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Nonlinear analysis of the Costas loop and phase-locked loop with squarer

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Draft [1] [2] [3]

Abstract: This work is devoted to the nonlinear analysis of the Costas loop and phase-locked loop with squarer. By using the special asymptotical methods for analysis of high-frequency harmonic and impulse oscillations, integro-differential and differential equations for the Costas loop and phase-locked loop with squarer are derived. The analytical methods for calculation of phase detector characteristics are suggested. For high-frequency clock oscillators new classes of such characteristics are described.

Keywords: Phase detector characteristics, nonlinear analysis, phase-locked loop, PLL, squarer, Costas loop, simulation

1 Introduction

The phase-locked loop (PLL) and Costas loop are widely used [1–3] in modern telecommunication and various devices of digital information processing [4–6]. Methods of analysis of these systems are well developed by engineers and considered in many publications (see e.g. [7–10]), but problems of construction of adequate nonlinear models and carrying out nonlinear analysis of such models are still far from being resolved [11–12] and require using special methods of qualitative theory of dynamical systems [13–23].

In the works [12, 16–23] description technique of PLL on three levels is suggested:

1) on the level of electronic realizations,

2) on the level of phase and frequency relations between inputs and outputs in block-diagrams,

3) on the level of differential and integro-differential equations.

The second level, involving the asymptotical analysis of high-frequency oscillations, is necessary for the correct derivation of equations and the passage onto the third level of description. For example, in PLL theory the fundamental notion of phase detector is formed exactly on the second level of description. In such consideration, *a phase detector characteristic depends on the class of considered oscillations*. While in the classical PLL the multipliers of oscillation are used, for harmonic oscillations the phase detector characteristic is also harmonic, and for impulse oscillations (for the same electronic realization of feedback loop) this characteristic is a continuous piecewise-linear periodic function [16–18].

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[2] PDF slides http://www.math.spbu.ru/user/nk/PDF/Nonlinear-analysis-of-Phase-locked-loop-PLL.pdf

[3] Extended material:

Leonov G.A., Kuznetsov N.V., Yuldahsev M.V., Yuldashev R.V., Analytical method for computation of phase-detector characteristic, IEEE Transactions on Circuits and Systems Part II, vol. 59, num. 10, 2012, pp. 633-637 (doi:10.1109/TCSII.2012.2213362)

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In the present work the development of the above-mentioned technique for analysis of PLL with squarer and phase quadrature components is being continued [4,9]. Here for standard electronic realizations the phase detector characteristics are computed and the differential equations are derived. From this consideration the essential conclusion is that PLL with integrator and harmonic oscillators tunes to a double frequency of the master oscillator, PLL with clock oscillators — to a frequency of the master oscillator, the Costas loop with harmonic oscillators tunes to a frequency of the master oscillator, and the Costas loop with clock oscillators — to a half frequency of the master oscillator.

2 A system with squarer

Consider a block-diagram of phase-locked loop with squarer (Fig. 1)

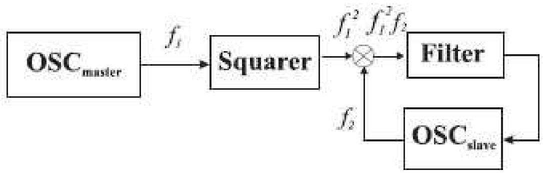

Figure 1: System with squarer

Here the **OSC**$_{\text{master}}$ is a master oscillator, the **Squarer** is an ideal squarer, the **OSC**$_{\text{slave}}$ is a slave oscillator (voltage-controlled oscillator), and $\bigotimes$ is a multiplier.

Consider first the case when the master and slave oscillators generate "almost harmonic oscillations" $f_1(t)$ and $f_2(t)$

(1) $$f_j(t) = A_j \sin(\omega_j(t)t + \Psi_j).$$

Here $A_j > 0, \Psi_j$ are certain constants, $j = 1, 2$.

Block $\bigotimes$ is a multiplier of oscillations: an output of this block is the oscillation $f_1^2(t)f_2(t)$, where $f_1^2(t)$ and $f_2(t)$ are inputs. For output of block **Squarer** we have $f_1(t)^2$, where $f_1(t)$ is an input of the block.

The relation between the input $\xi(t)$ and the output $\sigma(t)$ of linear filter **Filter** is as follows

$$\sigma(t) = a\xi(t) + \alpha(t) + \int_0^t \gamma(t-\tau)\xi(\tau)d\sigma.$$

Here a is a certain number, $\gamma(t)$ is an impulse transition function, and $\alpha(t)$ is an exponentially decreasing function, linearly depending on the initial state of filter at the moment $t = 0$.

A high-frequency property of oscillations can be reformulated in the following condition.

Consider the great fixed time interval $[0, T]$. It can be partitioned into small intervals of the form $[\tau, \tau + \delta]$ where the following relations

(2) $$\begin{array}{l} |\gamma(t) - \gamma(\tau)| \le C\delta, \\ |\omega_j(t) - \omega_j(\tau)| \le C\delta, \\ \forall\, t \in [\tau, \tau + \delta],\ \forall\, \tau \in [0, T], \end{array}$$

(3) $$|2\omega_1(\tau) - \omega_2(\tau)| \le C_1,\ \forall\, \tau \in [0, T],$$

(4) $$\omega_j(\tau) \ge R,\ \forall\, \tau \in [0, T]$$

are satisfied. Here we assume that δ is sufficiently small with respect to the fixed numbers T, C, C_1, the number R is sufficiently great with respect to the number δ. The latter means that on the small intervals $[\tau, \tau+\delta]$ the functions $\gamma(t)$ and $\omega_j(t)$ are "almost constants" and the functions $f_j(t)$ oscillate rapidly as harmonic functions. It is clear that such conditions occur for high-frequency oscillations.

Note specially condition (3). This condition is a requirement on phase-locked loop with squarer. The difference $2\omega_1(t)-\omega_2(t)$ is in a certain "capture range", which provides a capture of slave oscillator frequency (asymptotically, as a result of transient phenomenon) in such a way that this frequency coincides with double frequency of the master oscillator. In the case when $|2\omega_1(t)-\omega_2(t)|$ is less than or equal to a certain given number (i.e. $2\omega_1(0)-\omega_2(0)$ is situated in a capture range), for well-synthesized PLL of this kind we have the following stability property

$$\lim_{t\to+\infty}(2\omega_1(t)-\omega_2(t)) = 0. \tag{5}$$

Recall that for usual PLL (i.e. PLL without squarer), for which in place of (5) it is necessary to use the relation

$$\lim_{t\to+\infty}(\omega_1(t)-\omega_2(t)) = 0, \tag{6}$$

condition (3) must be replaced by the following condition [19]

$$|\omega_1(\tau)-\omega_2(\tau)| \le C_1, \quad \forall\tau\in[0,T]. \tag{7}$$

Consider now the block-diagram in Fig. 1. While deriving the differential equations of this PLL we apply the technique, developed in [15].

Consider two block-diagrams described in Fig. 2 and Fig. 3. Here $\theta_j(t) = \omega_j(t)t + \Psi_j$ are phases of the oscillations $f_j(t)$, **PD** is a nonlinear block with the characteristic $\varphi(\theta)$, being called a phase detector (discriminator), $\theta(t) = 2\theta_1(t)-\theta_2(t)$. The input $\xi(t)$ and the output $\sigma(t)$ of the **Filter** are related as

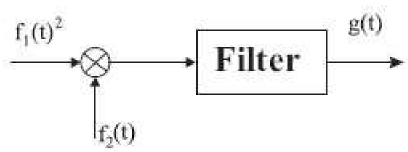

Figure 2: Multiplier and filter

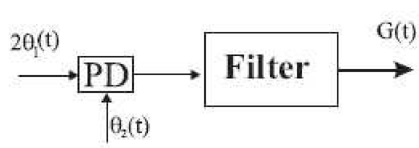

Figure 3: Phase detector and filter

$$\sigma(t) = \alpha(t) + \int_0^t \gamma(t-\tau)\xi(\tau)d\tau$$

Theorem 1. If conditions (2)–(4) are satisfied and

$$\varphi(\theta) = \frac{1}{4} A_1^2 A_2 \sin\theta,$$

then for the same initial states of the filter the following relation

$$|G(t) - g(t)| \le D\delta, \quad \forall\, t \in [0, T]$$

is valid. Here D is a certain number not depending on δ.

Proof.

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In the same way, we can prove the following

Theorem 2. If conditions (2)–(4) are satisfied and

$$\varphi(\theta) = \frac{1}{4} A_1^2 A_2 \sin\theta$$

then the following relation

$$\left| \int_0^t [(f_1(s))^2 f_2(s) - \varphi(2\theta_1(s) - \theta_2(s))]\, ds \right| \le$$
$$\le D\delta, \quad \forall\, t \in [0, T]$$

is valid. Here D is a certain number not depending on δ.

Further, we consider the impulse high-frequency oscillations of the type

$$\begin{aligned} f_1(t) &= A_1(1 + \text{sign}\, \sin(\omega_1(t)t + \psi_1), \\ f_2(t) &= A_2 \text{sign}\, \sin(\omega_2(t)t + \psi_2) \end{aligned} \tag{14}$$

and formulate the analogs of Theorems 1 and 2. For this purpose we introduce 2π-periodic function $F(\theta)$, which is also called a phase detector characteristic, in the following way

$$F(\theta) = \begin{cases} 2A_1^2 A_2 \left(1 + \frac{2\theta}{\pi}\right), & \text{for} \quad \theta \in [-\pi, 0], \\ 2A_1^2 A_2 \left(1 - \frac{2\theta}{\pi}\right), & \text{for} \quad \theta \in [0, \pi]. \end{cases}$$

Here we assume that in place of condition (3) condition (5) is satisfied.

Consider again block-diagrams in Fig. 2 and Fig. 3, where $f_j(t)$ have the form (14), the characteristic of PD is $F(\theta)$, and in Fig. 2 in place of the input $2\theta_1(t)$ we have the input $\theta_1(t)$.

Theorem 3. If conditions (2),(4), and (5) are satisfied, then for the same initial states of the filter we have the following relation

$$|G(t) - g(t)| \le D\delta, \quad \forall\, t \in [0, T].$$

Here D is a certain number not depending on δ.

Proof.

■

Similarly, we can prove the following

Theorem 4. If conditions (2), (4), and (5) are satisfied, then the following relation

$$\left| \int_0^t f_1^2(s) f_2(s) - F(\theta_1(s) - \theta_2(s)) ds \right| \le D\delta, \quad \forall\, t \in [0, T]$$

is valid. Here D is a certain number not depending on δ.

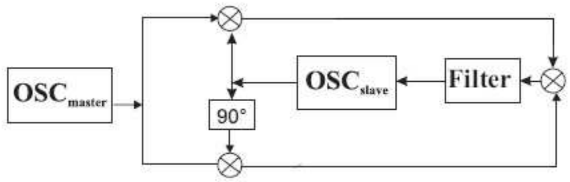

Figure 4: Costas loop

3 The Costas loop

Consider now a block-diagram of the Costas loop (Fig. 4)

Here all the denotations are the same as in Fig. 1. Block 90^o is a quadrature component. As before, we consider the case of the high-frequency harmonic and impulse signals.

However here together with the assumption that conditions (2) and (4) are satisfied we assume that (3) is satisfied for the harmonic signals of the type (1) and the relation

$$|\omega_1(\tau) - 2\omega_2(\tau)| \leq C_1, \quad \forall\, \tau \in [0, T], \tag{18}$$

is valid for the impulse signals of the type (14).

Then in place of the theorems on the asymptotic equivalence of block-diagrams in Fig. 2 and Fig. 3 we obtain the theorems on the asymptotic equivalence of block-diagrams in Fig. 5 and Fig. 6:

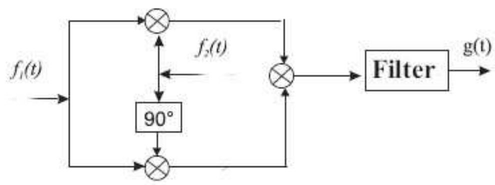

Figure 5: Three multipliers and filter

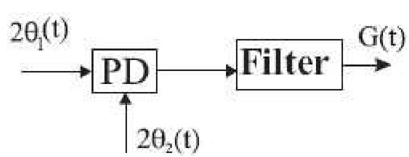

Figure 6: Phase detector and filter

To obtain the analogs of Theorems 1 and 2 for the block-diagrams in Fig. 5 and Fig. 6, in place of identity (8) we take the formula

$$(\sin\beta_1)(\sin(\beta_1 - \tfrac{\pi}{2}))(\sin\beta_2)^2 = -\tfrac{1}{4}\sin 2\beta_1 + \\ + \tfrac{1}{8}(\sin 2(\beta_1 + \beta_2) + \sin 2(\beta_1 - \beta_2)).$$

To obtain the analogs of Theorems 3 and 4, we take the relation

$$(\sin\beta_1)(\sin(\beta_1 - \tfrac{\pi}{2}))(\sin\beta_2)^2 = \\ = -(\cos(2\beta_1 - \beta_2) - \cos(2\beta_1 + \beta_2)).$$

By the scheme used in the proof of Theorem 1, for signals of the type (1) we can prove the following

Theorem 5. If conditions (1), (2), (4), and (5) are satisfied and

$$\varphi_1(\theta) = \frac{1}{8} A_1^2 A_2^2 \sin\theta,$$

then for the same initial states of the filter we have the relation

$$|G(t) - g(t)| \leq D\delta, \ \forall\, t \in [0, T].$$

Here D is a certain number not depending on δ, $\varphi_1(\theta)$ is a characteristic of PD.

For signals of the type (1), the analog of Theorem 2 is the following

Theorem 6. If conditions (2), (4), and (18) are satisfied and

$$\varphi_1(\theta) = \frac{1}{8} A_1^2 A_2^2 \sin\theta,$$

then the relation

$$\left| \int\limits_0^t f_1^2(s) f_2(s) f_2(s - \tfrac{\pi}{2}) - \varphi_1(2(\theta_1(s) - \theta_2(s))) ds \right| \leq$$
$$\leq D\delta \qquad \forall\, t \in [0, T]$$

is valid. Here D is a certain number not depending on δ.

For signals of the type (14) we introduce the phase detector characteristic $P(\theta)$ in the following way:

$$P(\theta) = -A_2 F(\theta).$$

Here $F(\theta)$ is a characteristic, introduced formally in order to formulate Theorems 3 and 4. In Fig. 6 we also change the input $2\theta_1(t)$ to the input $\theta_1(t)$.

Theorem 7. If conditions (2), (4), (14), and (18) are satisfied, then for the same initial states of the filter the relation

$$|G(t) - g(t)| \leq D\delta, \ \forall\, t \in [0, T]$$

is valid. The analog of Theorem 4 is the following

Theorem 8. If conditions (2), (4), (14), and (18) are satisfied, then the relation

$$\left| \int\limits_0^t f_1^2(s) f_2(s) f_2(s - \tfrac{\pi}{2}) - P((\theta_1(s) - 2\theta_2(s))) ds \right| \leq$$
$$\leq D\delta \qquad \forall\, t \in [0, T].$$

is valid.

4 Differential equations of phase-locked loop

Consider the quantity

$$\dot\theta_j(t) = \omega_j(t) + \dot\omega_j(t) t.$$

If PLL is well-synthesized, namely it has a property of global stability, then for this PLL we have the exponentially decreasing quantity $\dot\omega_j(t)$:

$$|\dot\omega_j(t)| \leq C e^{-\alpha t}. \tag{19}$$

Here C and α are certain positive numbers not depending on t. Therefore the quantity $\dot\omega_j(t) t$ is, as a rule, sufficiently small with respect to the number R (see condition (4)).

By the above relations we can conclude that there occurs the following approximate relation

(20) $$\dot{\theta}_j(t) = \omega_j(t).$$

In PLL theory this approximate relation is usually assumed to be exact in virtue of the above reasonings [15].

The control law of the slave oscillator is usually assumed to be linear [3]

(21) $$\omega_2(t) = \omega_2(0) + Lg(t).$$

Here $\omega_2(0)$ is the initial frequency of the slave oscillator, L is a certain number, $g(t)$ is a control signal, which is a filter output.

From relation (21) we have

(22) $$\begin{gathered}\theta_2(t) = \theta_2(0) + \omega_2(0)t + L(a\int\limits_0^t \xi(\tau)d\tau + \\ + \int\limits_0^t \alpha(\tau)d\tau + \int\limits_0^t\int\limits_0^\tau \gamma(\tau - s)\xi(s)dsd\tau),\end{gathered}$$

where $\xi(t)$ is a filter input.

From Theorems 2, 4, 6, and 8 it follows that the quantity

$$\int\limits_0^t \xi(\tau)d\tau$$

is asymptotically equivalent respectively to the quantities

$$\int\limits_0^t \varphi(2\theta_1(\tau) - \theta_2(\tau))d\tau, \quad \int\limits_0^t F(\theta_1(\tau) - \theta_2(\tau))d\tau,$$

$$\int\limits_0^t \varphi_1(2\theta_1(\tau) - 2\theta_2(\tau))d\tau, \quad \int\limits_0^t P(\theta_1(\tau) - 2\theta_2(\tau))d\tau.$$

From Theorems 1, 3, 5, and 7 it follows that the quantity

$$\int\limits_0^t \gamma(t-s)\xi(s)ds$$

are asymptotically equivalent respectively to the quantities

$$\begin{gathered}\int\limits_0^t \gamma(t-s)\varphi(2\theta_1(s) - \theta_2(s))ds, \\ \int\limits_0^t \gamma(t-s)F(\theta_1(s) - \theta_2(s))ds, \\ \int\limits_0^t \gamma(t-s)\varphi_1(2\theta_1(s) - 2\theta_2(s))ds, \\ \int\limits_0^t \gamma(t-s)P(\theta_1(s) - 2\theta_2(s))ds\end{gathered}$$

Using these asymptotical changes in (22) and assuming that the master oscillator is high-steady, i.e. $\omega_1(t) \equiv \omega_1(0)$, we obtain the following integro-differential equations for PLL.

For PLL with squarer and harmonic oscillator we have

(23)
$$\begin{array}{l}(2\theta_1(t)-\theta_2(t))^{\bullet}+L(a\varphi(2\theta_1(t)-\theta_2(t))+\alpha(t)+\\ +\int\limits_0^t \gamma(t-\tau)\varphi(2\theta_1(\tau)-\theta_2(\tau))d\tau)=2\omega_1(0)-\omega_2(0).\end{array}$$

For PLL with squarer and clock oscillator we have

(24)
$$\begin{array}{l}(\theta_1(t)-\theta_2(t))^{\bullet}+L(aF(\theta_1(t)-\theta_2(t))+\alpha(t)+\\ +\int\limits_0^t \gamma(t-\tau)F(\theta_1(\tau)-\theta_2(\tau))d\tau)=\omega_1(0)-\omega_2(0).\end{array}$$

For the Costas loop with harmonic oscillator we have

(25)
$$\begin{array}{l}(2\theta_1(t)-2\theta_2(t))^{\bullet}+2L(a\varphi_1(2\theta_1(t)-2\theta_2(t))+\\ +\alpha(t)+2\int\limits_0^t \gamma(t-\tau)\varphi_1(2\theta_1(\tau)-2\theta_2(\tau))d\tau)=\\ 2\omega_1(0)-2\omega_2(0).\end{array}$$

For the Costas loop with clock oscillator we have

(26)
$$\begin{array}{l}(\theta_1(t)-2\theta_2(t))^{\bullet}+2L(aP(\theta_1(t)-2\theta_2(t))+2\alpha(t)+\\ +2\int\limits_0^t \gamma(t-\tau)P(\theta_1(\tau)-2\theta_2(\tau))d\tau)=\omega_1(0)-2\omega_2(0).\end{array}$$

The block-diagram of PLL, which correspond to the above equations, is shown in Fig. 7.

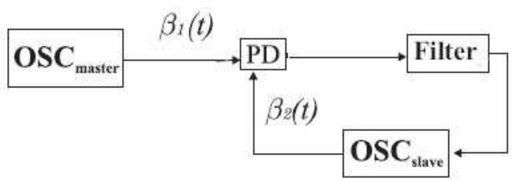

Figure 7: Block-diagram of PLL with squarer and the Costas loop

Here we have
for relation (23): $\beta_1(t)=2\theta_1(t)$, $\beta_2(t)=\theta_2(t)$;
for relation (24): $\beta_1(t)=\theta_1(t)$, $\beta_2(t)=\theta_2(t)$;
for relation (25): $\beta_1(t)=2\theta_1(t)$, $\beta_2(t)=2\theta_2(t)$;
for relation (26): $\beta_1(t)=\theta_1(t)$, $\beta_2(t)=2\theta_2(t)$.

In the case when transfer function of the filter is nondegenerate, i.e. its numerator and denominator have no common roots, equations (23)–(26) are equivalent to the following system of differential equations [4,13–15]:

(27)
$$\begin{array}{l}\dot z = Az + b\Psi(\sigma)\\ \dot\sigma = c^*z+\rho\Psi(\sigma).\end{array}$$

Here A is a constant $(n\times n)$-matrix, b and c are constant n-vectors, ρ is a number, $\Psi(\sigma)$ is 2π-periodic function, satisfying the following relations:

$$\rho = -aL, \quad W(p) = L^{-1}c^*(A-pI)^{-1}b$$
(in the case of relations (23) and (24));

$$\begin{cases} \psi(\sigma) = \varphi(\sigma) - \frac{2\omega_1(0)-\omega_2(0)}{L(a+W(0))} \\ \sigma = 2\theta_1 - \theta_2 \quad \text{(in the case of relation (23))}; \end{cases}$$

$$\begin{cases} \psi(\sigma) = F(\sigma) - \frac{\omega_1(0)-\omega_2(0)}{L(a+W(0))} \\ \sigma = \theta_1 - \theta_2 \quad \text{(in the case of relation (24))}; \end{cases}$$

$$\rho = -2aL, \quad W(p) = 2L^{-1}c^*(A-pI)^{-1}b$$
(in the case of relations (25) and (26));

$$\begin{cases} \psi(\sigma) = \varphi_1(\sigma) - \frac{\omega_1(0)-\omega_2(0)}{L(a+W(0))}, \\ \sigma = 2\theta_1 - 2\theta_2 \quad \text{(in the case of relation (25))}; \end{cases}$$

$$\begin{cases} \psi(\sigma) = P(\sigma) - \frac{\omega_1(0)-2\omega_2(0)}{2L(a+W(0))}, \\ \sigma = \theta_1 - 2\theta_2 \quad \text{(in the case of relation (26))}. \end{cases}$$

For system (27) with the nonlinearities of described type, the theory of global stability is well developed. Within the framework of this theory the different constructive conditions, under which estimate (19) is satisfied, are obtained. The obtained conditions permit us to conclude that the description of the considered PLL by equations (27) is correct.

From Theorems 1–8 it follows that for the deterministic (when a noise is lacking) description of the Costas loop and PLL with squarer, the conventional [9, 4] introduction of additional filters turns out unnecessary. Their role it plays a central filter. In this case the application of proportional-integral filters (when $a \neq 0$) it turns out possible since the slave oscillator itself has certain properties of filter (Theorems 2, 4, 6, 8).

5 Conclusion

On the basis of special asymptotical methods for analysis of high-frequency harmonic and impulse oscillations, the integro-differential and differential equations for the Costas loop and PLL with squarer are derived. The methods for the computation of phase detector characteristics are suggested. For high-frequency clock oscillators the new classes of such characteristics are described. It is mathematically rigorously shown that PLL with integrator and harmonic oscillators tunes to a double frequency of the master oscillator, PLL with clock oscillators — to a frequency of the master oscillator, the Costas loop with harmonic oscillators tunes to a frequency of the master oscillator, and the Costas loop with clock oscillators — to a half frequency of the master oscillator.

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