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STABILITY

Algorithms for Finding Hidden Oscillations in Nonlinear Systems. The Aizerman and Kalman Conjectures and Chua's Circuits

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Abstract—An algorithm for searching hidden oscillations in dynamic systems is developed to help solve the Aizerman's, Kalman's and Markus-Yamabe's conjectures well-known in control theory. The first step of the algorithm consists in applying modified harmonic linearization methods. A strict mathematical substantiation of these methods is given using special Poincare maps. Subsequent steps of the proposed algorithms rely on the modern applied theory of bifurcations and numerical methods of solving differential equations. These algorithms help find and localize hidden strange attractors (i.e., such that a basin of attraction of which does not contain neighborhoods of equilibria), as well as hidden periodic oscillations. One of these algorithms is used here to discover, for the first time, a hidden strange attractor in the dynamic system describing a nonlinear Chua's circuit, viz. an electronic circuit with nonlinear feedback.

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0. INTRODUCTION

M.A. Aizerman stated a conjecture in 1949 [1] that immediately attracted attention of many scientists prominent in the field of control theory and differential equations [2–4]. The conjecture is as follows. Consider a system with a scalar nonlinearity

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}\mathbf{x} + \mathbf{q}\psi(\mathbf{r}^*\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \tag{0.1}$$

where $\mathbf{P}$ is the constant $n \times n$-matrix, $\mathbf{q}$ and $\mathbf{r}$ are constant n-dimensional vectors, * is the transposition operation, and $\psi(\sigma)$ is a piecewise-continuous scalar function and $\psi(0) = 0$. Here, the solution of system (0.1) is taken in A.F. Filippov sense [5]. *Suppose all linear systems* (0.1) *with*

$$\psi(\sigma) = \mu\sigma, \quad \mu \in (\mu_1, \mu_2) \tag{0.2}$$

are asymptotically stable. Is system (0.1) *with any nonlinearity* $\psi(\sigma)$ *that satisfies the condition*

$$\mu_1 < \frac{\psi(\sigma)}{\sigma} < \mu_2, \quad \forall \sigma \neq 0,$$

globally stable (*i.e., the zero solution of system* (0.1) *is asymptotically stable and any solution tends to zero when* $t \to +\infty$)?

I.G. Malkin [6], N.P. Erugin [7], and N.N. Krasovskii [8] solved the Aizerman conjecture completely for $n = 2$ in 1952. Here, there is always a positive solution of the Aizerman conjecture for $n = 2$ except when the matrix $(\mathbf{P} + \mu_1\mathbf{q}\mathbf{r}^*)$ has a multiple double zero eigenvalue and

$$\int_0^{\infty} (\psi(\sigma) - \mu_1\sigma)d\sigma \neq +\infty \quad \text{or} \quad \int_0^{-\infty} (\psi(\sigma) - \mu_1\sigma)d\sigma \neq -\infty.$$

N.N. Krasovskii showed [8] that if all these conditions are met, system (0.1) possesses the solutions tending to infinity. It was the first counterexample to the Aizerman conjecture, which was then generalized to systems (0.1) of an arbitrary order [9–12].

The modification of M.A. Aizerman conjecture proposed by R.E. Kalman [13] in 1957 is as follows. *Suppose* $\psi(\sigma)$ *is a piecewise differentiable function*[1] *and*

$$\mu_1 < \psi'(\sigma) < \mu_2$$

at differentiability points. Is system (0.1) *globally stable if all linear systems* (0.1) *with* $\psi(\sigma) = \mu\sigma$, $\mu \in (\mu_1, \mu_2)$ *are asymptotically stable*?

Since the Krasovskii counterexample is obviously eliminated for $n = 2$, the Kalman conjecture has a positive solution for $n = 2$. It is shown in [14] that frequency stability criteria yield a positive solution to the Kalman conjecture for $n = 2$ and 3. Generalization of the R.E. Kalman conjecture for multidimensional nonlinearity is known as the Markus-Yamabe conjecture [15].

Suppose the Jacobian matrix $\left(\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right)$ *for the system*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n, \quad \mathbf{f} \in C^1, \quad \mathbf{f}(\mathbf{0}) = \mathbf{0} \tag{0.3}$$

has all eigenvalues with negative real components for any $\mathbf{x} \in \mathbb{R}^n$. *Is system* (0.3) *globally stable*? This conjecture has a positive solution for $n = 2$ [16–18] and a negative solution in the general case when $n \geq 3$. Thus, while it allows for the unbounded solution $(x_1(t), x_2(t), x_3(t)) = (18e^t, -12e^{2t}, e^{-t})$, the polynomial system

$$\dot{x}_1 = -x_1 + x_3(x_1 + x_2x_3)^2, \quad \dot{x}_2 = -x_2 - (x_1 + x_2x_3)^2, \quad \dot{x}_3 = -x_3$$

considered in [19] has the Jacobian matrix with three eigenvalues equivalent to –1.

In 1958, V.A. Pliss [9] developed a method for constructing three-dimensional nonlinear systems that meet the Aizerman condition and have periodic solutions. This method was then generalized to systems (0.1) of arbitrary dimension [20, 21]. However, classes of these systems did not meet the Kalman condition.

In this work, we describe the current situation in studying the Aizerman and Kalman conjectures and a new approach to solving them that is based on computational algorithms, where the first step consists in applying the modified harmonic linearization method. The classical harmonic linearization method (the method of harmonic balance, the method of describing functions) is quite common and frequently used to analyze nonlinear automatic control theory systems. It is not strictly mathematically substantiated and is an approximate method of analyzing control systems [22–32]. Ya.Z. Tsypkin proposed one of the first examples of the harmonic linearization method yielding incorrect results [33].

To describe a very simple connection between the harmonic linearization method and the Aizerman conjecture, we recall the standard procedure of this method for systems (0.1) and introduce the transfer function

$$W(p) = \mathbf{r}^*(\mathbf{P} - p\mathbf{I})^{-1}\mathbf{q},$$

where p is a complex variable. To find the harmonic oscillation $a\cos\omega_0 t$, which is an approximate solution

$$\sigma(t) = \mathbf{r}^*\mathbf{x}(t) \approx a\cos(\omega_0 t)$$

of system (0.1), we first give a harmonic linearization coefficient k so that the matrix of the linear system

$$\frac{d\mathbf{z}}{dt} = \mathbf{P}_0\mathbf{z}, \quad \mathbf{P}_0 = \mathbf{P} + k\mathbf{q}\mathbf{r}^* \tag{0.4}$$

has a pair of purely imaginary eigenvalues $\pm i\omega_0$ ($\omega_0 > 0$), with the rest of its eigenvalues having negative real components. Here, we assume that such k exists.

In practical problems, the transfer function $W(p)$ is used to find the values k and ω_0: first we find the variable ω_0 from the equation

$$\operatorname{Im} W(i\omega_0) = 0,$$

then we calculate k by the formula

$$k = -(\operatorname{Re} W(i\omega_0))^{-1}.$$

[1] In other words, it is the function that has a finite number of discontinuities of the first order on any interval and is differentiable on the continuity intervals.

If such k and ω_0 exist, the amplitude a can be found from the equation

$$\int_0^{2\pi/\omega_0} \psi(a\cos(\omega_0 t))\cos(\omega_0 t)dt = ka\int_0^{2\pi/\omega_0}(\cos(\omega_0 t))^2 dt.$$

We apply the described procedure to the Aizerman conjecture. Obviously, the condition $k \in (\mu_1, \mu_2)$ is not satisfied in this case. Then, for any nonzero values σ, one of the estimates

$$k\sigma^2 < \psi(\sigma)\sigma \quad \text{or} \quad \psi(\sigma)\sigma < k\sigma^2.$$

holds.

Hence, for all $a \neq 0$, the inequality

$$\int_0^{2\pi/\omega_0} (\psi(a\cos(\omega_0 t))a\cos(\omega_0 t) - k(a\cos(\omega_0 t))^2)dt \neq 0 \tag{0.5}$$

holds. Thus, by the harmonic linearization method, system (0.1) has no periodic solutions under the Aizerman (and Kalman) conditions. However, the results of V.A. Pliss and his followers [20, 21] contradict these non-strict conclusions.

The above mentioned facts have led to many decades of attempts to find classes of systems, where the harmonic linearization method (and its various generalizations) turned out to be accurate and yielded true results. Being among the first works in this direction, publications [34, 35] used the variant of the classical method of small parameter. In what follows, this direction was heavily criticized by saying that "*these methods of small parameter assume that the initial system hardly differs from a linear system with the proper generating frequency. One cannot make such assumptions within the automatic control theory since the system is a fortiori non-conservative and stability conditions are met in linear approximation with sufficient margin*" [36]. With the criticism taken into account, other methods of introducing the small parameter began to be developed based on the filter hypothesis [37–39].

Advances in numerical methods, computer science and applied bifurcation theory allow us to return to earlier ideas of applying methods of small parameter and method of harmonic linearization in dynamic systems and consider them from new points of view. Here, we modify and justify the harmonic linearization method to create search algorithms for oscillations in the Aizerman and Kalman conjectures. Such *oscillations are hidden in the sense that their attraction domains do not include neighborhoods of stationary states*. Therefore, one cannot set up computational procedures based on studying the transient process starting in the neighborhood of non-stable equilibrium state and coming to the attracting oscillating mode. It is this oscillation excitation that is typical of self-excited oscillators and other classical self-oscillating systems [40–44].

Harmonic linearization method, classical method of small parameter and numerical methods applied together allow reducing calculation of periodic modes to a multi-step procedure, with the harmonic linearization method applied at its first step. We describe this procedure, mainly following [45–48]. We rewrite system (0.1) as

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}_0\mathbf{x} + \mathbf{q}\varphi(\mathbf{r}^*\mathbf{x}), \tag{0.6}$$

where $\varphi(\sigma) = \psi(\sigma) - k\sigma$. Since we are interested in periodic solutions of system (0.6), it seems natural to introduce a finite sequence of functions $\varphi^0(\sigma), \varphi^1(\sigma), \dots, \varphi^m(\sigma)$ so that the graphs of neighboring functions $\varphi^j(\sigma)$ and $\varphi^{j+1}(\sigma)$ does not differ much in a sense, and the function $\varphi^0(\sigma)$ is small and $\varphi^m(\sigma) = \varphi(\sigma)$. The fact that the function $\varphi^0(\sigma)$ is small allows us to apply and justify, in this case, the harmonic linearization method for the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}_0\mathbf{x} + \mathbf{q}\varphi^0(\mathbf{r}^*\mathbf{x}), \tag{0.7}$$

finding the stable periodic solution $\mathbf{x}^0(t)$ close to the harmonic solution. All points of this stable periodic solution are either in the attraction domain of the stable periodic solution $\mathbf{x}^1(t)$ of system

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}_0\mathbf{x} + \mathbf{q}\varphi^j(\mathbf{r}^*\mathbf{x}) \tag{0.8}$$

with $j = 1$ or there is bifurcation of loss of stability and the periodic solution disappears when passing from system (0.7) to (0.8) with $j = 1$. In the first case, we can numerically obtain $\mathbf{x}^1(t)$ by originating the trajectory of system (0.8) with $j = 1$ from the initial point $\mathbf{x}^0(0)$. Starting at the point $\mathbf{x}^0(0)$, after the transient process the computational procedure comes at the periodic solution $\mathbf{x}^1(t)$ and calculates it. This requires rather big interval $[0, T]$, on which calculation is performed.

After $\mathbf{x}^1(t)$ is calculated, we can move to system (0.8) with $j = 2$ and set up a procedure to find the periodic solution $\mathbf{x}^2(t)$ by starting the trajectory of system (0.8) with $j = 2$ from the initial point $\mathbf{x}^1(T)$ that approaches the periodic trajectory $\mathbf{x}^2(t)$ as t grows. Going on with this procedure and subsequently obtaining $\mathbf{x}^j(t)$ using the trajectory of system (0.8) with the initial conditions $\mathbf{x}^{j-1}(T)$, we either arrive at calculation of the periodic solution of system (0.8) with $j = m$ (i.e., the initial system (0.6)) or observe that there is bifurcation of loss of stability and the periodic solution disappears at the certain step.

Note that, in addition to *hidden periodic oscillations*, the described algorithm allows us to find *hidden strange attractors a basin of attraction of which also does not contain neighborhoods of equilibria* (unlike classical attractors such as, for instance, in Lorenz [49], Rossler [50], etc. systems, where linearized systems, in the neighborhoods of equilibrium states, have eigenvalues with positive real component and trajectories going out of these equilibrium states and attracted by the attractor). In this work, we obtain hidden strange attractors as we analyze nonlinear Chua's circuits.

1. SUBSTANTIATION FOR THE HARMONIC LINEARIZATION METHOD

1.1. Reducing the System

Using the nonsingular transformation $\mathbf{x} = \mathbf{S}\mathbf{y}$, we can reduce system (0.7) with the nonlinearity $\varphi^0(\sigma)$ to the form (see, for instance, [48, 51])

$$\begin{aligned} \dot{y}_1 &= -\omega_0 y_2 + b_1\varphi^0(y_1 + \mathbf{c}_3{}^*\mathbf{y}_3), \\ \dot{y}_2 &= \omega_0 y_1 + b_2\varphi^0(y_1 + \mathbf{c}_3{}^*\mathbf{y}_3), \\ \dot{\mathbf{y}}_3 &= \mathbf{A}_3\mathbf{y}_3 + \mathbf{b}_3\varphi^0(y_1 + \mathbf{c}_3{}^*\mathbf{y}_3), \end{aligned} \tag{1.1}$$

where y_1, y_2 are scalar variables, $\mathbf{y}_3$ is the $(n-2)$-dimensional vector, $\mathbf{b}_3$ and $\mathbf{c}_3$ are $(n-2)$-dimensional vectors, b_1 and b_2 are real numbers, and $\mathbf{A}_3$ is a matrix of dimension $(n-2) \times (n-2)$ with all its eigenvalues having negative real components.

Without loss of generality, we assume that for the matrix $\mathbf{A}_3$ there exists a positive number $d > 0$ such that

$$\mathbf{y}_3^*(\mathbf{A}_3 + \mathbf{A}_3^*)\mathbf{y}_3 \le -2d|\mathbf{y}_3|^2, \quad \forall \mathbf{y}_3 \in \mathbb{R}^{n-2}. \tag{1.2}$$

We write the transfer function of system (0.7)

$$\mathbf{r}^*(\mathbf{P}_0 - p\mathbf{I})^{-1}\mathbf{q} = \frac{\eta p + \theta}{p^2 + \omega_0^2} + \frac{R(p)}{Q(p)} \tag{1.3}$$

and the transfer function of system (1.1)

$$\frac{-b_1 p + b_2\omega_0}{p^2 + \omega_0^2} + \mathbf{c}_3^*(\mathbf{A}_3 - p\mathbf{I})^{-1}\mathbf{b}_3, \tag{1.4}$$

where $\mathbf{I}$ is the unit matrix, η and θ are some real numbers, $Q(p)$ is a stable polynomial of the power $(n-2)$, $R(p)$ is the polynomial of the power less than $(n-2)$. We assume that the polynomials $R(p)$ and $Q(p)$ have no common roots. Since systems (0.7) and (1.1) are equivalent, their transfer functions coincide. Hence, we have

$$\eta = -b_1, \quad \theta = b_2\omega_0, \quad \mathbf{c}_3^*\mathbf{b}_3 + b_1 = \mathbf{r}^*\mathbf{q}, \quad \frac{R(p)}{Q(p)} = \mathbf{c}_3^*(\mathbf{A}_3 - p\mathbf{I})^{-1}\mathbf{b}_3. \tag{1.5}$$

CONCLUSIONS

In this work, we described the algorithms for finding oscillations in nonlinear dynamic systems. The first step of these algorithms consists in applying modified harmonic linearization methods (method of harmonic balance, method of describing functions). Special Poincare maps were used to give their strict mathematical substantiation. Subsequent steps of the proposed algorithms rely on the modern applied theory of bifurcations and numerical methods of solving differential equations. Efficiency of the algorithms was demonstrated by constructing counterexamples to the Aizerman, Kalman, and Markus-Yamabe conjectures. Such algorithms help find and localize hidden strange attractors [88] as well as hidden periodic oscillations. One of these algorithms was used to discover, for the first time, a hidden strange attractor in the dynamic system describing a nonlinear Chua's circuit, viz. an electronic circuit with nonlinear feedback. Note that L. Chua stated in [70] that there are no such attractors.

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