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Stabilization of unstable control system via design of delayed feedback

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Abstract:- The problem of stabilization of three-dimensional time-invariant controllable systems by means of delayed output feedback is considered. It is shown the potential of delayed output feedback. The theorems proved in this paper show that such a delayed feedback approach allows to extend the possibilities available with static time-invariant output feedback for stabilizability.

Keywords: Stabilization, unstable control system, unstable periodic orbits, UPO, delayed feedback, time-invariant system, static time-invariant output feedback stabilization, asymptotic stability.

1 Introduction

Stabilization of dynamical systems is one of the main problems in control theory. The different questions of stabilization have been studied intensively in the past decade. The interest to the problems of stabilization is motivated by as the requirements of a practice of control, as the open problems of control theory [1]. At present they remain, as before, at the focus of attention of researchers.

As it was shown in pioneer works of K. Pyragas [2-4] special stabilization methods (delayed feedback) allow one to control a chaos in different physical and chemical systems, in particular, in electronic oscillators and laser systems. Note that a delayed feedback stabilization of systems is of practical importance.

In these works stabilization of unstable periodic orbit (UPO) with period T

$$\mathbf{x_{upo}}(t+T) = \mathbf{x_{upo}}(t),$$

embedded in a strange attractor of system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, is realized by introducing a delayed feedback in system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{b}K(t)[y(t-T) - y(t)], \quad y(\mathbf{x}(t)) = \mathbf{c}^*\mathbf{x}(t).$$

By using this technique for Rössler system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_1 + ax_2 \\ b + (x_1 - c)x_3 \end{pmatrix}$$

one can stabilize UPO embedded into attractor: Fig. 1 shows the simulation of stabilized embedded period-one UPO (here $T \approx 5.88$) for the Rössler system with parameters $a = b = 0.2, c = 5.7$, and for

$$K(t) = 0.3, \quad \mathbf{b}^* = \begin{pmatrix} 0, 1, 0 \end{pmatrix}, \mathbf{c}^* = \begin{pmatrix} 0, 1, 0 \end{pmatrix}.$$

Consider the reduction of the stabilization of UPO to the problem of linear system stabilization.

Carrying out the linearization along UPO with period T

$$\mathbf{x}^\delta(t) = \mathbf{x}(t) - \mathbf{x}_{upo}(t),$$
$$y^\delta(t) = \mathbf{c}^*\mathbf{x}^\delta(t) = y(t) - y_{upo}(t)$$

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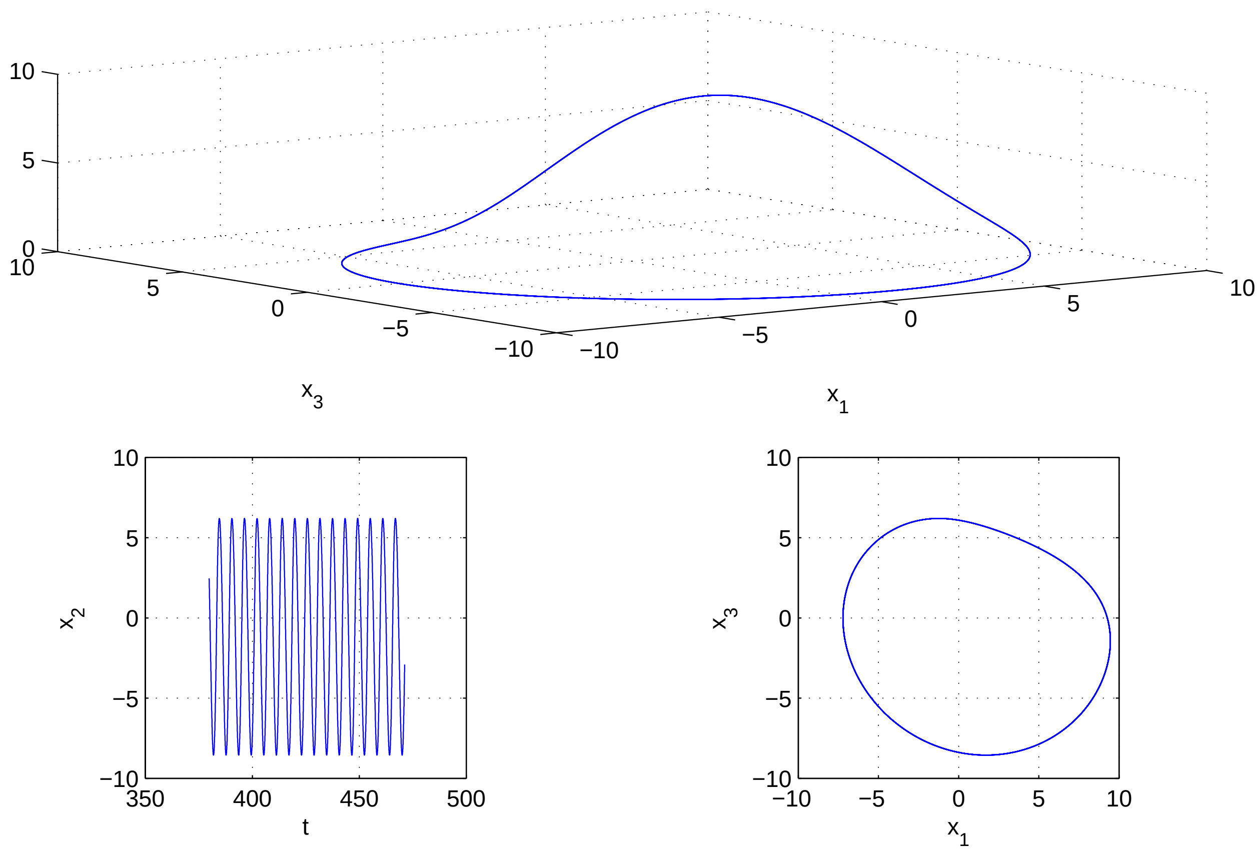

Fig. 1: UPO embedded into chaotic Rössler attractor

we obtain

$$\dot{\mathbf{x}}^{\delta}(t) = \mathbf{A}(t)\mathbf{x}_L^{\delta}(t) + \mathbf{b}K(t)[y^{\delta}(t-T) - y^{\delta}(t)]$$

where linear part is T-periodic — $\mathbf{A}(t+T) = \mathbf{A}(t)$.

Consider fundamental matrix $\mathbf{X}(t)$ of the linear part of the system and apply classical Floquet reduction. Further we use standard notations

$$\mathbf{\Lambda} = \frac{1}{T}\mathrm{Ln}\mathbf{X}(T),\ \mathbf{\Phi}(t) = \mathbf{X}(t)e^{\mathbf{\Lambda}t}\ (\mathbf{\Phi}(t+T) = \mathbf{\Phi}(t)).$$

Introducing new variable $\mathbf{x}_L^{\delta} = \mathbf{\Phi}(t)\mathbf{v}(t)$, one can pass from the system with periodic coefficients to the system with constant coefficients

$$\dot{\mathbf{v}}(t) = \mathbf{\Lambda}\mathbf{v}(t) + \mathbf{C}[\mathbf{v}(t-T) - \mathbf{v}(t)].$$

By the Lyapunov theorem the asymptotical stability of these linear stationary delay systems implies asymptotical stability of periodic orbit of the corresponding original nonlinear delay system. Thus, we arrive at a problem of stabilization of linear stationary systems by the feedback with delay (i.e. to find suitable matrix C).

A great number of publications is devoted to the questions of stabilization of linear stationary systems. A survey on the results of these works can be found, for example, in [5-9]. It should be noted that now the flow of publications on the problem of stabilization (and related topics) of linear stationary systems remains the same.

In the present work new approach to stabilization problem is discussed in the case of three-dimensional linear controllable systems.

2 Three-dimensional problem statement

Consider three-dimensional linear stationary system with scalar input and output:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \quad y(t) = \mathbf{c}x(t). \tag{1}$$

Here $\mathbf{x}(t) \in \mathbb{R}^3$ is a state vector, $u(t) \in \mathbb{R}$ is a control (input), $y(t) \in \mathbb{R}$ is an output, $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{c}$ are real constant (3×3)-matrix, (3×1)-matrix, and (1×3)-matrix, respectively.

We assume that system (1) is controllable and observable.

Introduce a delayed feedback in system (1). Consider two approaches:

$$u(t) = ky(t - \tau) \tag{2}$$

and

$$u(t) = k[y(t - \tau) - y(t)], \tag{3}$$

where $k \neq 0$ and $\tau > 0$ are constants.

A main problem is as follows: to determine values of parameters k and $\tau > 0$ such that system (1) with feedbacks (2) or (3) is asymptotically stable. Closed system (1)–(3) is a linear stationary vector delay differential equation.

In the scalar case when $x(t) \in \mathbb{R}$, A and b are constants and $k = c = 1$ the asymptotic stability of the corresponding delay differential equation (1), (2) is considered in [10].

2.1 Problem reformulation

By nondegenerate linear transformation system (1) can be reduced to canonical form:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = -a_1x_1(t) - a_2x_2(t) - a_3x_3(t) - u(t), \\ y(t) = c_1x_1(t) + c_2x_2(t) + c_3x_3(t), \end{cases} \tag{4}$$

where $a_1, a_2, a_3; c_1, c_2, c_3$ are real numbers.

System (4) with feedbacks (2) or (3) is reduced to the following third-order scalar delay differential equations

$$\begin{aligned} &\dddot{x}(t) + a_3\ddot{x}(t) + a_2\dot{x}(t) + a_1x(t) + \\ &+ kc_3\ddot{x}(t - \tau) + kc_2\dot{x}(t - \tau) + kc_1x(t - \tau) = 0, \end{aligned} \tag{5}$$

$$\begin{aligned} &\dddot{x}(t) + (a_3 - kc_3)\ddot{x}(t) + (a_2 - kc_2)\dot{x}(t) + \\ &+ (a_1 - kc_1)x(t) + kc_3\ddot{x}(t - \tau) + \\ &+ kc_2\dot{x}(t - \tau) + kc_1x(t - \tau) = 0, \end{aligned} \tag{6}$$

respectively. It is known [10] that a necessary and sufficient condition of asymptotic stability of solutions of equations (5) and (6) is negativeness of real parts of all roots of the characteristic quasipolynomials ($z \in \mathbb{C}$)

$$\begin{aligned} F_1(z) = z^3 + a_3z^2 + a_2z + a_1 + \\ + ke^{-\tau z}(c_3z^2 + c_2z + c_1) \end{aligned}, \tag{7}$$

$$\begin{aligned} &F_2(z) = z^3 + (a_3 - kc_3)z^2 + (a_2 - kc_2)z + \\ &+ (a_1 - kc_1) + ke^{-\tau z}(c_3z^2 + c_2z + c_1), \end{aligned} \tag{8}$$

respectively. Thus, a main problem can be reformulated in the following way:

To determine values of parameters k and $\tau > 0$ such that the real parts of all roots of characteristic quasipolynomials (7) or (8) have negative real parts.

2.2 Results formulation

Consider three cases:

1)$c_1 \neq 0; c_2 = c_3 = 0,$

2)$c_2 \neq 0; c_1 = c_3 = 0,$

3)$c_3 \neq 0; c_1 = c_2 = 0.$

Case 1). Without loss of generality, one can assume that

$$c_1 = 1.$$

From the Routh–Hurwitz conditions it follows that a stationary stabilization of system (4) by the feedback

$$u(t) = ky(t) \ (k = \text{const})$$

without delay is possible if and only if

$$a_2 > 0, \ a_3 > 0.$$

Introduce in system (4) the feedback with delay of forms (2) or (3).

Then we have the following theorems:

Theorem 1. *Let in system (4) be*

$$c_1 \neq 0 (c_1 := 1), c_2 = c_3 = 0.$$

Then system (4) is stabilized by feedback (2) if and only if at least one of inequalities sets

a) $a_2 > 0, \ a_3 > 0$

or

b) $a_2 < 0, \ a_3 > 0, \ a_2^2 < 2a_1a_3$ *is satisfied.*

Theorem 2. *Let in system (4) be*

$$c_1 \neq 0 (c_1 := 1), c_2 = c_3 = 0.$$

Then system (4) is stabilized by feedback (3) if and only if

$$a_1 > 0, \ a_3 > 0.$$

Case 2). Without loss of generality, it can be assumed that

$$c_2 = 1.$$

Stationary stabilization of systems (4) by the feedback $u(t) = ky(t)$ is possible if and only if $a_1 > 0, \ a_3 > 0.$

Introduce in system (4) the feedback of forms (2) or (3).

The following theorems can be proved:

Theorem 3. *Let in system (4) be*

$$c_2 \neq 0 (c_2 := 1), c_1 = c_3 = 0.$$

Then system (4) is stabilized by feedback (2) if and only if

$$a_1 > 0, \ a_3 > 0.$$

Theorem 4. *Let in system (4) be*

$$c_2 \neq 0 (c_2 := 1), c_1 = c_3 = 0.$$

In this case if $a_3 > 0$ *and*

$$a)\ 0 < a_1 < \frac{\pi^2}{\pi^2 - 8} a_2 a_3 + \frac{8\pi}{(\pi^2 - 8)\sqrt{\pi^2 - 8}} a_2 \sqrt{a_2}$$
$$a_2 > 0, a_3 > 0$$

or

$$b)\ a_1 > \frac{\pi}{4 - \pi} a_3(-a_2) + \frac{4\sqrt{\pi}}{(4 - \pi)\sqrt{4 - \pi}} (-a_2)\sqrt{-a_2},$$
$$a_2 < 0, a_3 > 0,$$

then system (4) is stabilized by feedback (3).

Case 3). Without loss of generality, it can be assumed

$$c_3 = 1.$$

Stationary stabilization of system (4) by the feedback without delay ($\tau = 0$) is possible if and only if

$$a_1 > 0,\ a_2 > 0.$$

Introduce in system (4) the feedback of forms (2) or (3).

Then the following theorems are valid:

Theorem 5. *Let in system (4) be*

$$c_3 \neq 0 (c_3 := 1),\ c_1 = c_2 = 0.$$

Then system (4) is stabilized by feedback (2) if

$$a) \quad a_1 > 0,\ a_2 > 0$$

or

$$b)\ a_2 < 0,\ a_3 > 0,$$
$$a_1 > \frac{\pi + 3\sqrt{3}}{\pi - 3\sqrt{3}} a_2 \left(a_3 + 6\sqrt{\frac{a_2}{\pi^2 - 27}} \right),$$

or

$$c) \quad a_3 < 0,\ a_2 < \frac{\pi^2 - 27}{(6 - \pi\sqrt{3})^2} a_3^2,$$
$$a_1 > \frac{\pi + 3\sqrt{3}}{\pi - 3\sqrt{3}} a_2 \left(a_3 + 6\sqrt{\frac{a_2}{\pi^2 - 27}} \right).$$

Theorem 6. *Let in system (4) be*

$$c_3 \neq 0 (c_3 := 1),\ c_1 = c_2 = 0.$$

Then system (4) is stabilized by feedback (3) if at least one of inequalities sets:

a) $\qquad a_2 > 0,\ a_3 > 0,$

$$0 < a_1 < a_2 \left(a_3 + \frac{4}{(-\sigma_0)\pi} \sqrt{a_2} \right),$$

or

b) $a_1 > 0,\ a_3 > 0,$

$$a_2 > \frac{\pi + 12(2-\sqrt{3})}{\pi}\frac{a_1}{a_2} - \frac{12\sqrt{a_1 a_3}}{\sqrt{\pi[\pi + 12(2-\sqrt{3})]}},$$

or

$$c) \quad a_1 > \frac{\pi + 3\sqrt{3}}{\pi - \sqrt{3}} a_2 \left(a_3 - 6\sqrt{\frac{a_2}{\pi^2 - 27}}\right),$$

$$\frac{\pi^2}{(6+\pi\sqrt{3})^2} a_3^2 < a_2 < 0$$

is satisfied.

Here in *a*) $\sigma_0 = \min_{\alpha\in[0,2\pi]}(\cos\alpha + \sin\alpha/\alpha)$.

The proofs of Theorems 1-6 are based on the method of D - decomposition [11] of a coefficients space of the characteristic quasipolynomials F_1 and F_2 and on the subsequent use of Rouche's theorem on zeros of analytic functions [12].

3 Conclusion

In the work the problem on stationary stabilization by an output of three-dimensional linear stationary controllable delayed feedback system is considered. The possibilities of stabilization of three-dimensional systems by the delayed feedbacks of forms (2) and (3) are shown. Also possibilities of stabilization of unstable periodic solutions of three-dimensional dynamical is demonstrated.

Theorems 1, 2, 5, and 6 imply that the introduction of delay in a feedback enlarges possibilities of usual stationary stabilization $u(t) = ky(t)$. From Theorem 3 it follows that the domains of system stabilization by a feedback with delay and without delay coincide. Theorem 4 implies that in the case when a system is stabilized by feedback (3) the enlargement of domain of system stabilization by feedback without delay is impossible.

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