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MATHEMATICS

On Problems of Aizerman and Kalman

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Abstract—The problems of the stability of nonlinear control systems posed by Aizerman and Kalman have stimulated the development of methods for detecting hidden periodic oscillations in multidimensional dynamical systems.

In the 1950s, Pliss developed an analytical method for detecting periodic oscillations in third-order systems satisfying the generalized Routh–Hurwitz conditions.

It has turned out that this generalized method of Pliss can be regarded as a special version of the describing function method in the critical case. Being combined with computational procedures based on applied bifurcation theory, this method makes it possible to obtain new classes of systems for which the conjectures of Aizerman and Kalman are false.

The known approaches to constructing counterexamples to Aizerman's and Kalman's conjectures proposed by Fitts, Barabanov, and Llibre are reviewed. A new effective analytical-numerical method for constructing such counterexamples is presented. The method is based on combining the classical theory of small parameter, bifurcation theory, and the method of harmonic linearization. It is applied to numerically construct a series of counterexamples to Aizerman's and Kalman's conjectures.

Keywords: Aizerman conjecture, Kalman conjecture, absolute stability, periodic solutions, hidden oscillations.

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1. INTRODUCTION

The problems of the stability of nonlinear control systems posed by Aizerman [1] and Kalman [2] have stimulated the development of methods for detecting hidden periodic oscillations in multidimensional dynamical systems.

In the 1950s, Pliss [3] developed an analytical method making it possible to reveal periodic oscillations in third-order systems satisfying the generalized Routh–Hurwitz conditions. Later, this method was extended to multidimensional systems [4–7].

It turned out that the generalized method of Pliss can be regarded as a special version of the describing function method in the critical case [8]. Being combined with computational procedures based on applied bifurcation theory, this method has made it possible to obtain new classes of systems for which Aizerman's and Kalman's conjectures are false. This paper is devoted to describing the corresponding procedures.

Consider the system

$$\frac{dx}{dt} = Px + q\varphi(r^*x), \quad x \in \mathbb{R}^n, \tag{1}$$

where P is a constant $n \times n$ matrix, q and r are constant n-vectors, $*$ denotes the transposition operation, and $\varphi(\sigma)$ is a piecewise differentiable continuous scalar function such that $\varphi(0) = 0$. Suppose that, for any $k \in (\mu_1, \mu_2)$, the zero solution of system (1) with $\varphi(\sigma) = k\sigma$ is globally asymptotically stable (i.e., the zero solution is Lyapunov stable and any solution of system (1) tends to zero as $t \longrightarrow \infty$, or, is other words, the zero solution is a global attractor of system (1)).

This notation for nonlinear dynamical systems with one nonlinearity is traditionally used in the theory of absolute stability of nonlinear control systems [9].

In 1949, Aizerman [1] stated the conjecture that all systems of the form (1) satisfying the condition

$$k_1\sigma < \varphi(\sigma) < k_2\sigma, \quad \sigma \neq 0, \tag{2}$$

are globally stable.

Necessary conditions for absolute stability obtained in [5–7] disprove this conjecture.

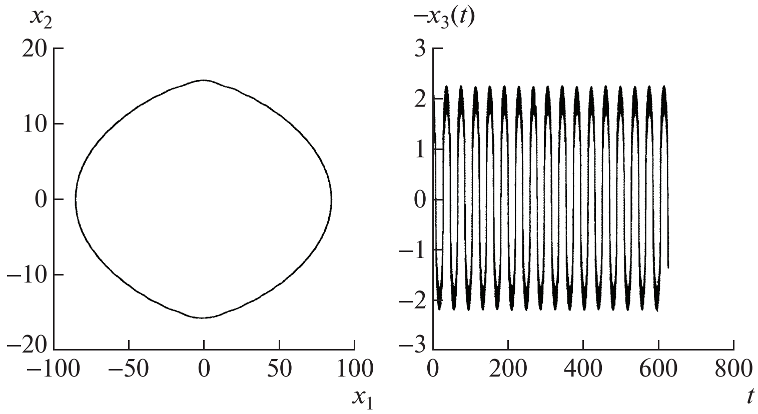

Fig. 1. The projection of the trajectory of system (5) with initial data $x_1(0) = 85.1189$, $x_2(0) = 0.9222$, $x_3(0) = -2.0577$, $x_4(0) = -2.6850$ on the plane (x_1, x_2).

In 1957, Kalman stated a similar conjecture [2] under a more severe constraint; namely, he conjectured that if the function $\varphi(\sigma)$ satisfies the condition

$$\mu_1 < \varphi'(\sigma) < \mu_2 \tag{3}$$

at its differentiability points, then system (1) is globally stable.

It is well known that this conjecture is true for $n = 2$ and $n = 3$ [7].

The only counterexample to this conjecture extensively cited in the literature was given by Fitts in [10], where a computer modeling of system (1) for $n = 4$ with the transfer function

$$W(p) = \frac{p^2}{[(p+\beta)^2 + 0.9^2][(p+\beta)^2 + 1.1^2]} \tag{4}$$

and the cubic nonlinearity $\varphi(\sigma) = k\sigma^3$ was performed.

Below, we describe a computer modeling of the Fitts system. Reconstructing the system from transfer function (4) for $\beta = 0.01$ and $k = 10$, we obtain

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -0.9803x_1 - 0.0404x_2 - 2.0206x_3 - 0.0400x_4 + \varphi(-x_3) \end{aligned} \tag{5}$$

with the nonlinearity $\varphi(\sigma) = 10\sigma^3$.

Modeling this system with initial data $x_1(0) = 85.1189$, $x_2(0) = 0.9222$, $x_3(0) = -2.0577$, and $x_4(0) = -2.6850$, we obtain a "periodic" solution (see Fig. 1).

Thus, Fitts' experiments have revealed a periodic solution of system (1) for some values of the parameters β and k. However, it was shown in [11, 12] that the results of experiments performed by Fitts for some of the parameter values $\beta \in (0.572, 0.75)$ are incorrect.

In [12], a proof of the existence of a system of the form (1) for $n = 4$ for which Kalman's conjecture is false was presented; this is an existence theorem and needs to be carefully checked.

In [12], the system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_4, \\ \dot{x}_3 &= x_1 - 2x_4 - \varphi(x_4), \\ \dot{x}_4 &= x_1 + x_3 - x_4 - \varphi(x_4) \end{aligned} \tag{6}$$

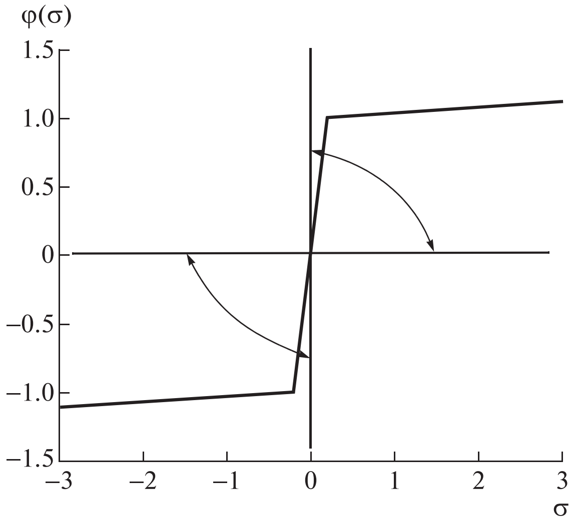

Fig. 2. The graph of $\varphi(\sigma)$ at $i = 5$ and the stability sector.

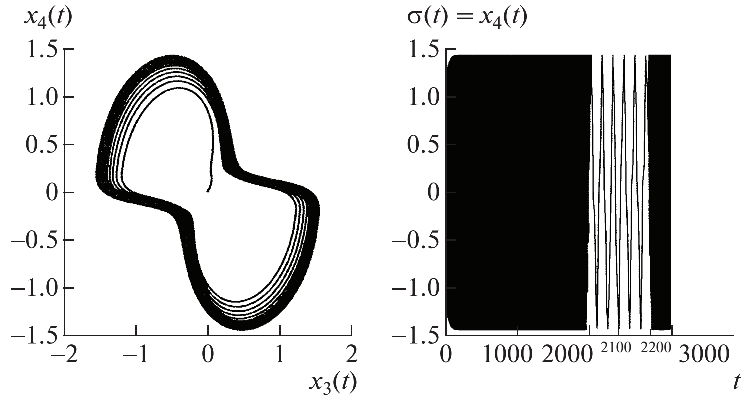

Fig. 3. The projection of the trajectory of system (5) with initial data $x_1(0) = 0$, $x_2(0) = 1/2$, $x_3(0) = 0$, $x_4(0) = 0$ on the plane (x_3, x_4).

with the nonlinearity $\varphi(\sigma) = \mathrm{sgn}(\sigma)$ was considered. We change the form of nonlinearity as

$$\varphi(\sigma) = \begin{cases} 5\sigma, & \forall|\sigma| \le \frac{1}{5} \\ \mathrm{sgn}(\sigma) + \frac{1}{25}\left(\sigma - \mathrm{sgn}(\sigma)\frac{1}{5}\right), & \forall|\sigma| > \frac{1}{5}. \end{cases} \tag{7}$$

The graph of this nonlinearity is shown in Fig. 2.

Let us find a periodic solution to system (6) with nonlinearity (7). Modeling this system with initial data $x_1(0) = 0$, $x_2(0) = 1/2$, $x_3(0) = 0$, and $x_4(0) = 0$, we obtain the periodic solution shown in Fig. 3.

In [13–15], mistakes made in [12] were pointed out, thus, in [13], we read: "He tried to prove that this system and systems close to this have a periodic orbit. But his arguments are not complete, and we checked numerically that in the region where he tries to find the periodic orbit all the solutions have ω-limit equal to the origin"; in [14], we read: "In 1988 Barabanov gave ideas for constructing a class C^1 MY-SYSTEM (Markus–Yamabe system) in 4 dimensions with a nonconstant periodic orbit,—and hence a counterexample to MYC (Markus–Yamabe Conjecture) in $\mathbb{R}^4$. But the details of his paper were in some doubt";

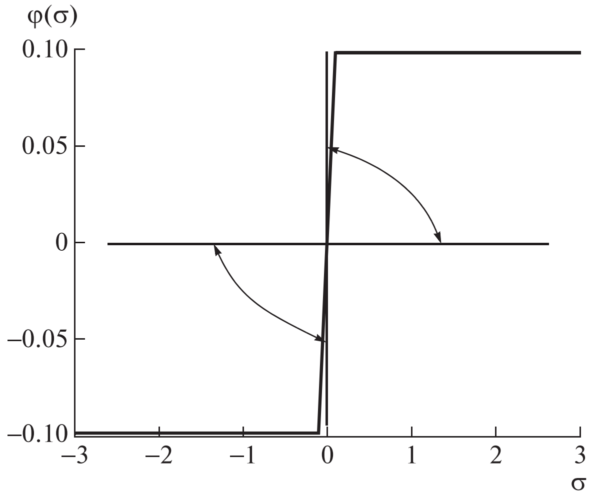

Fig. 4. The graph of $\varphi(\sigma)$ for $i = 5$ and the stability sector.

in [15], it was written that, "in 1988, Barabanov made an attempt to construct a counterexample to the Markus–Yamabe theorem in $\mathbb{R}^n$ for $n \geq 4$, and that, recently, mistakes in his paper have been found."

In [13], an attempt to overcome difficulties which had emerged in [12] by analytical-numerical methods was made.

Let us model the system proposed in [13], namely,

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_4, \\ \dot{x}_3 &= x_1 - 2x_4 - \frac{9131}{900}\varphi(x_4), \\ \dot{x}_4 &= x_1 + x_3 - x_4 - \frac{1837}{180}\varphi(x_4), \end{aligned} \tag{8}$$

where

$$\varphi(\sigma) = \begin{cases} \sigma, & \forall|\sigma| \leq \dfrac{900}{9185} \\ \operatorname{sgn}(\sigma)\dfrac{900}{9185}, & \forall|\sigma| > \dfrac{900}{9185}. \end{cases} \tag{9}$$

The graph of such a nonlinearity is shown in Fig. 4.

Modeling the solution of system (8) with initial data $x_1(0) = 0$, $x_2(0) = 1/2$, $x_3(0) = 0$, and $x_4(0) = 0$, we obtain the periodic solution shown in Fig. 5.

It should be mentioned that, in the examples considered above, the search procedure for systems with periodic solutions is empirical. The search for systems themselves, as well as for their solutions, is very time- and labor-consuming.

In this paper, we propose a constructive algorithm for obtaining classes of systems (1) for which Kalman's conjecture is false.

2. AN ANALYTICAL-NUMERICAL METHOD FOR CALCULATING PERIODIC SOLUTIONS

The analytical-numerical procedure for calculating periodic solutions used in this paper follows that proposed in [8, 16–18].