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Differential equations of Costas loop

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Draft [1] [2]

Abstract:- It is well known, that to effectively simulate and investigate PLL systems, nonlinear mathematical model has to be derived. This article provides mathematical model of Costas Loop circuit for wide range of signal waveforms. Investigation of Costas Loop in signal domain is reduced to nonlinear phase domain model of PLL with special phase detector characteristics. This allows to avoid a number of numerical problems in the simulation of Costas Loop in signal domain.

Keywords: Phase detector characteristics, nonlinear analysis, phase-locked loop, PLL, Costas loop, simulation

1 Introduction

The Costas loop was invented in the 1950s [1] and is intended for carrier phase recovery from signals widely applied in communications and control systems [2–4].

To describe a mathematical model of the Costas loop in the form of differential equations, it was necessary to develop asymptotic methods for analyzing high-frequency oscillations propagating in nonlinear electronic circuits of special form [5, 6].

In this paper, we generalize the approach described in [5–9]. Specifically, it is proved that the Costas loop in the signal space is asymptotically equivalent to the proposed scheme in the phase-frequency space. These results are used to derive differential equations that describe the dynamics of the Costas loop and generalize the results in [9, 10]

2 Statement Of The Problem

Consider the product of high-frequency oscillations passing through a linear filter (Fig.1).

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[2] PDF slides http://www.math.spbu.ru/user/nk/PDF/Nonlinear-analysis-of-Phase-locked-loop-PLL.pdf

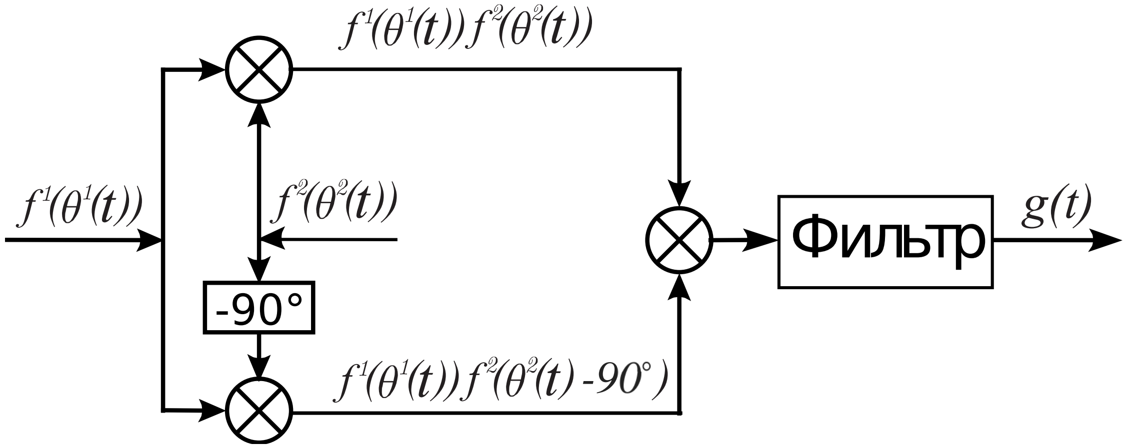

Figure 1: Multiplier and the filter.

Here, $\otimes$ is a multiplier, the -90^o unit shifts the signal phase by $-\frac{\pi}{2}$, $f^1\big(\theta^1(t)\big)$ and $f^2\big(\theta^2(t)\big)$ are high-frequency oscillations (signals from a reference oscillator and a voltage controlled oscillator (VCO), respectively) [10–13], and $g(t)$ is the output of the filter (low-frequency filter).

Assume that $f^1(\theta)$ is a bounded 2π-periodic piecewise differentiable function (i.e., a function that has a finite number of jumps and is differentiable on its intervals of continuity). Then, according to the Lipschitz criterion [14], the Fourier series corresponding to $f^1(\theta)$ converges to function values at continuity points and to the half-sum of the left and right limits at discontinuity points. Note that functions different at a finite number of points are equivalent in $L^1_{[-\pi,\pi]}$.

Therefore, while analyzing the synchronization of the oscillators, we consider $f^1(\theta)$ with the values at discontinuity points indicated by the Lipschitz criterion; i.e.,

$$f^1(\theta) = \sum_{i=1}^{\infty} \big(a_i^1 \cos(i\theta) + b_i^1 \sin(i\theta)\big), \tag{1}$$

$$a_i^1 = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f^1(x)\cos(ix)dx, \quad b_i^1 = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f^1(x)\sin(ix)dx, \quad i \in \mathbb{N}. \tag{2}$$

The properties of Fourier coefficients for piecewise differentiable functions [14] imply the estimates

$$a_i^1 = O\left(\frac{1}{i}\right), b_i^1 = O\left(\frac{1}{i}\right). \tag{3}$$

The VCO signal is assumed to be harmonic:

$$f^2(\theta) = b_1^2 \sin(\theta). \tag{4}$$

The input $\xi(t)$ and the output $\psi(t)$ of the linear filter are related by the formula

$$\psi(t) = \alpha_0(t) + \int\limits_0^t \gamma(t-\tau)\xi(\tau)d\tau, \tag{5}$$

where $\alpha_0(t)$ is an exponentially decaying function depending linearly on the initial state of the filter at $t = 0$ and $\gamma(t)$ is the impulsive transition function of the linear filter. In what follows, $\alpha_0(t)$ and $\gamma(t)$ are assumed to be differentiable functions with bounded derivatives. According to (5), the function $g(t)$ has the form

$$g(t)=\alpha_0(t)+\int_0^t \gamma(t-\tau)f^1\big(\theta^1(\tau)\big)f^2\big(\theta^2(\tau)\big)f^1\big(\theta^1(\tau)\big)f^2\big(\theta^2(\tau)-\frac{\pi}{2}\big)d\tau. \tag{6}$$

The high-frequency property of the signals can be reformulated as follows.

Suppose that $\theta^1(t)$ and $\theta^2(t)$ are smooth functions and there exists a sufficiently large number ω_{min} such that the condition

$$\dot\theta^p(\tau) \geq \omega_{min} > 0, \quad p = 1,2 \tag{7}$$

holds on a sufficiently long time interval $[0,T]$, where T is independent of ω_{min} ($\dot\theta^p(t)$ are the frequencies of the corresponding signals). Assume that the difference of the frequencies is uniformly bounded on the considered time interval:

$$\big|\dot\theta^1(\tau)-\dot\theta^2(\tau)\big| \leq \Delta\omega,\ \forall\tau\in[0,T], \tag{8}$$

where $\Delta\omega$ is a constant.

The interval $[0,T]$ is divided into short intervals of length δ:

$$\delta = \frac{1}{\sqrt{\omega_{min}}}. \tag{9}$$

Assume that

$$|\dot\theta^p(\tau)-\dot\theta^p(t)| \leq \Delta\Omega,\ p=1,2, \quad |t-\tau|\leq\delta, \quad \forall\tau,t\in[0,T], \tag{10}$$

where $\Delta\Omega$ is a constant independent of t or τ. It follows from (10) and (9) that $\dot\theta^p(t)$ is almost a constant on short time intervals.

Since $\gamma(t)$ is bounded, there exists a constant C such that

$$|\gamma(\tau)-\gamma(t)| \leq C\delta, \quad |t-\tau|\leq\delta, \quad \forall\tau,t\in[0,T]. \tag{11}$$

Consider a 2π-periodic function $\varphi(\theta)$ of the form

$$\begin{aligned}
&\varphi(\theta) = \\
&= \frac{(b_1^2)^2}{8}\Bigg[(a_1^1)^2\sin(2\theta) + 2\sum_{q=1}^{\infty} a_q^1 a_{q+2}^1\sin(2\theta) - \\
&- 2a_1^1 b_1^1\cos(2\theta) + 2\sum_{q=1}^{\infty} a_{q+2}^1 b_q^1\cos(2\theta) - 2\sum_{q=1}^{\infty} a_q^1 b_{q+2}^1\cos(2\theta) - \\
&- (b_1^1)^2\sin(2\theta) + 2\sum_{q=1}^{\infty} b_q^1 b_{q+2}^1\sin(2\theta)\Big)\Bigg].
\end{aligned} \tag{12}$$

It follows from (3) that this series converges uniformly, while $\varphi(\theta)$ is continuous and bounded together with its derivative. Consider the block diagram shown in Fig. 2.

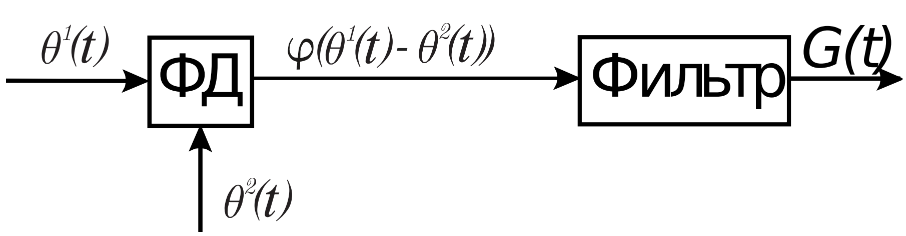

Figure 2: Phase detector and the filter

Here, PD is a nonlinear phase detector with the output $\varphi\big(\theta^1(t)-\theta^2(t)\big)$ (PD characterizes the performance of all the intermediate elements in Fig. 1 situated between the inputs and the filter) and $G(t)$ is the output of the filter, which, according to (5), is given by

$$G(t) = \alpha_0(t) + \int\limits_0^t \gamma(t-\tau)\varphi\big(\theta^1(\tau)-\theta^2(\tau)\big)d\tau. \tag{13}$$

We assume that the characteristics and the initial data of the filters in Figs. 1 and 2 coincide.

3 Main Result

Theorem 1 *If conditions* (8) – (10) *hold, then*

$$|G(t)-g(t)| \le C_1\delta, \quad \forall t \in [0,T]. \tag{14}$$

Proof. Let $t \in [0,T]$. Consider the difference

$$\begin{aligned} g(t)-G(t) = \int\limits_0^t \gamma(t-s)\Big[& f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)\big)f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)-\frac{\pi}{2}\big)- \\ & -\varphi\big(\theta^1(s)-\theta^2(s)\big)\Big]ds. \end{aligned} \tag{15}$$

Let $m \in \mathbb{N}\cup\{0\}$ be such that $t \in [m\delta,(m+1)\delta]$. According to (9)

$$m < \frac{T}{\delta} + 1. \tag{16}$$

The continuity condition implies that $\gamma(t)$ is bounded on $[0,T]$; moreover, $f^1(\theta), f^2(\theta)$, and $\varphi(\theta)$ are bounded on $\mathbb{R}$. Then the following estimates hold:

$$\begin{aligned} &\int\limits_t^{(m+1)\delta} \gamma(t-s)f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)\big)f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)-\frac{\pi}{2}\big)ds = O(\delta), \\ &\int\limits_t^{(m+1)\delta} \gamma(t-s)\varphi\big(\theta^1(s)-\theta^2(s)\big)ds = O(\delta). \end{aligned} \tag{17}$$

It follows that (15) can be represented as

$$\begin{aligned} & g(t)-G(t) = \\ & \sum_{k=0}^{m} \int\limits_{[k\delta,(k+1)\delta]} \gamma(t-s)\Big[f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)\big)f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)-\frac{\pi}{2}\big)- \\ & -\varphi\big(\theta^1(s)-\theta^2(s)\big)\Big]ds + O(\delta). \end{aligned} \tag{18}$$

Let us show that, on each of the intervals $[k\delta,(k+1)\delta]$, the corresponding integrals are $O(\delta^2)$, which, in view of (16), implies the assertion of the theorem.

It follows from conditions (11) that, on each of the intervals $[k\delta,(k+1)\delta]$, we have

$$\gamma(t-s)=\gamma(t-k\delta)+O(\delta),\ t>s,\ s\in[k\delta,(k+1)\delta]. \tag{19}$$

which holds uniformly in t and $O(\delta)$ is independent of k. Then, combining (18) with (19) and recalling that $f^1(\theta), f^2(\theta), \varphi(\theta)$ are bounded, we obtain

$$\begin{aligned} g(t)-G(t)=\sum_{k=0}^{m}\gamma(t-k\delta)\int\limits_{[k\delta,(k+1)\delta]}\Big[f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)\big)f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)-\frac{\pi}{2}\big)-\\ -\varphi\big(\theta^1(s)-\theta^2(s)\big)\Big]ds+O(\delta). \end{aligned} \tag{20}$$

Define

$$\theta^p_k(s)=\theta^p(k\delta)+\dot\theta^p(k\delta)(s-k\delta),\ p=1,2. \tag{21}$$

Then condition (10) for $s\in[k\delta,(k+1)\delta]$ implies that

$$\theta^p(s)=\theta^p_k(s)+O(\delta). \tag{22}$$

From (8) and the boundedness of the derivative of $\varphi(\theta)$ on $\mathbb{R}$, we have

$$\int\limits_{[k\delta,(k+1)\delta]}\big|\varphi\big(\theta^1(s)-\theta^2(s)\big)-\varphi\big(\theta^1_k(s)-\theta^2_k(s)\big)\big|ds=O(\delta^2). \tag{23}$$

By assumption, $f^1(\theta)$ is piecewise differentiable and bounded, while $f^2(\theta)$ is smooth. If $f^1(\theta)$ is additionally continuous on $\mathbb{R}$, then $f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)\big)f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)-\frac{\pi}{2}\big)$ satisfies

$$\begin{aligned} \int\limits_{[k\delta,(k+1)\delta]} f^1(\theta^1(s))f^2(\theta^2(s))f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)-\frac{\pi}{2}\big)ds=\\ =\int\limits_{[k\delta,(k+1)\delta]} f^1\big(\theta^1_k(s)\big)f^2\big(\theta^2_k(s)\big)f^1\big(\theta^1_k(s)\big)f^2\big(\theta^2_k(s)-\frac{\pi}{2}\big)ds+O(\delta^2). \end{aligned} \tag{24}$$

Since conditions (7) and (8) hold and the function $\theta^1(s)$ is differentiable and satisfies (10), we conclude that, for all $k=0...m$, there are sets E_k (the union of sufficiently small neighborhoods of the discontinuity points of $f^1(\theta^1(s))$, $s\in[k\delta,(k+1)\delta]$) such that

$$\int\limits_{E_k}ds=O(\delta^2), \tag{25}$$

which holds uniformly in k. From this and the piecewise differentiability and boundedness of $f^1(\theta)$, we obtain (24). Then, using (24) and (23), we can rewrite (20) as

$$\begin{aligned} g(t)-G(t)=\sum_{k=0}^{m}\gamma(t-k\delta)\int\limits_{[k\delta,(k+1)\delta]}\Big[f^1\big(\theta^1_k(s)\big)f^2\big(\theta^2_k(s)\big)f^1\big(\theta^1_k(s)\big)f^2\big(\theta^2_k(s)-\frac{\pi}{2}\big)-\\ -\varphi\big(\theta^1_k(s)-\theta^2_k(s)\big)\Big]ds+O(\delta)=\sum_{k=0}^{m}\gamma(t-k\delta)\int\limits_{[k\delta,(k+1)\delta]}\Big[-b^2_1\sin\big(\theta^2_k(s)\big)b^2_1\cos\big(\theta^2_k(s)\big)\\ \Big(\sum_{j=1}^{\infty}a^1_j\sin\big(j\theta^1_k(s)\big)+b^1_j\cos\big(j\theta^1_k(s)\big)\Big)\Big(\sum_{j=1}^{\infty}a^1_j\sin\big(j\theta^1_k(s)\big)+b^1_j\cos\big(j\theta^1_k(s)\big)\Big)\\ -\varphi\big(\theta^1_k(s)-\theta^2_k(s)\big)\Big]ds+O(\delta)\ . \end{aligned} \tag{26}$$

By the Jordan the uniform convergence of Fourier series [14], on each of the intervals free of the discontinuity points, the Fourier series of $f^1(\theta)$ converges uniformly. Since the number of discontinuity points of $f^1(\theta)$ is finite, there exists a number $M = M(\delta) > 0$ such that the remainders of the series of $f^1(\theta)$ and $\varphi(\theta)$ do not exceed δ outside a sufficiently small neighborhoods of the discontinuity points:

$$
\begin{aligned}
&f^1(\theta) = \sum_{i=1}^{\infty}\big(a_i^1\cos(i\theta) + b_i^1\sin(i\theta)\big) = \sum_{i=1}^{M}\big(a_i^1\cos(i\theta) + b_i^1\sin(i\theta)\big) + O(\delta),\\
&\varphi(\theta) = \frac{(b_1^2)^2}{8}\Bigg[(a_1^1)^2\sin(2\theta) + 2\sum_{l=1}^{\infty}a_l^1a_{l+2}^1\sin(2\theta) -\\
&- 2a_1^1b_1^1\cos(2\theta) + 2\sum_{l=1}^{\infty}a_{l+2}^1b_l^1\cos(2\theta) - 2\sum_{l=1}^{\infty}a_l^1b_{l+2}^1\cos(2\theta) -\\
&- (b_1^1)^2\sin(2\theta) + 2\sum_{l=1}^{\infty}b_l^1b_{l+2}^1\sin(2\theta)\Big)\Bigg] =\\
&= \frac{(b_1^2)^2}{8}\Bigg[(a_1^1)^2\sin(2\theta) + 2\sum_{l=1}^{M}a_l^1a_{l+2}^1\sin(2\theta) -\\
&- 2a_1^1b_1^1\cos(2\theta) + 2\sum_{l=1}^{M}a_{l+2}^1b_l^1\cos(2\theta) - 2\sum_{l=1}^{M}a_l^1b_{l+2}^1\cos(2\theta) -\\
&- (b_1^1)^2\sin(2\theta) + 2\sum_{l=1}^{M}b_l^1b_{l+2}^1\sin(2\theta)\Big)\Bigg] + O(\delta).
\end{aligned}
$$

Consequently, using the above-introduced neighborhoods E_k, equalities (25) and (26), and the boundedness of $f^1(\theta)$ and $f^2(\theta)$ on $\mathbb{R}$, we have

$$
\begin{aligned}
&g(t) - G(t) = \sum_{k=0}^{m}\gamma(t-k\delta)\int\limits_{[k\delta,(k+1)\delta]}\Bigg[-b_1^2\sin\big(\theta_k^2(t)\big)b_1^2\cos\big(\theta_k^2(t)\big)\sum_{i=1}^{M}\sum_{j=1}^{M}\Bigg\{\\
&\Big(a_i^1\cos\big(i\theta_k^1(s)\big) + b_i^1\sin\big(i\theta_k^1(s)\big)\Big)\Big(a_j^1\cos\big(j\theta_k^1(s)\big) + b_j^1\sin\big(j\theta_k^1(s)\big)\Big)\Bigg\} -\\
&-\varphi\big(\theta_k^1(s) - \theta_k^2(s)\big)\Bigg]ds + O(\delta).
\end{aligned}
\tag{27}
$$

Using the formulas for the product of sines and cosines and the estimate

$$
\int\limits_{[k\delta,(k+1)\delta]}\sin\big(p\theta_k^1(s)\big)ds = \frac{O(\delta^2)}{p},\ p\in\mathbb{N},
\tag{28}
$$

which follows from (7) and (9), we derive the assertion of the theorem.

4 EXAMPLES

Let us give examples of computing the characteristic of a phase detector with the use of formula (12) for some types of signals.

Corollary 1.

$$f^1(t) = b^1 \sin(\theta^1(t)), \quad f^2(t) = b^2 \sin(\theta^2(t)),$$
$$\varphi(\theta^1 - \theta^2) = -\frac{(b^1 b^2)^2}{8} \sin(2\theta^1 - 2\theta^2). \tag{29}$$

Corollary 2.

$$f^1(t) = b^1 \text{sign} \sin(\theta^1(t)) = \frac{4b^1}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\big((2n-1)(\omega^1(t)t + \psi^1)\big),$$
$$f^2(t) = b^2 \sin(\theta^2(t)), \tag{30}$$
$$\varphi(\theta^1 - \theta^2) = \frac{(b^1 b^2)^2}{8} \frac{16}{\pi^2} \Big[-1 + 2 \sum_{l=1}^{\infty} \frac{1}{2l-1} \frac{1}{2l+1} \Big] \sin(2\theta^1 - 2\theta^2), \tag{31}$$

but

$$2 \sum_{l=1}^{\infty} \frac{1}{2l-1} \frac{1}{2l+1} = 1, \tag{32}$$

i.e.,

$$\varphi(\theta^1 - \theta^2) = 0. \tag{33}$$

This result is also easy to verify without using the theorem. The signal from the reference oscillator is squared and multiplied by $-(a^2)^2 \sin(\theta^2(s)) \sin(\theta^2(s) - \frac{\pi}{2})$. The squaring of a pulse signal yields a constant signal. The signal $-(a^2)^2 \sin(\theta^2(s)) \sin(\theta^2(s) - \frac{\pi}{2})$ is of high frequency; therefore, it is subject to filtration.

Corollary 3. Consider a sawtooth signal (Fig. 3):

$$f^1(t) = \frac{2}{\pi} \sum_{l=1}^{\infty} \frac{1}{l} \sin\big(l\theta^1(t)\big). \tag{34}$$

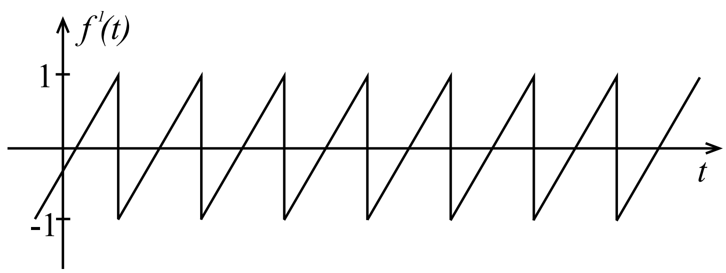

Figure 3: Sawtooth.

Then

$$\varphi(\theta^1 - \theta^2) = \frac{1}{2\pi^2} \Big[-\sin(2\theta^1 - 2\theta^2) + 2 \sum_{l=1}^{\infty} \frac{1}{l(l+2)} \sin(2\theta^1 - 2\theta^2) \Big) \Big]. \tag{35}$$

From this, since

$$2 \sum_{l=1}^{\infty} \frac{1}{l(l+2)} = \frac{3}{2}, \tag{36}$$

we finally obtain

$$\varphi(\theta^1 - \theta^2) = \frac{1}{4\pi^2} \sin(2\theta^1 - 2\theta^2). \tag{37}$$

Due to the characteristics obtained in this work, we can proceed from the model of the Costas loop in the signal space to a dynamic model in the phase-frequency space.

Equations for the Costas loop. According to the above theorem, the outputs of the schemes depicted in Figs. 1 and 2 are asymptotically equivalent. The VCO frequency is usually assumed to vary according to a linear law [2]:

$$\dot{\theta}^2(t) = \omega^2_{free} + LG(t), \ t \in [0, T], \tag{38}$$

where ω^2_{free} is the natural frequency of the VCO. Assume that the frequency of the reference oscillator is a constant:

$$\dot{\theta}^1(t) \equiv \omega^1. \tag{39}$$

Then, combining law (38) with filter equation (5) gives

$$\dot{\theta}^2(t) = \omega^2_{free} + L\Big(\alpha_0(t) + \int\limits_0^t \gamma(t-\tau)\varphi\big(\theta^1(\tau) - \theta^2(\tau)\big)d\tau\Big), \ t \in [0, T]. \tag{40}$$

Taking into account (39), we derive the equation

$$\begin{aligned} &\omega^1 - \omega^2_{free} = \\ &= (\dot{\theta}^1(t) - \dot{\theta}^2(t)) + L\Big(\alpha_0(t) + \int\limits_0^t \gamma(t-\tau)\varphi\big(\theta^1(\tau) - \theta^2(\tau)\big)d\tau\Big), \ t \in (0, T]. \end{aligned} \tag{41}$$

It can be rewritten as a system of differential equations. For this purpose, the filter can be described by the system

$$\frac{dx}{dt} = Ax + b\xi(t), \ \psi(t) = c^* x, \tag{42}$$

whose solution at $\rho = 0$ is given by (5), where

$$\gamma(t-\tau) = c^* e^{A(t-\tau)} b, \quad \alpha_0(t) = c^* e^{At} x_0. \tag{43}$$

Here, A is a constant matrix; $x(t)$ is the state vector of the filter; b and c are constant vectors; $\xi(t)$ and $\psi(t)$ are scalar functions characterizing the input and output of the filter, respectively; and x_0 is the initial state of the filter.

Then, in view of (38), we obtain a system of differential equations describing the Costas loop in the space:

$$\begin{aligned} \dot{x} &= Ax + b\varphi(\Delta\theta), \\ \Delta\dot{\theta} &= \Delta\omega_0 - Lc^* x, \end{aligned} \tag{44}$$

where

$$\Delta\theta(t) = \theta^1(t) - \theta^2(t), \quad \Delta\omega_0 = \omega^1 - \omega^2_{free}, \tag{45}$$

Here, $\Delta\theta(t)$ is the phase difference and $\Delta\omega_0$ is the initial difference between the frequencies of the oscillators.

It should be noted, that instead of conditions (8) and (11) for simulations of real system one have to consider the following conditions

$$|\Delta\omega| << \omega_{min}, \ |\lambda_A| << \omega_{min},$$

where λ_A is the largest (in modulus) eigenvalue of the matrix A. Also, for correctness of transition from equation (15) to (20) one have to consider $T << \omega_{min}$. Theoretical results are justified by simulation of Costas Loop model in phase-frequency space and signal space (Fig. 4).

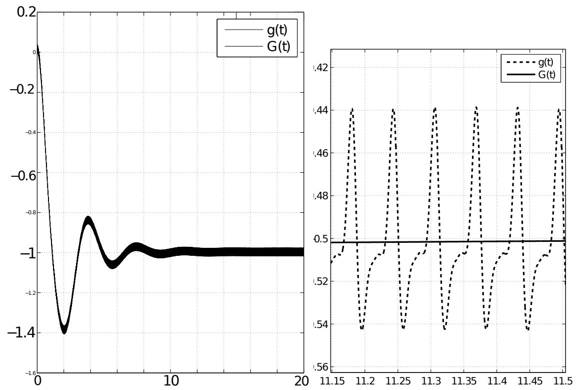

Figure 4: $\omega^2_{free} = 100$, $\omega^1 = 101$, $L = 30$, filter transfer functions $\frac{1}{s+1}$, sine and sawtooth waveforms

Unlike the filter output for the phase-frequency model, the output of the filter for signal space Costas Loop model contains additional high-frequency oscillations. These high-frequency oscillations interfere with qualitative analysis and efficient simulation of Costas Loop.

References

[1] *J. Costas* // Synchoronous communications, Proc.IRE, 1956, V. 44, P. 1713–1718.

[2] *E. Ronald*. Phase-lock loops: Design, simulation and application, Oberwill, Switzerland: McGraw Hill, 2003.

[3] *D. Abramovitch* // Phase-locked loops: A control centric tutorial, Proceedings of the American Control Conference, 2002, V. 1, P. 1–15.

[4] *W. Lindsey*. Synchronization systems in communication and control, New Jersey: Prentice-Hall, 1972.

[5] *G. Leonov, S. Seledghi* // Stability and bifurcations of phase-locked loops for digital signal processors, *International journal of bifurcation and chaos*, 2005, V. 15, no. 4, P. 1347–1360.

[6] *G. A. Leonov // Autom. Remote Control*, 2006, no. 10, P. 47–55.

[7] *N. Kuznetsov, G. Leonov, S. Seledzhi* // Phase locked loops design and analysis, International Conference on Informatics in Control, Automation and Robotics, INSTICC Press, 2008 doi:10.5220/0001485401140118, V. 1, P. 114–118.

[8] *G. Leonov, S. Seledzhi, N. Kuznetsov, P. Neittaanmaki* // Asymptotic analysis of phase control system for clocks in multiprocessor arrays, International Conference on Informatics in Control, Automation and Robotics, Proceedings, INSTICC Press, 2010 doi:10.5220/0002938200990102, V. Signal processing, Systems Modeling and Control (ICINCO-SPSMC), P. 99–102.

[9] *G. A. Leonov // Autom. Remote Control*, 2008, no. 9, P. 33–43.

[10] *N. Kuznetsov, G. Leonov, S. Seledzhi* // Nonlinear analysis of the Costas loop and phase-locked loop with squarer, Proceedings of the IASTED International Conference on Signal and Image Processing, ACTA Press, 2009, V. 654, P. 1–7.

[11] *N. Kuznetsov, G. Leonov, M. Yuldashev, R. Yuldashev* // Analytical methods for computation of phase-detector characteristics and pll design, International Symposium on Signals, Circuits and Systems, IEEE press, 2011 doi:10.1109/ISSCS.2011.5978639, P. 7–10.

[12] *N. Kuznetsov, G. Leonov, P. Neittanmäki, S. Seledzhi, M. Yuldashev, R. Yuldashev* // High-frequency analysis of phase-locked loop and phase detector characteristic computation, 8th International Conference on Informatics in Control, Automation and Robotics (ICINCO 2011), INSTICC Press, 2011 doi:10.5220/0003522502720278, P. 272–278.

[13] *G. A. Leonov, N. V. Kuznetsov, M. V. Yuldashev, and M. V. Yuldashev*, Computation of Phase Detector Characteristics in Synchronization Systems, Doklady Mathematics, 84(1), 2011, P. 459–463.

[14] *G. M. Fikhtengolts*. Course in Differential and Integral Calculus, Nauka, Moscow, 1962. P. 436, 490, 502–503.