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Algorithm for Localizing Chua Attractors Based on the Harmonic Linearization Method.

G.A. Leonov, V.I. Vagaitsev, N.V. Kuznetsov,

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Abstract. The method of harmonic linearization, numerical methods, and the applied bifurcation theory together discover new opportunities for analysis of hidden attractors of control systems. In the present paper these opportunities are demonstrated. New analytical-numerical method based on the above-mentioned technique is discussed. Application of this technique for hidden attractor localization of generalized Chua systems is given.

Keywords: chaotic attractor, Chua system, Chua circuits, hidden attractor, hidden oscillation

1 Introduction

In the works Leonov (2009, 2010) the methods of search of attractors of multidimensional nonlinear dynamic systems with scalar nonlinearity were suggested. In the present work the approach suggested is generalized on the systems of the form

$$\frac{dx}{dt} = Px + \psi(x), \tag{1.1}$$

where P is a constant $n \times n$-matrix, $\psi(x)$ is a continuous vector-function, and $\psi(0) = 0$. It is shown that modification of the used algorithms permits one to fulfil localization of attractor of system (1.1), which were obtained by Chua and his progeny in studying nonlinear electrical circuits with feedback [Chua (1992); Bilotta & Pantana (2008)].

For the search of periodic solution close to harmonic oscillation, we consider matrix K such that the matrix $P_0 = P + K$ has a pair of purely imaginary eigenvalues $\pm i\omega_0 (\omega_0 > 0)$ and the rest of its eigenvalues have negative real parts. Then system (1.1) can be rewritten as

$$\frac{dx}{dt} = P_0 x + \varphi(x), \tag{1.2}$$

where $\varphi(x) = \psi(x) - Kx$.

Since we are interested in attractor of system (1.2), it is natural to introduce a finite sequence of continuous functions $\varphi^0(x), \varphi^1(x), ..., \varphi^m(x)$ in such a way that the graphs of neighboring functions φ^j and φ^{j+1}, in a sense, are slightly differed from each other, the function $\varphi^0(x)$ is small, and $\varphi^m(x) = \varphi(x)$.

In this case the smallness of function $\varphi^0(x)$ permits one to apply and justify the method of harmonic linearization for the system

$$\frac{dx}{dt} = P_0 x + \varphi^0(x), \tag{1.3}$$

if the stable periodic solution $x(t) = x^0(t)$ close to harmonic one is determined. All the points of this stable periodic solution are located in the domain of attraction of stable periodic solution $x(t) = x^1(t)$ of the system

$$\frac{dx}{dt} = P_0 x + \varphi^j(x) \tag{1.4}$$

[1]Nikolay V. Kuznetsov, nkuznetsov239 at gmail.com (correspondence author)
[2]PDF slides http://www.math.spbu.ru/user/nk/PDF/Hidden-attractor-localization-Chua-circuit.pdf

with $j = 1$ or in the change from system (1.3) to system (1.4) with $j = 1$ we observe the instability bifurcation destroying periodic solution. In the first case it is possible to find $x^1(t)$ numerically, starting a trajectory of system (1.4) with $j = 1$ from the initial point $x^0(0)$.

Starting from the point $x^0(0)$, after transient process the computational procedure reaches to the periodic solution $x^1(t)$ and computes it. In this case the interval $(0, \widetilde{T})$, on which the computation is carried out, must be sufficiently large.

After the computation of $x^1(t)$ it is possible to obtain the following system (1.4) with $j = 2$ and to organize a similar procedure of computing the periodic solution $x^2(t)$, starting a trajectory, which with increasing t approaches to periodic trajectory $x^2(t)$, from the initial point $x^2(0) = x^1(\widetilde{T})$.

Proceeding this procedure and computing $x^j(t)$, using trajectories of system (1.4) with the initial data $x^j(0) = x^{j-1}(\widetilde{T})$, we either arrive at periodic solution of system (1.4) with $j = m$ (i.e. at original system (1.2)) either observe, at a certain step, the instability bifurcation destroying periodic solution.

In the described procedure the simplest and the most natural class of functions φ^j are the following functions: $\varphi^0(x) = \varepsilon\varphi(x), \varphi^1(x) = \varepsilon^1\varphi(x), ..., \varphi^{m-1}(x) = \varepsilon^{m-1}\varphi(x), \varphi^m(x) = \varphi(x)$, where ε is a "classical" small positive parameter and, for example, $\varepsilon^j = j/m, j = 1, ..., m$.

For system (1.3) with such function $\varphi^0(x)$ it turns out that it is possible to justify rigorously the method of harmonic linearization and to determine the initial conditions, for which system (1.3) has a stable periodic solution close to harmonic one.

2 Poincare map for harmonic linearization

Suppose, for the vector-function $\varphi(x)$ the estimate

$$|\varphi(x') - \varphi(x'')| \le L|x' - x''|, \quad \forall\, x', x'' \in \mathbb{R}^n \tag{2.5}$$

is satisfied. By nonsingular linear transformation system (1.3) can be reduced to the form

$$\begin{aligned} &\dot{x}_1 = -\omega_0 x_2 + \varepsilon\varphi_1(x_1, x_2, x_3),\ \dot{x}_2 = \omega_0 x_1 + \varepsilon\varphi_2(x_1, x_2, x_3) \\ &\dot{x}_3 = Ax_3 + \varepsilon\varphi_3(x_1, x_2, x_3) \end{aligned} \tag{2.6}$$

Here A is a constant $(n-2) \times (n-2)$ matrix, all eigenvalues of which have negative real parts, φ_3 is an $(n-2)$-dimensional vector-function, φ_1, φ_2 are certain scalar functions. Without loss of generality, it may be assumed that for the matrix A there exists positive number α such that

$$x_3^*(A + A^*)x_3 \le -2\alpha|x_3|^2, \quad \forall\, x_3 \in \mathbb{R}^{n-2} \tag{2.7}$$

Here $*$ is a transposition operation.

In a phase space of system (2.6) we introduce the following set

$$\Omega = \{|x_3| \le D\varepsilon, \quad x_2 = 0, \quad x_1 \in [a_1, a_2]\}.$$

Here D, a_1, a_2 are certain positive numbers, which will be determined below. From condition (2.5) and the form of system (2.6) for solutions with initial data from Ω we obtain the following relations

$$\begin{aligned} &x_1(t) = \cos(\omega_0 t)x_1(0) + O(\varepsilon),\ x_2(t) = \sin(\omega_0 t)x_1(0) + O(\varepsilon), \\ &x_3(t) = \exp(At)x_3(0) + O(\varepsilon) \end{aligned} \tag{2.8}$$

From formulas (2.8) it follows that for any point $x_1(0), x_2(0) = 0, x_3(0)$, belonging to Ω, there exists a number

$$T = T(x_1(0), x_3(0)) = 2\pi/\omega_0 + O(\varepsilon)$$

such that the relation

$$x_1(T) > 0, \quad x_2(T) = 0, \tag{2.9}$$

is satisfied and the conditions $\forall t \in (0, T)$

$$x_1(t) > 0, \ x_2(t) = 0$$

are not satisfied.

Construct a Poincare map F of the set Ω for the trajectories of system (2.6):

$$F \begin{Vmatrix} x_1(0) \\ 0 \\ x_3(0) \end{Vmatrix} = \begin{Vmatrix} x_1(T) \\ 0 \\ x_3(T) \end{Vmatrix}. \tag{2.10}$$

Introduce the describing function

$$\Phi(a) = \int\limits_0^{2\pi/\omega_0} \Big[\varphi_1 \left((\cos \omega_0 t) a, (\sin \omega_0 t) a, 0 \right) \cos \omega_0 t +$$
$$+ \varphi_2 \left((\cos \omega_0 t) a, (\sin \omega_0 t) a, 0 \right) \sin \omega_0 t. \Big] dt$$

From estimates (2.8) and condition on nonlinearity (2.5) for solutions of system (2.6) we obtain the following relations

$$\begin{aligned} &|x_3(T)| \leq D\varepsilon \\ &x_1^2(T) - x_1^2(0) = 2x_1(0)\varepsilon\Phi(x_1(0)) + O(\varepsilon^2) \end{aligned} \tag{2.11}$$

Theorem 1 *If the inequalities*

$$\Phi(a_1) > 0, \quad \Phi(a_2) < 0 \tag{2.12}$$

are satisfied, then for small enough $\varepsilon > 0$ the Poincare map F of the set Ω into itself is as follows

$$F\Omega \subset \Omega.$$

From this theorem and the Brouwer fixed point theorem we have the following

Corollary 1 *If the inequalities (2.12) are satisfied, then for small enough $\varepsilon > 0$ system (2.6) has a periodic solution with the period*

$$T = \frac{2\pi}{\omega_0} + O(\varepsilon)$$

This solution is stable in the sense that its neighborhood Ω is mapped into itself: $F\Omega \subset \Omega$.

Theorem 2 *If the inequalities (2.12) have opposite sign, then for small enough $\varepsilon > 0$ Poincare map (2.10) of the set Ω has hyperbolic character: there occurs the contraction with respect to x_3 (the estimate (2.11) is satisfied) and the stretching with respect to x_1 : $Fa_1 < a_1, \ Fa_2 > a_2$.*

3 Algorithm for determination of stable periodic solutions of generating systems for systems with scalar nonlinearity

Consider the case of scalar nonlinearity. In this case system (1.3) takes the form

$$\frac{dx}{dt} = P_0 x + q\varepsilon\varphi(r^* x), \tag{3.13}$$

where $P_0 = P + kqr^*$ is a constant $n \times n$-matrix, r and q are n-dimensional vectors, $\varphi(\sigma)$ is a continuous scalar function ($\varphi(0) = 0$), k is a coefficient of harmonic linearization (which is chosen in such a way that matrix P_0 has a pair of purely imaginary eigenvalues $\pm i\omega_0$, $\omega_0 > 0$ and the rest of its eigenvalues have negative real parts).

By nonsingular linear transformation, system (3.13) can be reduced to the form

$$\begin{aligned} \dot{x}_1 &= -\omega_0 x_2 + b_1\varepsilon\varphi(x_1 + c^* x_3) \\ \dot{x}_2 &= \omega_0 x_1 + b_2\varepsilon\varphi(x_1 + c^* x_3) \\ \dot{x}_3 &= A x_3 + b\varepsilon\varphi(x_1 + c^* x_3). \end{aligned} \tag{3.14}$$

Here A is a constant $(n-2) \times (n-2)$-matrix, all eigenvalues of which have negative real parts, b and c are $(n-2)$-dimensional vectors, b_1 and b_2 are certain real numbers.

Now we write the transfer function of system (3.13):

$$W_1(p) = r^*(P_0 - pI)^{-1} q = \frac{\eta p + \theta}{p^2 + \omega_0^2} + \frac{R(p)}{Q(p)}, \tag{3.15}$$

and the transfer function of system (3.14):

$$W_2(p) = \frac{-b_1 p + b_2\omega_0}{p^2 + \omega_0^2} + c^*(A - pI)^{-1} b. \tag{3.16}$$

Here η, θ are certain real numbers, $Q(p)$ is a stable polynomial of degree $(n-2)$, $R(p)$ is a polynomial of degree smaller than $(n-2)$. Suppose, the polynomials $R(p)$ and $Q(p)$ have no common roots. By equivalence of systems (3.13) and (3.14) the transfer functions of these systems coincide. This implies the relations

$$\eta = -b_1, \quad \theta = b_2\omega_0, \quad \frac{R(p)}{Q(p)} = c^*(A - pI)^{-1} b. \tag{3.17}$$

In the works Leonov (2009, 2010) the following theorems are obtained.

Theorem 3 *If the conditions*

$$\Phi(a) = 0, \eta\frac{\mathrm{d}\Phi(a)}{\mathrm{d}a} > 0$$

are satisfied, then for small enough $\varepsilon > 0$ system (3.13) with transfer function (3.15) has T-periodic solution such that

$$r^* x(t) = a\cos(\omega_0 t) + O(\varepsilon), \quad T = \frac{2\pi}{\omega_0} + O(\varepsilon).$$

This periodic solution is stable in the sense of that there exists its certain ε-neighborhood such that all solutions with the initial data from this ε-neighborhood remain in it with increasing time t.

Theorem 4 *If the conditions*

$$\Phi(a) = 0, \eta \frac{\mathrm{d}\Phi(a)}{\mathrm{d}a} < 0$$

are satisfied, then for small enough $\varepsilon > 0$ *system (3.13) with transfer function (3.15) has the solution of the form*

$$r^* x(t) = a\cos(\omega_0 t) + O(\varepsilon), \quad t \in \big[0, \frac{2\pi}{\omega_0}\big]$$

and in the neighborhood of this solution the behavior of trajectories has hyperbolic character.

Theorems 3 and 4 coincide with the procedure of search of stable and unstable periodic solutions by means of the harmonic linearization method. Khalil (2002).

4 Algorithm for hidden attractor localization of Chua system

The systems of differential equations, describing the behavior of Chua circuits Chua (1992); Bilotta & Pantana (2008), are three-dimensional dynamical systems with scalar nonlinearity. Let us apply the above mentioned method to generalized Chua system, represented in dimensionless quantities.

$$\begin{aligned} \dot{x} &= \alpha(y - x) - f(x) \\ \dot{y} &= (x - y + z) \\ \dot{z} &= -(\beta y + \gamma z) \end{aligned} \tag{4.18}$$

Here the function

$$\begin{aligned} f(x) = \quad & \alpha\big(bx + \frac{1}{2}(a-b)(|x+1| - |x-1|) + \\ & + \frac{1}{2}(s-a)(|x+\delta_0| - |x-\delta_0|)\big) \end{aligned} \tag{4.19}$$

describes a nonlinear element of system, $\alpha, \beta, \gamma, a, b$ are parameters of the classical Chua system, and δ_0 and s are parameters that determine the stability of zero equilibrium.

Consider an example $s = -0.31, \delta_0 = 0.2, a = 0.1691, b = -0.4768, \alpha = -1.398, \beta = -0.0136, \gamma = -0.0297$. In this case the procedure described can be used for numerical localization of attractor of system.

For this purpose we introduce the coefficient k, the parameter ε and transform system (4.18) to the form (2.6). Then we construct solutions of varied system (4.18) (transformed to the form (2.6)) with the nonlinearity $\varepsilon(f(x) - kx)$ by means of sequential increasing ε with the step 0.1 from the value $\varepsilon_1 = 0.1$ to $\varepsilon_{10} = 1$.

Firstly, we compute the coefficient of harmonic linearization $k = -0.3067$ and the value of "start" frequency $\omega_0 = -0.6436$. By theorem 3 we obtain initial data $x(0) = -1.1061$, $y(0) = 0$, $z(0) = 0$ for the first step of multistage procedure of construction of solutions. For $\varepsilon_1 = 0.1$ after transient process the computational procedure arrives at a periodic solution close to harmonic one. Further, with increasing parameter ε this periodic solution close to harmonic one is transformed into chaotic attractor of the type "double-scroll", Chua (1992); Bilotta & Pantana (2008).

In the classical Chua system there occurs classical excitation of oscillations in the case when a trajectory from the neighborhood of unstable zero equilibrium reaches the attractor. In this system, in despite of the existence of stable zero equilibrium, the described procedure also allows one to go on "hidden" attractor by means of sequential approximations. In Fig. 1 are shown the projections of solutions on the plane $\{x, y\}$ for the values $\varepsilon_1 = 0.1$, $\varepsilon_3 = 0.3$, $\varepsilon_7 = 0.7$, and $\varepsilon_{10} = 1$, respectively.

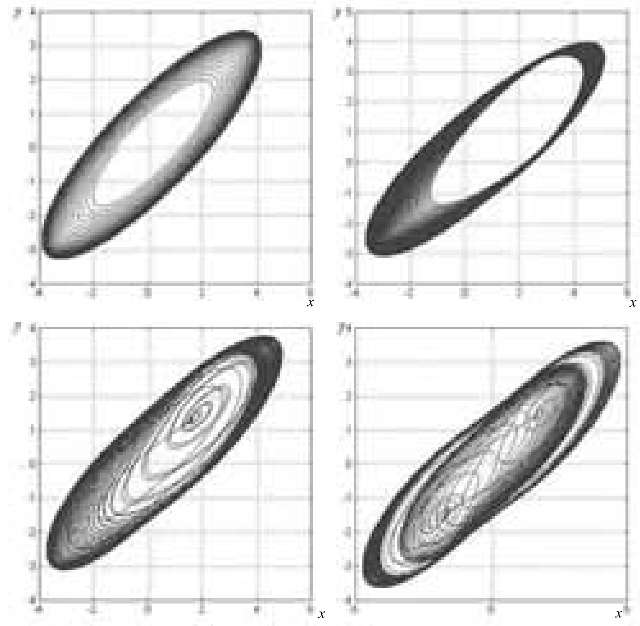

Figure 1: $\varepsilon = 0.1$, $\varepsilon = 0.3$, $\varepsilon = 0.7$, $\varepsilon = 1$

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