [3]. Consider one of the

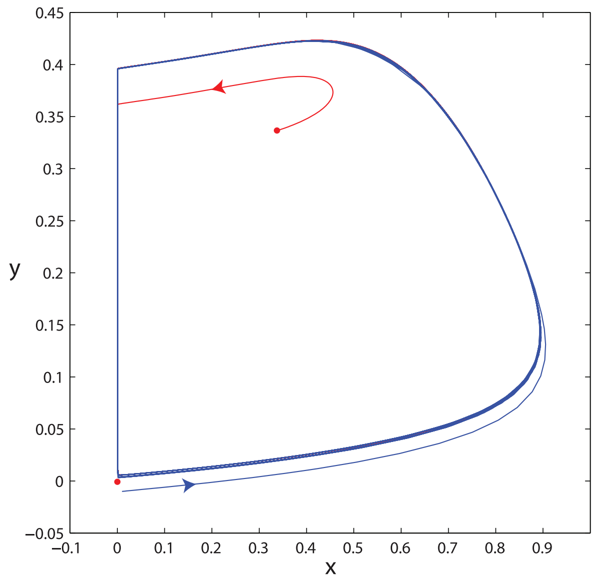

Fig. 3.2 Numerical localization of the limit cycle in the Belousov-Zhabotinsky model

Belousov-Zhabotinsky dynamical models

$$\begin{aligned}\varepsilon\dot{x} &= x(1-x) + \frac{f(q-x)}{q+x}z,\\ \dot{z} &= x - z,\end{aligned} \tag{3.2}$$

and perform its simulation, using standard parameters: $f = 2/3$, $q = 8 \times 10^{-4}$, $\varepsilon = 4 \times 10^{-2}$ (see Fig. 3.2).

Consider now the examples of numerical localization of well-known chaotic attractors in three-dimensional dynamical models.

Example 3.3 (The Lorenz system) Consider the Lorenz system [27]

$$\begin{aligned}\dot{x} &= \sigma(y-x),\\ \dot{y} &= x(\rho - z) - y,\\ \dot{z} &= xy - \beta z,\end{aligned} \tag{3.3}$$

and carry out its simulation with standard parameters $\sigma = 10$, $\beta = 8/3$, $\rho = 28$ (see Fig. 3.3). Here the computed trajectory is started from a small neighborhood of an unstable zero stationary point.

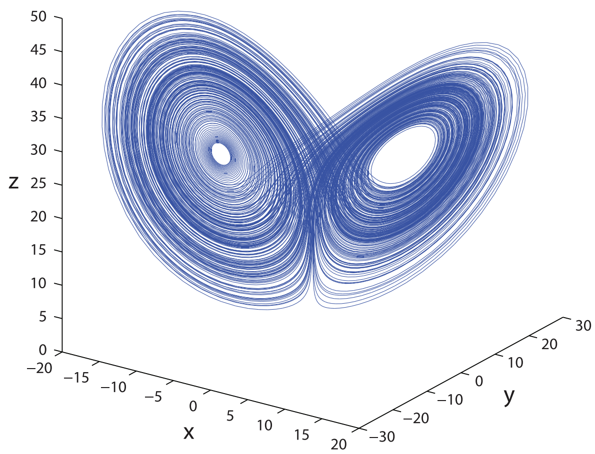

Fig. 3.3 Numerical localization of a chaotic attractor in the Lorenz system

Example 3.4 (The Chua system) Consider the classical Chua circuit [7] and its dynamical model in dimensionless coordinates

$$\begin{aligned}\dot{x} &= \alpha(y - x) - \alpha f(x),\\ \dot{y} &= x - y + z,\\ \dot{z} &= -(\beta y + \gamma z).\end{aligned} \tag{3.4}$$

Here the function

$$f(x) = m_1 x + (m_0 - m_1)\mathrm{sat}(x) \tag{3.5}$$

characterizes a nonlinear element called the Chua diode. In this system, strange attractors [29] then called the Chua attractors were discovered. To date all the known classical Chua attractors are those excited from unstable equilibria. This makes it possible to compute different Chua attractors with relative ease [5]. Perform the simulation of the Chua attractor with the following parameters: $\alpha = 9.35$, $\beta = 14.79$, $\gamma = 0.016$, $m_0 = -1.1384$, $m_1 = 0.7225$ (see Fig. 3.4).

Here, in all examples, the limit cycles and attractors are those excited from unstable equilibria (i.e., self-excited attractors).

3.2 Hidden Oscillations and Hidden Attractors

In the middle of the last century, oscillations of another type were found, so-called *hidden oscillations*: the oscillations, the existence of which is not obvious. They are not "connected" with equilibrium (i.e. in this case it is impossible to localize a periodic solution by the computing of trajectory with the initial data from a small

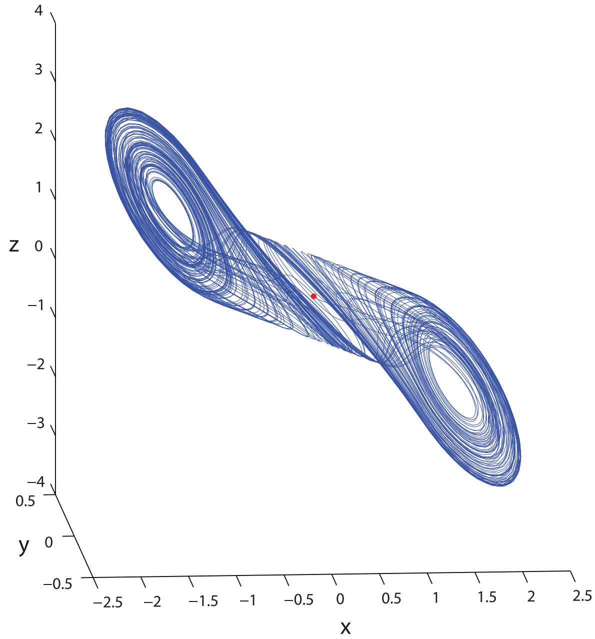

Fig. 3.4 Numerical localization of a chaotic attractor in the Chua circuit

neighborhood of equilibrium). In addition, in this case it is unlikely that the integration of trajectories with random initial data will lead to localization of such hidden oscillation since the basin of attraction can be very small and the considered system dimension can be large.

For the first time the problem of finding hidden oscillations arose in the 16th Hilbert problem (1900) for two-dimensional polynomial systems. For more than a century, in the framework of the solution of this problem, the numerous theoretical and numerical results were obtained. However, the problem is still far from being resolved even for the simple class of quadratic systems. In the 1940s and 1950s, academician A.N. Kolmogorov became the initiator of a few hundred of the following computational experiments [16]: he asked students (at Moscow State University) to find limit cycles in two-dimensional quadratic systems with randomly chosen coefficients. The result was absolutely unexpected: limit cycles were not found in any of the experiments, though it is known that quadratic systems with limit cycles form open domains in the space of coefficients and, therefore, for a random choice of polynomial coefficients, the probability of hitting in these sets is positive.

Note that numerical localization of small and nested limit cycles [13, 16, 20, 22, 24, 25] is a difficult problem.

Example 3.5 (Four limit cycles in a quadratic system) Nowadays the application of special analytical-numerical methods [17, 25] allows one to visualize four limit cycles in a quadratic system [12].