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Localization of the Attractor of the Differential Equations for the Solar Wind–Magnetosphere–Ionosphere Model

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Abstract. The problem of localization of attractor of “solar wind-magnetosphere-ionosphere” (WINDMI) three-dimensional model has stimulated further development of method of conical nets. In the paper the development of this method is carried out and analytical localization of the attractor of WINDMI model is performed.

Keywords: frequency method of positively invariant cone grids, attractor localization, WINDMI model, solar wind-magnetosphere-ionosphere.

1 Introduction: method of conical nets

The problem of localization of the attractor of a three-dimensional solar wind-magnetosphere-ionosphere model
(WINDMI model [1,2]) has stimulated the development of the method of conical nets to solve this problem. In this communication, we describe this development and obtain analytical estimates for the attractor of the model.

Consider a system

$$\frac{dx}{dt} = Px + q\varphi(r^*x), \quad x \in \mathbb{R}^n, \tag{1}$$

where P is a constant singular $(n \times n)$ matrix, q and r are n-dimensional vectors, $*$ is the transposition operation, and $\varphi(\sigma)$ is a differentiable scalar function that satisfies the sector conditions

$$\varphi(\sigma) < \mu(\sigma - \alpha), \quad \forall \sigma \geq \alpha, \tag{2}$$

$$\mu(\sigma - \beta) < \varphi(\sigma), \quad \forall \sigma \leq \beta. \tag{3}$$

Here μ, α, β are some numbers such that

$$\mu > 0, \quad \alpha < \beta.$$

We will assume that the pair (P, q) is fully controllable, the pair (P, r) is fully observable, and system (1) has only one equilibrium state.

Theorem 1. Let $r^*q \leq 0$. There exists a number $\lambda > 0$ such that the matrix $P + \lambda I$ has $(n-1)$ eigenvalues with negative real parts and the following inequality holds:

$$\operatorname{Re} W(i\omega - \lambda) + \mu |W(i\omega - \lambda)|^2 \leq 0, \quad \forall \omega \in \mathbb{R}. \tag{4}$$

Then, for any solution $x(t)$ of system (1), there exists a number T such that

$$r^*x(t) \in (\alpha, \beta), \quad \forall t > T. \tag{5}$$

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Here, $W(p) = r^*(P - pI)^{-1}q$ is the transfer function of system (1), I is a unit $(n \times n)$ matrix, and i is the imaginary unit.

Below, we present a scheme for proving Theorem 1. The existence of s symmetric $n \times n$ matrix H for which the following conditions are met follows from condition (4) (a detailed proof is a available in [3]):

1) the matrix H has one negative and $(n-1)$ positive eigenvalues;

2) for all $z \in \mathbb{R}^n$ and $\xi \in \mathbb{R}$, the following tqnequality holds:

$$2z^*H[(P + \lambda I)z + q\xi] + r^*z(r^*z - \mu^{-1}\xi) \leq 0. \tag{6}$$

Note that the equility

$$2Hq = \mu^{-1}r.$$

follows from (6).

It follows from this and from the condition of the theorem than $r^*H^{-1}r = 2\mu r^*q \leq 0$ and, hence [3],

$$z^*Hz \geq 0, \quad \forall z \in \{z^*r = 0\}. \tag{7}$$

Let $d \in \mathbb{R}^n$ such that $d \neq 0$ and $Pd = 0$, $r^*d = 1$. It then follows from (6)that $d^*Hd < 0$.

Let us now consider Lyapunow functions

$$\begin{aligned} V_1(x) &= V(x - \alpha d) = (x - \alpha d)^*H(x - \alpha d), \\ V_2(x) &= V(x - \beta d) = (x - \beta d)^*H(x - \beta d). \end{aligned}$$

It follows from Eqs. (2), (3) and (6) that

$$\begin{aligned} \dot{V}_1(x(t)) + 2\lambda V_1(x(t)) < 0 \quad &\text{for } r^*x(t) > \alpha, \\ \dot{V}_2(x(t)) + 2\lambda V_2(x(t)) < 0 \quad &\text{for } r^*x(t) < \beta. \end{aligned}$$

It follows this and from (7) that the set

$$\begin{aligned} \Omega(\alpha) &= \{(x - \alpha d)^*H(x - \alpha d) < 0, \quad r^*x \geq \alpha\}, \\ \Omega(\beta) &= \{(x - \beta d)^*H(x - \beta d) < 0, \quad r^*x \leq \beta\}. \end{aligned}$$

are positively invariant [3].It is easy to see that the closures $\overline{\Omega}(\alpha)$ and $\overline{\Omega}(\beta)$. In this case, the boundaries $\partial\Omega(\alpha)$ and $\partial\Omega(\beta)$ contain no whole trajectories and are almost everywhere transversal to the vector field of system (1). These boundsries form a continuum set of surfaces (conical net) in the phase space of system (1) that is shown in Fig. 1 The structure constructed here “drives in” any solution in the set $\Omega(\alpha) \cap \Omega(\beta)$ as the

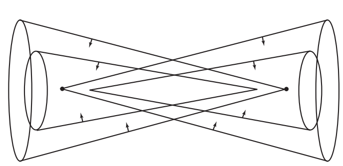

Figure 1: Conical net.

time t increases. The latter proves the theorem.

Note that the estimate obtained is unimprovable in the class of nonlinearities under consideration, because if $\varphi(\sigma) = 0$ for all $\sigma \in (\alpha, \beta)$, then $x = \nu d$ for $\nu \in (\alpha, \beta)$ is a stationary solution of the system.

2 Solar wind-magnetosphere-ionosphere model (WINDMI) attractor localization

Consider the WINDMI model

$$\dddot{x} + b\ddot{x} + c_1\dot{x} + \varphi(x) = 0,\ \ \varphi(x) = (c_2 + c_3\tanh(x)),$$
$$b > 0, c_1 > 0, c_3 > c_2 > 0. \tag{8}$$

It describes the energy flow dynamics in the solar wind-magnetosphere-ionosphere model and is used to analyze geomagnetic storms and substorms [1,2].

Let us apply the theory developed above to investigate Eq. (8). Here, the maximum value of coefficient μ can be calculated [6] from the formula

$$\mu = \left[\begin{array}{ll} \dfrac{b}{3}\left(c_1 - \dfrac{2}{9}b^2\right), & b^2 \leq 3c_1 \\ \dfrac{b}{3}\left(c_1 - \dfrac{2}{9}b^2\right) + 2\left(\dfrac{b^2}{9} - \dfrac{c_1}{3}\right)^{3/2}, & b^2 > 3c_1 \end{array}\right.$$

Let as define a point $x_0 > 0$ such that $\varphi'(x_0) = \mu$,

$$x_0 = \operatorname{argcosh}\sqrt{\frac{c_3}{\mu}} \qquad \text{for } \frac{c_3}{\mu} > 1,$$

and, given the equality $\varphi'(x_0) = \varphi'(-x_0) = \mu$, we will obtain constants (2) and (3) for the nonlinearity $\varphi(x)$ (Fig. 2). Denote

$$\alpha_0 = -\frac{\varphi(x_0)}{\mu} + x_0, \quad -\frac{\varphi(-x_0)}{\mu} - x_0 = \beta_0.$$

According to (2), (3) and (5), we will obtain

$$\alpha_0 \leq \lim_{t\to+\infty}\inf x(t), \ \lim_{t\to+\infty}\sup x(t) \leq \beta_0. \tag{9}$$

To estimate $\dot{x}$ and $\ddot{x}$ note that, in view of the conditions for the coefficients being positive, the characteristic polynomial of the linear part of Eq. (8) has eigenvalues to the left of the imaginary axis and the nonlinearity $\varphi(x)$ is limited:

$$|\varphi(x)| \leq c_2 + c_3.$$

Using the Cauchy formula, estimates can then be obtained [7–9] for $|\dot{x}(t)|$

$$y_{att} = \left[\begin{array}{ll} \lim\limits_{t\to+\infty}\sup|\dot{x}(t)| \leq \dfrac{1}{c_1}(c_2 + c_3), & b^2 \geq 4c_1, \\ \lim\limits_{t\to+\infty}\sup|\dot{x}(t)| \leq \dfrac{2}{b\sqrt{c_1 - \frac{b^2}{4}}}(c_2 + c_3), & b^2 < 4c_1 \end{array}\right.$$

and for $|\ddot{x}(t)|$

$$\lim_{t\to+\infty}\sup|\ddot{x}(t)| \leq \frac{(c_1 y_{att} + c_2 + c_3)}{b}.$$

Together with estimate (9) for $|x(t)|$, localize the attractor of Eq. (8).

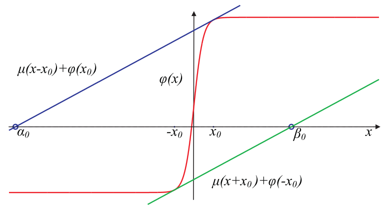

Figure 2: Nonlinearity estimation

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