Cycles of Two-Dimensional Systems: Computer Calculations, Proofs, and Experiments

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Abstract. The study of cycles of two-dimensional dynamical systems was stimulated by as purely mathematical problems (Hilbert's sixteenth problem, the center-and-focus problem) as many applied problems (the oscillations of electronic generators and electrical machines, the dynamics of populations, the critical and permissible boundaries of stability, see. for example, [Bautin, 1952; Andronov & Leontovich, 1956; Andronov et al., 1959; Cherkas, 1976; Perko, 1990] and others).

One of the central problems in studying small cycles in the neighborhood of equilibrium is a computation of the Lyapunov quantities (Lyapunov values of coefficients, Poincare-Lyapunov constant, focus values) [Lyapunov, 1882; Bautin & Leontovich 1976; Marsden & McCracken, 1976; Lloyd & Pearson, 1990; Yu, 1998; Yu & Han, 2005; Lynch, 2005].

While a general form of the first and second Lyapunov quantities were computed for two-dimensional real autonomous system (in terms of coefficients of the right-hand side of system) in the 40-50s of the last century [Bautin, 1949; Bautin, 1952; Serebryakova, 1959], the third Lyapunov quantity was computed only in certain special cases [Lloyd & Pearson, 1997; Yu & Han, 2005; Lynch, 2005].

In the present work the general expression of the third Lyapunov quantity is given. In this case together with the classical Lyapunov method for computation of Lyapunov quantities, based on the passage to the polar coordinates [Lyapunov, 1882], *a direct method for Lyapunov quantities computation in Euclidian coordinates and in the time domain*, is described and used. The first steps in the development of this method were made in [Kuznetsov & Leonov 2007].

The computation of Lyapunov quantities by two different analytic methods with applying the modern software tools of symbolic computing permits us to show that the obtained formulas for the third Lyapunov quantity are correct.

This results are applied to quadratic systems and the experiments for computing large cycles are performed. Our experience of computations shows that it is practically impossible to trace small cycles in the neighborhood of equilibrium, where the zero and first Lyapunov quantities are equal to zero. However we can distinctly see the large cycles in a series of computer experiments.

In other experiments it occurs the interesting phenomenon of the existence of certain "domains of flattening", i.e. the limit points and limit segments of nonperiodic trajectories. The latter substantially involves the qualitative analysis of quadratic systems.

For the analysis of quadratic systems it is often useful to reduce them to the Lienard equations of special type. In this work such a reduction is used and with its help the sets in the space of parameters, corresponding the existence of four cycles in quadratic systems, are evaluated.

Keywords: large (normal amplitude) and small limit cycle, Lienard equation, quadratic system, Lyapunov focus values (Lyapunov quantities, Poincare-Lyapunov constants, Lyapunov coefficients), 16th Hilbert problem

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²PDF slides <http://www.math.spbu.ru/user/nk/PDF/Limit-cycles-Focus-values-Lyapunov-quantity-16th-Hilbert.pdf>

1 Calculation of Lyapunov quantities in the Euclidean system of coordinates and in the time domain

1.1 Approximation of solution in the Euclidean system of coordinates

Consider a system of two autonomous differential equations

$$\begin{aligned}\frac{dx}{dt} &= -y + f(x, y), \\ \frac{dy}{dt} &= x + g(x, y),\end{aligned}\tag{1}$$

where $x, y \in \mathbb{R}$ and the functions $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$ have continuous partial derivatives of $(n + 1)$ th order in the open neighborhood U of radius R_U of the point $(x, y) = (0, 0)$

$$f(\cdot, \cdot), g(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \quad \in \mathbb{C}^{(n+1)}(U).\tag{2}$$

Suppose, the expansion of the functions f, g begins with the terms of at least the second order and therefore we have

$$f(0, 0) = g(0, 0) = 0, \quad \frac{df}{dx}(0, 0) = \frac{df}{dy}(0, 0) = \frac{dg}{dx}(0, 0) = \frac{dg}{dy}(0, 0) = 0.\tag{3}$$

Further we will use a smoothness of the functions f and g and will follow the first Lyapunov method on finite time interval (see f.e. classical works [Lindelof, 1894; Lefschetz, 1957; Cesari, 1959] and others). By assumption on smoothness (2) in the neighborhood U we have

$$\begin{aligned}f(x, y) &= \sum_{k+j=2}^n f_{kj} x^k y^j + o((|x| + |y|)^n) = f_n(x, y) + o((|x| + |y|)^n), \\ g(x, y) &= \sum_{k+j=2}^n g_{kj} x^k y^j + o((|x| + |y|)^n) = g_n(x, y) + o((|x| + |y|)^n).\end{aligned}\tag{4}$$

The existence condition of $(n + 1)$ partial derivatives with respect to x and y for f and g is used for simplicity of exposition and can be weakened.

Let $x(t, x(0), y(0)), y(t, x(0), y(0))$ be a solution of system (1) with the initial data

$$x(0) = 0, \quad y(0) = h.\tag{5}$$

Denote

$$x(t, h) = x(t, 0, h), \quad y(t, h) = y(t, 0, h).$$

Below a time derivative will be denoted by x' and $\dot{x}$.

Lemma 1 *A positive number $H \in (0, R_U)$ exists such that for all $h \in [0, H]$ the solution $(x(t, h), y(t, h))$ is defined for $t \in [0, 4\pi]$.*

The validity of lemma follows from condition (3) and the existence of two purely imaginary eigenvalues of the matrix of linear approximation of system (1).

This implies [Hartman, 1984] the following

Lemma 2 *If smoothness condition (2) is satisfied, then*

$$x(\cdot, \cdot), y(\cdot, \cdot) \in \mathbb{C}^{(n+1)}([0, 4\pi] \times [0, H])\tag{6}$$

Further we will consider the sufficiently small initial data $h \in [0, H]$, a finite time interval $t \in [0, 4\pi]$ and use a uniform boundedness of the solution $(x(t, h), y(t, h))$ and its mixed partial derivatives with respect to h and t up to the order $(n + 1)$ inc in the set $[0, 4\pi] \times [0, H]$.

We apply now a well-known linearization procedure [Leonov & Kuznetsov, 2007]. From Lemma 2 it follows that for each fixed t the solution of system can be represented by the Taylor formula

$$\begin{aligned} x(t, h) &= h \frac{\partial x(t, \eta)}{\partial \eta} \Big|_{\eta=0} + \frac{h^2}{2} \frac{\partial^2 x(t, \eta)}{\partial \eta^2} \Big|_{\eta=h\theta_x(t, h)} \quad 0 \leq \theta_x(t, h) \leq 1, \\ y(t, h) &= h \frac{\partial y(t, \eta)}{\partial \eta} \Big|_{\eta=0} + \frac{h^2}{2} \frac{\partial^2 y(t, \eta)}{\partial \eta^2} \Big|_{\eta=h\theta_y(t, h)} \quad 0 \leq \theta_y(t, h) \leq 1. \end{aligned} \quad (7)$$

Note that by Lemma 2 and relation (7), the functions

$$\frac{h^2}{2} \frac{\partial^2 x(t, \eta)}{\partial \eta^2} \Big|_{\eta=h\theta_x(t, h)}, \quad \frac{h^2}{2} \frac{\partial^2 y(t, \eta)}{\partial \eta^2} \Big|_{\eta=h\theta_y(t, h)}$$

and their time derivatives are smooth functions of t and have the order of smallness $o(h)$ uniformly with respect to t on a considered finite time interval $[0, 4\pi]$.

Introduce the following denotations

$$\tilde{x}_{h^k}(t) = \frac{\partial^k x(t, \eta)}{\partial \eta^k} \Big|_{\eta=0}, \quad \tilde{y}_{h^k}(t) = \frac{\partial^k y(t, \eta)}{\partial \eta^k} \Big|_{\eta=0}.$$

We shall say that the sums

$$\begin{aligned} x_{h^m}(t, h) &= \sum_{k=1}^m \tilde{x}_{h^k}(t) \frac{h^k}{k!} = \sum_{k=1}^m \frac{\partial^k x(t, \eta)}{\partial \eta^k} \Big|_{\eta=0} \frac{h^k}{k!}, \\ y_{h^m}(t, h) &= \sum_{k=1}^m \tilde{y}_{h^k}(t) \frac{h^k}{k!} = \sum_{k=1}^m \frac{\partial^k y(t, \eta)}{\partial \eta^k} \Big|_{\eta=0} \frac{h^k}{k!} \end{aligned}$$

are the m th approximation of solution of system with respect to h . Substitute representation (7) in system (1). Then, equating the coefficients of h^1 and taking into account (3), we obtain

$$\begin{aligned} \frac{d\tilde{x}_{h^1}(t)}{dt} &= -\tilde{y}_{h^1}(t), \\ \frac{d\tilde{y}_{h^1}(t)}{dt} &= \tilde{x}_{h^1}(t). \end{aligned} \quad (8)$$

Hence, by conditions on initial data (5) for the first approximation with respect to h of the solution $(x(t, h), y(t, h))$, we have

$$x_{h^1}(t, h) = \tilde{x}_{h^1}(t)h = -h \sin(t), \quad y_{h^1}(t, h) = \tilde{y}_{h^1}(t)h = h \cos(t). \quad (9)$$

Similarly, to obtain the second approximation $(x_{h^2}(t, h), y_{h^2}(t, h))$, we substitute representation

$$\begin{aligned} x(t, h) &= x_{h^2}(t, h) + \frac{h^3}{3!} \frac{\partial^3 x(t, \eta)}{\partial \eta^3} \Big|_{\eta=h\theta_x(t, h)}, \\ y(t, h) &= y_{h^2}(t, h) + \frac{h^3}{3!} \frac{\partial^3 y(t, \eta)}{\partial \eta^3} \Big|_{\eta=h\theta_y(t, h)}. \end{aligned} \quad (10)$$

in formula (4) for $f(x, y)$ and $g(x, y)$. Note that in expressions for f and g by virtue of (3) the coefficients of h^2 (denote them by $u_{h^2}^f$ and $u_{h^2}^g$, respectively) depend only on $\tilde{x}_{h^1}(t)$ and $\tilde{y}_{h^1}(t)$, i.e., by (9) they are known functions of time and are independent of the unknown functions $\tilde{x}_{h^2}(t)$ and $\tilde{y}_{h^2}(t)$. Thus, we have

$$\begin{aligned} f(x_{h^2}(t, h) + o(h^2), y_{h^2}(t, h) + o(h^2)) &= u_{h^2}^f(t)h^2 + o(h^2), \\ g(x_{h^2}(t, h) + o(h^2), y_{h^2}(t, h) + o(h^2)) &= u_{h^2}^g(t)h^2 + o(h^2). \end{aligned}$$

Substituting (10) in system (1), for the determination of $\tilde{x}_{h^2}(t)$ and $\tilde{y}_{h^2}(t)$ we obtain

$$\begin{aligned}\frac{d\tilde{x}_{h^2}(t)}{dt} &= -\tilde{y}_{h^2}(t) + u_{h^2}^f(t), \\ \frac{d\tilde{y}_{h^2}(t)}{dt} &= \tilde{x}_{h^2}(t) + u_{h^2}^g(t).\end{aligned}\tag{11}$$

Lemma 3 *For solutions of the system*

$$\begin{aligned}\frac{d\tilde{x}_{h^k}(t)}{dt} &= -\tilde{y}_{h^k}(t) + u_{h^k}^f(t), \\ \frac{d\tilde{y}_{h^k}(t)}{dt} &= \tilde{x}_{h^k}(t) + u_{h^k}^g(t)\end{aligned}\tag{12}$$

with the initial data

$$\tilde{x}_{h^k}(0) = 0, \quad \tilde{y}_{h^k}(0) = 0\tag{13}$$

we have

$$\begin{aligned}\tilde{x}_{h^k}(t) &= u_{h^k}^g(0) \cos(t) + \cos(t) \int_0^t \cos(\tau) ((u_{h^k}^g(\tau))' + u_{h^k}^f(\tau)) d\tau + \\ &+ \sin(t) \int_0^t \sin(\tau) ((u_{h^k}^g(\tau))' + u_{h^k}^f(\tau)) d\tau - u_{h^k}^g(t), \\ \tilde{y}_{h^k}(t) &= u_{h^k}^g(0) \sin(t) + \sin(t) \int_0^t \cos(\tau) ((u_{h^k}^g(\tau))' + u_{h^k}^f(\tau)) d\tau - \\ &- \cos(t) \int_0^t \sin(\tau) ((u_{h^k}^g(\tau))' + u_{h^k}^f(\tau)) d\tau.\end{aligned}\tag{14}$$

The relations (14) can be verified by direct differentiation.

Repeating this procedure for the determination of the coefficients $\tilde{x}_{h^k}$ and $\tilde{y}_{h^k}$ of the functions $u_{h^k}^f(t)$ and $u_{h^k}^g(t)$, by formula (14) we obtain sequentially the approximations $(x_{h^k}(t, h), y_{h^k}(t, h))$ for $k = 1, \dots, n$. For $h \in [0, H]$ and $t \in [0, 4\pi]$ we have

$$\begin{aligned}x(t, h) &= x_{h^n}(t, h) + \frac{h^{n+1}}{(n+1)!} \frac{\partial^{n+1} x(t, \eta)}{\partial \eta^{n+1}} \Big|_{\eta=h\theta_x(t, h)} = \\ &= x_{h^n}(t, h) + o(h^n) = \sum_{k=1}^n \tilde{x}_{h^k}(t) \frac{h^k}{k!} + o(h^n), \\ y(t, h) &= y_{h^n}(t, h) + \frac{h^{n+1}}{(n+1)!} \frac{\partial^{n+1} y(t, \eta)}{\partial \eta^{n+1}} \Big|_{\eta=h\theta_y(t, h)} = \\ &= y_{h^n}(t, h) + o(h^n) = \sum_{k=1}^n \tilde{y}_{h^k}(t) \frac{h^k}{k!} + o(h^n), \\ 0 &\leq \theta_x(t, h) \leq 1, \quad 0 \leq \theta_y(t, h) \leq 1.\end{aligned}\tag{15}$$

Here by Lemma 2

$$\tilde{x}_{h^k}(\cdot), \tilde{y}_{h^k}(\cdot) \in \mathbb{C}^n([0, 4\pi]), \quad k = 1, \dots, n\tag{16}$$

and the estimate $o(h^n)$ is uniform $\forall t \in [0, 4\pi]$. From (13) and by the choice of initial data in (8) we obtain

$$x_{h^k}(0, h) = x(0, h) = 0, \quad y_{h^k}(0, h) = y(0, h) = h, \quad k = 1, \dots, n.$$

1.2 Computation of Lyapunov quantities in the time domain

Consider for the initial datum $h \in (0, H]$ the time $T(h)$ of first crossing of the solution $(x(t, h), y(t, h))$ of the half-line $\{x = 0, y > 0\}$. Complete a definition (by continuity) of the function $T(h)$ in zero: $T(0) = 2\pi$. Since by (9) the first approximation of solution crosses the half-line $\{x = 0, y > 0\}$ at the time 2π , then the crossing time can be represented as

$$T(h) = 2\pi + \Delta T(h),$$

where $\Delta T(h) = O(h)$. We shall say that $\Delta T(h)$ is a residual of crossing time.

By definition of $T(h)$ we have

$$x(T(h), h) = 0. \quad (17)$$

Since by (6), $x(\cdot, \cdot)$ has continuous partial derivatives with respect to either arguments up to the order n inc and $\dot{x}(t, h) = \cos(t)h + o(h)$, by the theorem on implicit function [Zorich, 2002], the function $T(\cdot)$ is n times differentiable. It is possible to show (for example, considering the function $z(t, h) = x(t, h)/h$ and completing its definition in zero by the function $x_{h^1}(t)$ or making use of special theorems of mathematical analysis) that $T(h)$ is also differentiable n times in zero. By the Taylor formula we have

$$T(h) = 2\pi + \sum_{k=1}^n \tilde{T}_k h^k + o(h^n), \quad (18)$$

where $\tilde{T}_k = \frac{1}{k!} \frac{d^k T(h)}{dh^k}$ (usually called period constants [Gine, 2007]). We shall say that the sum

$$\Delta T_k(h) = \sum_{j=1}^k \tilde{T}_j h^j \quad (19)$$

is the k th approximation of the residual of the time $T(h)$ of the crossing of the solution $(x(t, h), y(t, h))$ of the half-line $\{x = 0, y > 0\}$. Substituting relation (18) for $t = T(h)$ in the right-hand side of the first equation of (15) and denoting the coefficient of h^k by $\tilde{x}_k$, we obtain the series $x(T(h), h)$ in terms of powers of h :

$$x(T(h), h) = \sum_{k=1}^n \tilde{x}_k h^k + o(h^n). \quad (20)$$

In order to express the coefficients $\tilde{x}_k$ by the coefficients $\tilde{T}_k$ of the expansion of residual of crossing time we assume that in (15) $t = 2\pi + \tau$:

$$x(2\pi + \tau, h) = \sum_{k=1}^n \tilde{x}_{h^k}(2\pi + \tau) \frac{h^k}{k!} + o(h^n). \quad (21)$$

By smoothness condition (16) we have

$$\tilde{x}_{h^k}(2\pi + \tau) = \tilde{x}_{h^k}(2\pi) + \sum_{m=1}^n \tilde{x}_{h^k}^{(m)}(2\pi) \frac{\tau^m}{m!} + o(\tau^n), \quad k = 1, \dots, n.$$

Substitute this representation in (21) for the solution $x(2\pi + \tau, h)$ for $\tau = \Delta T(h)$, and bring together the coefficients of the same exponents h . Since $(\Delta T(h))^n = O(h^n)$, by (17) and taking into account (18) for $T(h)$, we obtain

$$\begin{aligned} h : 0 &= \tilde{x}_1 = \tilde{x}_{h^1}(2\pi), \\ h^2 : 0 &= \tilde{x}_2 = \tilde{x}_{h^2}(2\pi) + \tilde{x}_{h^1}'(2\pi) \tilde{T}_1, \\ h^3 : 0 &= \tilde{x}_3 = \tilde{x}_{h^3}(2\pi) + \frac{1}{2} \tilde{x}_{h^2}'(2\pi) \tilde{T}_1 + \tilde{x}_{h^1}''(2\pi) \tilde{T}_2 + \frac{1}{2} \tilde{x}_{h^1}''(2\pi) \tilde{T}_1^2, \\ &\dots \\ h^n : 0 &= \tilde{x}_n = \dots \end{aligned}$$

From the above we sequentially find $\tilde{T}_j$. The coefficients $T_{k=1,\dots,n-1}$ can be determined sequentially since the expression for $\tilde{x}_k$ involves only the coefficients $T_{m < k}$ and the factor $\tilde{x}'_{h^1}(2\pi)$ multiplying T_{k-1} is equal to -1 .

We apply a similar procedure to determine the coefficients $\tilde{y}_k$ of the expansion

$$y(T(h), h) = \sum_{k=1}^n \tilde{y}_k h^k + o(h^n).$$

Substitute the representation

$$\tilde{y}_{h^k}(2\pi + \Delta T(h)) = \tilde{y}_{h^k}(2\pi) + \sum_{m=1}^n \tilde{y}_{h^k}^{(m)}(2\pi) \frac{\Delta T(h)^m}{m!} + o(h^n), \quad k = 1, \dots, n$$

in the expression

$$y(2\pi + \Delta T(h), h) = \sum_{k=1}^n \tilde{y}_{h^k}(2\pi + \Delta T(h)) \frac{h^k}{k!} + o(h^n).$$

Equating the coefficients of the same exponents h , we obtain the following relations

$$\begin{aligned} h : \quad & \tilde{y}_1 = \tilde{y}_{h^1}(2\pi), \\ h^2 : \quad & \tilde{y}_2 = \tilde{y}_{h^2}(2\pi) + \tilde{y}'_{h^1}(2\pi) \tilde{T}_1, \\ h^3 : \quad & \tilde{y}_3 = \tilde{y}_{h^3}(2\pi) + \frac{1}{2} \tilde{y}''_{h^2}(2\pi) \tilde{T}_1 + \tilde{y}'_{h^1}(2\pi) \tilde{T}_2 + \frac{1}{2} \tilde{y}''_{h^1}(2\pi) \tilde{T}_1^2, \\ \dots & \\ h^n : \quad & \tilde{y}_n = \dots \end{aligned}$$

for the sequential determination of $\tilde{y}_{i=1,\dots,n}$. Here $\tilde{y}_{h^{k=1,\dots,n}}(\cdot)$ and $\tilde{T}_{k=1,\dots,n-1}$ are the obtained above quantities.

Thus, for $n = 2m + 1$ under the condition $f(\cdot, \cdot), g(\cdot, \cdot) \in \mathbb{C}^{(2m+2)}(U)$ we sequentially obtained the approximations of the solution $(x(t, h), y(t, h))$ at the time $t = T(h)$ of the first crossing of the half-line $\{x = 0, y > 0\}$ accurate to $o(h^{2m+1})$ and the approximation of the time $T(h)$ itself accurate to $o(h^{2m})$. If in this case $\tilde{y}_k = 0$ for $k = 2, \dots, 2m$, then $\tilde{y}_{2m+1}$ is called the m th Lyapunov quantity L_m . Note, that, according to the Lyapunov theorem, the first nonzero coefficient of the expansion $\tilde{y}_i$ is always of an odd number and for sufficiently small initial data h the sign of $\tilde{y}_i$ (of the Lyapunov quantity) designates a qualitative behavior (winding or unwinding) of the trajectory $(x(t, h), y(t, h))$ on plane [Lyapunov, 1892].

Expression for the first, second, and third Lyapunov quantities in the general form can be found in Appendix 1.

1.3 Duffing equation

Consider the Duffing equation $\ddot{x} + x + x^3 = 0$ written as the system

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + x^3. \end{aligned} \tag{22}$$

Suppose that $x_0 = 0, y_0 = h_y$. Then

$$y(t)^2 + x(t)^2 + \frac{1}{2}x(t)^4 = y_0^2, \tag{23}$$

and, therefore,

$$y(t) = \pm \sqrt{h_y^2 - x(t)^2 - \frac{x(t)^4}{2}}, \quad x(t)^2 = -1 + \sqrt{1 + 2h_y^2 - 2y(t)^2}.$$

By virtue of system (22), this gives us

$$\frac{dt}{dy} = \frac{1}{x(1+x^2)} = \frac{1}{\sqrt{-1 + \sqrt{1 + 2h_y^2 - 2y^2}} \sqrt{1 + 2h_y^2 - 2y^2}}.$$

Then, for the intersection time $T(h_y)$, we have

$$T(h_y) = 4 \int_0^{h_y} \frac{dy}{\sqrt{-1 + \sqrt{1 + 2h_y^2 - 2y^2}} \sqrt{1 + 2h_y^2 - 2y^2}}.$$

Having changed the variables

$$y = h_y \cos(z), z = \arccos \frac{y}{h_y}, y = h_y \Rightarrow z = \frac{\pi}{2}, dy = -h_y \sin(z) dz$$

we obtain

$$T(h_y) = \int_0^{\pi/2} \frac{-h_y \sin(z) dz}{\sqrt{-1 + \sqrt{1 + 2h_y^2 \sin^2 z}} \sqrt{1 + 2h_y^2 \sin^2 z}}.$$

Expanding $T(h)$ in powers of h gives an estimate for the time when the trajectory meets the vertical half-line ($x = 0, y > 0$)

$$T(h_y) = 2\pi - \frac{3\pi}{4} h_y^2 + \frac{105\pi}{128} h_y^4 - \frac{1155\pi}{1024} h_y^6 + o(h_y^6),$$

this corresponds to the values obtained by the above algorithm. Below, we list the solutions obtained by the solutions approximation algorithm:

$$\tilde{x}_{h^1}(t) = -\sin(t), \quad \tilde{y}_{h^1}(t) = \cos(t);$$

$$\tilde{x}_{h^2}(t) = \tilde{y}_{h^2}(t) = 0;$$

$$\tilde{x}_{h^3}(t) = \frac{1}{8} \cos(t)^2 \sin(t) - \frac{3}{8} t \cos(t) + \frac{1}{4} \sin(t),$$

$$\tilde{y}_{h^3}(t) = -\frac{3}{8} t \sin(t) + \frac{3}{8} \cos(t) - \frac{3}{8} \cos(t)^3;$$

$$\tilde{x}_{h^4}(t) = \tilde{y}_{h^4}(t) = 0;$$

$$\tilde{x}_{h^5}(t) = -\frac{1}{64} \sin(t) \cos(t)^4 - \frac{45}{256} \cos(t)^2 \sin(t) + \frac{69}{256} \cos(t) t +$$

$$+ \frac{9}{128} \sin(t) t^2 - \frac{7}{32} \sin(t) + \frac{9}{64} t \cos(t)^3,$$

$$\tilde{y}_{h^5}(t) = \frac{33}{256} \sin(t) t + \frac{5}{64} \cos(t)^5 + \frac{27}{64} t \cos(t)^2 \sin(t) -$$

$$- \frac{9}{128} \cos(t) t^2 + \frac{83}{256} \cos(t)^3 - \frac{103}{256} \cos(t),$$

Thus, a periodic solution is approximated by a series with nonperiodic coefficients.

Note that Lyapunov quantities computed above are equal to zero,

$$L_1 = L_2 = \dots = 0,$$

which agrees with condition (23).

1.4 Computation of the first, second, and third Lyapunov quantities in general form

Consider a complete system with the expansion of the right-hand side up to the seventh order

$$\begin{aligned}
\dot{x} = & -y + f_{20}x^2 + f_{11}xy + f_{02}y^2 + f_{30}x^3 + f_{21}x^2y + f_{12}xy^2 + f_{03}y^3 + \\
& + f_{40}x^4 + f_{31}x^3y + f_{22}x^2y^2 + f_{13}xy^3 + f_{04}y^4 + \\
& + f_{50}x^5 + f_{41}x^4y + f_{32}x^3y^2 + f_{23}x^2y^3 + f_{14}xy^4 + f_{05}y^5 + \\
& + f_{60}x^6 + f_{51}x^5y + f_{42}x^4y^2 + f_{33}x^3y^3 + f_{24}x^2y^4 + f_{15}xy^5 + f_{06}y^6 \\
& + f_{70}x^7 + f_{61}x^6y + f_{52}x^5y^2 + f_{43}x^4y^3 + f_{34}x^3y^4 + f_{25}x^2y^5 + f_{16}xy^6 + f_{07}y^7, \\
\dot{y} = & x + g_{20}x^2 + g_{11}xy + g_{02}y^2 + g_{30}x^3 + g_{21}x^2y + g_{12}xy^2 + g_{03}y^3 + \\
& + g_{40}x^4 + g_{31}x^3y + g_{22}x^2y^2 + g_{13}xy^3 + g_{04}y^4 + \\
& + g_{50}x^5 + g_{41}x^4y + g_{32}x^3y^2 + g_{23}x^2y^3 + g_{14}xy^4 + g_{05}y^5 + \\
& + g_{60}x^6 + g_{51}x^5y + g_{42}x^4y^2 + g_{33}x^3y^3 + g_{24}x^2y^4 + g_{15}xy^5 + g_{06}y^6 \\
& + g_{70}x^7 + g_{61}x^6y + g_{52}x^5y^2 + g_{43}x^4y^3 + g_{34}x^3y^4 + g_{25}x^2y^5 + g_{16}xy^6 + g_{07}y^7.
\end{aligned} \tag{24}$$

For the first Lyapunov quantity we have [Bautin, 1952]

$$L_1 = \frac{\pi}{4}(g_{21} + f_{12} + 3f_{30} + 3g_{03} + f_{20}f_{11} + f_{02}f_{11} - g_{11}g_{20} + 2g_{02}f_{02} - 2f_{20}g_{20} - g_{02}g_{11}).$$

Note that since $\tilde{T}_1 = 0$, the residual of crossing time does not influence L_1 .

To compute the second Lyapunov quantity, we obtain the coefficients $\tilde{T}_2$ and $\tilde{T}_3$ of the expansion of crossing time residual.

$$\begin{aligned}
\tilde{T}_2 = & \frac{\pi}{12}(-9g_{30} + 4f_{20}^2 + 9f_{03} - 3g_{12} + 10g_{20}^2 + 10f_{02}^2 + 4g_{02}^2 + g_{11}^2 + f_{11}^2 + 3f_{21} - \\
& - 5f_{20}g_{11} - f_{11}g_{20} - 5f_{11}g_{02} + 10g_{02}g_{20} - f_{02}g_{11} + 10f_{20}f_{02}),
\end{aligned}$$

$$\begin{aligned}
\tilde{T}_3 = & -\frac{\pi}{18}(2f_{20} + f_{02} + g_{11})(-9g_{30} + 4f_{20}^2 + 9f_{03} - 3g_{12} + 10g_{20}^2 + 10f_{02}^2 + 4g_{02}^2 \\
& + g_{11}^2 + f_{11}^2 + 3f_{21} - 5f_{20}g_{11} - f_{11}g_{20} - 5f_{11}g_{02} + 10g_{02}g_{20} - f_{02}g_{11} + 10f_{20}f_{02}).
\end{aligned}$$

We obtain the coefficient g_{03} from the condition $L_1 = 0$:

$$g_{03} = -\frac{1}{3}(g_{21} + f_{12} + 3f_{30} + f_{20}f_{11} + f_{02}f_{11} - g_{11}g_{20} + 2g_{02}f_{02} - 2f_{20}g_{20} - g_{02}g_{11})$$

and get the expression for the second Lyapunov quantity

$$\begin{aligned}
L_2 = & -\frac{\pi}{72}(-66f_{20}g_{04} - 3f_{11}g_{30}f_{20} - 24g_{20}g_{02}g_{21} + 12f_{30}g_{11}f_{02} + 4f_{11}f_{20}^2g_{11} - 12f_{11}f_{21}f_{20} + 2g_{20}g_{11}^3 - 9g_{11}g_{02}g_{12} - \\
& 12f_{20}f_{11}f_{03} - 12g_{11}g_{02}f_{03} + 3g_{20}f_{12}f_{11} + 9g_{21}g_{30} - 6f_{02}f_{11}g_{12} + 9g_{20}g_{11}g_{02}^2 + 30f_{20}g_{02}g_{12} + 30g_{02}f_{21}f_{20} - 60g_{04}f_{02} + \\
& g_{11}^2f_{11}f_{20} - 5f_{11}f_{20}^3 - 21f_{20}f_{13} - 3f_{11}^3f_{20} - 9g_{02}g_{21}f_{11} + 7g_{11}g_{21}f_{02} - 5f_{11}g_{11}f_{02}^2 + 5f_{02}^2f_{11}f_{20} - 3g_{11}g_{20}f_{21} + 6g_{02}f_{20}f_{11}^2 + \\
& 9g_{21}f_{03} - 3f_{30}f_{11}^2 + 15f_{11}f_{40} - 21g_{11}g_{30}g_{02} - 6g_{11}f_{03}f_{11} + f_{11}f_{02}g_{11}^2 - 18g_{20}f_{03}f_{20} - 42g_{20}g_{02}f_{30} - 6g_{11}g_{12}g_{20} - \\
& 30f_{02}^2g_{20}f_{20} + 3f_{11}^2g_{02}f_{02} + 60f_{40}g_{20} + 9g_{11}g_{40} + 24f_{20}g_{20}f_{21} - 9g_{11}g_{20}f_{03} - 10g_{11}f_{20}^2g_{02} + 18g_{02}g_{12}f_{02} - 6g_{11}f_{11}g_{30} - \\
& 24f_{20}f_{03}g_{02} - 30f_{03}f_{02}g_{02} - 24g_{11}g_{20}g_{30} - 12f_{11}f_{30}g_{02} - 3g_{12}f_{11}f_{20} + f_{12}g_{11}^2 - 9f_{21}f_{30} + 27f_{30}g_{30} + 3f_{30}g_{11}^2 + 15f_{30}g_{02}^2 - \\
& 9f_{02}f_{31} - 28g_{20}f_{02}f_{20}^2 - 2g_{11}^2g_{21} - 3f_{22}f_{11} - 14f_{12}f_{20}^2 - 6f_{12}g_{12} + 27g_{13}g_{02} - 3f_{02}f_{11}^3 + 7f_{20}g_{21}g_{11} + 3g_{20}^2f_{11}f_{20} - \\
& 10g_{02}g_{11}^2f_{20} - 10f_{02}f_{12}f_{20} - 12g_{20}f_{30}f_{11} + 6f_{02}f_{21}g_{20} + 18f_{02}f_{11}g_{02}^2 + 3f_{12}g_{02}f_{11} + 6g_{20}g_{02}f_{12} + 18g_{02}^2f_{12} + 9g_{13}g_{20} - \\
& 3f_{12}f_{11}^2 - 45g_{20}^2f_{30} - 15f_{13}f_{02} + 30f_{20}g_{20}^3 - 18g_{02}f_{04} + 18f_{20}g_{40} - 21g_{20}f_{20}^2g_{11} + 2g_{02}g_{11}^3 + 3f_{02}g_{20}f_{11}^2 + 20f_{02}f_{20}^2g_{02} - \\
& 9g_{02}^2g_{21} - 9g_{21}g_{20}f_{11} - 9f_{04}f_{11} + 6f_{22}g_{20} + 45f_{30}f_{02}^2 + 15g_{11}g_{20}^3 - 15g_{11}g_{04} + 12f_{02}g_{02}f_{21} - 5f_{12}g_{11}f_{20} + 18g_{12}g_{20}f_{20} - \\
& 5f_{12}g_{11}f_{02} + 20f_{02}g_{21}f_{20} + 21g_{02}g_{31} - 30g_{20}g_{02}^2f_{20} + 6g_{20}g_{12}f_{02} + 12f_{22}g_{02} + 3f_{21}g_{21} + 18f_{20}^3g_{02} + 24g_{11}g_{20}^2g_{02} + \\
& 18f_{20}g_{02}g_{20}^2 + 6f_{11}g_{31} - 6g_{22}f_{02} + 15g_{31}g_{20} + 3g_{22}g_{11} - 12g_{22}f_{20} - 9f_{30}g_{12} - 18f_{20}g_{02}^3 - 24f_{20}g_{02}g_{30} + 15f_{20}f_{11}g_{02}^2 - \\
& 7g_{20}f_{02}g_{11}^2 + 6g_{20}f_{02}f_{11}g_{02} - 6g_{11}f_{13} - 28f_{02}g_{11}g_{20}f_{20} - 12g_{11}f_{20}g_{02}f_{02} + 9g_{02}g_{11}f_{11}g_{20} - 9f_{02}f_{21}f_{11} - f_{11}g_{11}f_{02}f_{20} - \\
& 15f_{30}f_{20}^2 + 10f_{02}^2f_{20}g_{02} - 8g_{11}^2f_{02}g_{02} + 42f_{20}f_{30}f_{02} - 15g_{11}g_{20}f_{02}^2 + 6f_{20}g_{20}f_{11}g_{02} - 6f_{21}f_{12} + 6g_{20}f_{11}^2f_{20} + 66f_{40}g_{02} + \\
& 27f_{30}f_{03} - 45g_{05} - 9g_{23} + 15g_{21}f_{02}^2 - 27f_{20}f_{31} - 9g_{41} + 3g_{21}g_{12} + 9g_{11}g_{20}^2f_{11} - 15f_{11}f_{02}f_{03} - 45f_{50} + 12f_{30}g_{11}f_{20} + \\
& 10g_{20}f_{20}^3 - 48f_{20}g_{20}g_{30} - 10g_{02}f_{02}^2g_{11} - 9g_{11}^2g_{20}f_{20} + 13g_{21}f_{20}^2 - 9f_{14} - 9f_{32} - 15g_{21}g_{20}^2).
\end{aligned}$$

For the first time this result was apparently obtained by Serebryakova in 1959 [Serebryakova, 1959].

To compute the third Lyapunov quantity, we obtain first $\tilde{T}_4$:

$$\begin{aligned} \tilde{T}_4 = & \frac{\pi}{1152} (784f_{20}^4 + 1540g_{20}^4 + 49g_{11}^4 - 352g_{21}g_{20}g_{11} + 48g_{21}^2 - 336f_{40}g_{11} + 2616f_{20}^2f_{03} + 480g_{22}g_{02} + 200g_{20}^2f_{02}^2 + \\ & 700f_{11}g_{20}^3 + 270f_{03}g_{30} - 154f_{20}g_{11}^3 + 1728g_{40}g_{02} + 424g_{11}^2g_{02}^2 - 48f_{22}g_{11} + 54f_{03}f_{21} + 453f_{02}^2g_{11}^2 - 2184f_{20}^2g_{30} + 400g_{02}^4 + \\ & 864f_{30}^2 + 1540f_{02}^4 + 556g_{20}^2f_{02}g_{11} + 945g_{30}^2 + 240f_{12}^2 + 768f_{20}f_{40} + 2352f_{20}f_{02}g_{02}^2 - 320g_{11}g_{21}g_{02} - 180g_{20}^2f_{20}g_{11} - \\ & 1134f_{11}g_{30}g_{20} + 5172f_{20}f_{03}f_{02} + 513f_{03}^2 + 153g_{12}^2 + f_{11}^4 - 762f_{11}f_{03}g_{02} - 1800f_{20}^2g_{20}f_{11} - 708f_{20}g_{20}g_{11}g_{02} - 888g_{12}g_{02}^2 - \\ & 84f_{21}g_{20}^2 + 1040f_{02}^2g_{02}^2 + 432g_{40}f_{11} + 4692f_{20}^2f_{02}^2 + 648f_{21}g_{02}^2 + 1180f_{02}^3g_{11} - 96g_{21}f_{02}g_{02} - 198f_{11}g_{30}g_{02} - 480f_{12}g_{20}f_{02} - \\ & 1500g_{12}g_{20}g_{02} + 228f_{20}^2f_{02}g_{11} + 912f_{02}f_{12}f_{11} - 1944g_{30}g_{02}^2 + 672f_{02}f_{40} - 48g_{22}f_{11} + 402f_{03}g_{11}^2 + 2772g_{20}^2g_{02}^2 - 63f_{20}^2g_{11}^2 + \\ & 150f_{21}g_{11}^2 + 444f_{21}f_{02}^2 + 3300f_{03}f_{02}^2 + 384f_{22}f_{02} + 432g_{11}f_{04} - 712f_{11}g_{02}^3 + 880f_{20}f_{12}f_{11} - 18f_{11}f_{03}g_{20} + 1992f_{20}f_{02}^2g_{11} + \\ & 162f_{20}f_{21}g_{11} - 2080f_{20}f_{12}g_{02} - 150f_{11}f_{02}g_{11}g_{02} - 828f_{11}g_{20}g_{02}^2 - 96f_{11}f_{30}g_{11} + 1392f_{11}f_{30}f_{02} + 980g_{11}^2g_{20}g_{02} + \\ & 64f_{11}f_{20}g_{21} - 64f_{12}f_{11}g_{11} - 6f_{11}f_{21}g_{20} - 1812g_{30}f_{20}f_{02} + 112g_{11}f_{12}g_{02} - 2744f_{11}f_{20}g_{02} - 1280f_{20}g_{20}f_{12} + 680f_{02}^2g_{20}g_{02} + \\ & 80f_{12}g_{20}g_{11} - 4164f_{20}f_{02}f_{11}g_{02} - 1212f_{20}f_{02}g_{12} - 408f_{20}g_{11}g_{02}^2 + 102f_{11}g_{20}g_{12} - 3552f_{20}f_{30}g_{20} + 22f_{11}^2f_{20}g_{11} - \\ & 1128f_{20}^2g_{12} - 48g_{20}f_{13} + 696f_{11}^2f_{20}^2 + 672g_{04}g_{20} + 2336f_{20}^2g_{02}^2 - 162f_{21}g_{12} + 96g_{31}g_{11} + 50f_{11}^2g_{11}^2 + 1128f_{03}g_{02}^2 - \\ & 3780g_{30}g_{20}^2 + 21f_{11}^2g_{20}^2 + 480f_{20}f_{22} - 432g_{02}f_{31} + 2016g_{40}g_{20} + 3080g_{30}^2g_{02} + 66f_{11}^2f_{03} - 48f_{02}g_{31} + 768g_{04}g_{02} + \\ & 1728f_{04}f_{20} - 54f_{11}^2g_{12} + 3984g_{20}^2f_{20}^2 - 2f_{11}^3g_{20} - 198f_{03}g_{12} - 432f_{20}g_{13} + 630g_{30}g_{12} - 46f_{11}f_{20}g_{11}g_{02} + 10f_{11}g_{20}f_{20}g_{11} + \\ & 30f_{20}g_{12}g_{11} + 318f_{20}f_{02}g_{11}^2 + 168f_{11}g_{20}^2g_{02} - 4788g_{30}g_{20}g_{02} - 2400f_{02}f_{30}g_{02} + 1734f_{20}f_{03}g_{11} + 4168f_{02}f_{20}g_{20}g_{02} - \\ & 62f_{02}f_{11}g_{20}g_{11} + 816f_{30}g_{20}g_{11} - 2260f_{11}g_{20}f_{20}f_{02} - 82f_{11}g_{20}g_{11}^2 + 2792g_{20}^2f_{20}f_{02} + 96f_{13}f_{11} + 9f_{21}^2 - 1536f_{02}f_{12}g_{02} + \\ & 928f_{20}g_{20}g_{21} + 620f_{02}g_{20}g_{11}g_{02} - 924g_{12}g_{20}^2 + 1424g_{20}g_{02}^3 - 396f_{02}^2g_{12} - 546g_{30}g_{11}^2 + 804g_{20}^2g_{11}^2 - 180f_{02}^2g_{30} - \\ & 2f_{02}g_{11}^3 - 528f_{13}g_{02} - 18g_{30}f_{11}^2 + 6f_{21}f_{11}^2 + 864f_{30}f_{12} - 198g_{12}g_{11}^2 + 4040f_{02}^3f_{20} + 369f_{11}^2g_{02}^2 - 330f_{11}g_{11}^2g_{02} - \\ & 1728g_{20}f_{30}f_{02} + 90f_{21}f_{02}g_{11} + 216f_{02}g_{11}g_{02}^2 + 126f_{11}^2g_{20}g_{02} - 90f_{02}g_{11}g_{12} + 1392f_{20}f_{30}f_{11} + 1614f_{03}f_{02}g_{11} + \\ & 5488f_{20}^2g_{20}g_{02} + 32f_{11}g_{21}g_{11} - 294g_{30}f_{20}g_{11} + 1332f_{11}^2f_{20}f_{02} - 1876f_{11}f_{02}^2g_{02} + 1308f_{02}f_{20}f_{21} - 3840f_{20}f_{30}g_{02} + \\ & 800f_{20}g_{21}g_{02} + 692f_{11}^2f_{02}^2 - 300f_{03}g_{20}^2 - 288f_{30}g_{21} + 2768f_{20}^3f_{02} - 58f_{11}^3g_{02} - 144g_{20}f_{31} - 528g_{31}f_{20} - 616f_{20}^3g_{11} + \\ & 984f_{20}^2f_{21} + 384g_{20}g_{22} - 144f_{02}g_{13} + 90f_{21}g_{30} + 2016f_{02}f_{04} - 336f_{11}g_{04} - 366f_{02}g_{30}g_{11} + 96f_{02}f_{11}g_{21} - 18f_{02}f_{11}^2g_{11} + \\ & 468f_{03}g_{20}g_{02} - 96f_{02}g_{20}g_{21} + 144f_{41} - 720g_{50} + 720f_{05} + 144f_{23} - 144g_{14} - 144g_{32} + 816f_{30}g_{11}g_{02} + 510f_{11}g_{12}g_{02} + \\ & 444f_{21}g_{20}g_{02} - 500g_{20}f_{11}f_{02}^2 - 222f_{11}f_{21}g_{02}). \end{aligned}$$

From condition $L_2 = 0$ one can get g_{05} and compute expression for L_3 .

$$\begin{aligned} L_3 = & \frac{\pi}{1728} (1062g_{12}g_{02}^2f_{30} + 18g_{12}g_{20}f_{22} + 108f_{02}g_{20}g_{12}^2 + 70f_{02}^2f_{20}f_{11} + 3000f_{02}^2g_{20}f_{40} + 210f_{02}^3f_{21}g_{02} - 96g_{21}^2g_{02}g_{11} - \\ & 70f_{02}^3g_{11}f_{12} - 198f_{02}^2f_{30}f_{21} - 63f_{02}^2g_{11}g_{22} + 1500g_{20}^3f_{02}^2f_{20} + 1350g_{20}^3g_{02}g_{21} - 30g_{11}^3f_{03}f_{11} - 225g_{20}^3f_{11}^2f_{20} - 1305g_{20}^2g_{02}g_{31} - \\ & 9g_{20}^2f_{11}f_{22} - 1908g_{20}^2g_{40}f_{20} + 180f_{11}g_{50}g_{11} + 2228f_{20}^3f_{02}^2g_{02} - 1052f_{20}^3f_{02}f_{12} + 72f_{03}g_{02}f_{22} + 1008f_{03}f_{20}g_{04} + \\ & 540f_{20}^2g_{02}^2f_{30} - 144f_{03}f_{11}g_{30}g_{11} + 9f_{11}^5f_{20} + 432f_{03}g_{40}f_{20} + 252f_{20}f_{11}f_{03}^2 + 171g_{20}g_{11}^2f_{04} + 522f_{20}f_{12}f_{40} - 135g_{20}f_{12}f_{31} + \\ & 432f_{30}^2g_{20}g_{11} + 12g_{21}^2f_{20}f_{11} + 1890f_{02}g_{06} - 180g_{20}f_{02}f_{41} + 1170g_{30}g_{20}g_{02}f_{20}f_{11} - 648f_{20}f_{02}g_{20}f_{11}g_{21} + 1158f_{20}f_{02}g_{02}g_{11}g_{12} + \\ & 930f_{02}^2f_{11}f_{40} + 30f_{12}^2f_{20}g_{20} + 27f_{41}g_{20}g_{11} - 99g_{02}f_{12}f_{13} + 972g_{02}g_{20}g_{41} + 270g_{02}^2f_{02}g_{22} + 573f_{20}^2f_{02}f_{11}^2g_{20} + 36f_{12}^2f_{11}g_{11} - \\ & 1206f_{20}g_{20}f_{41} + 270f_{20}g_{20}f_{05} + 360f_{02}^2g_{02}^2g_{20}g_{11} + 180f_{20}f_{11}f_{23} - 315g_{02}^2f_{11}g_{31} + 540g_{20}^4g_{21} + 594f_{20}^4g_{02}g_{11} - \\ & 1665f_{02}g_{11}f_{50} + 1287f_{20}^2g_{02}g_{31} - 198f_{20}^2f_{04}f_{11} + 48g_{11}^4f_{02}g_{20} + 80g_{11}^4f_{20}g_{20} - 438g_{11}^3f_{20}^2g_{02} - 135f_{03}g_{23} + 198g_{11}^3f_{02}f_{30} + \\ & 75g_{11}^3f_{11}f_{40} + 312g_{11}^2g_{02}f_{40} - 886g_{11}^2f_{20}^2g_{02} + 81f_{02}g_{21}g_{13} - 273g_{11}^2f_{20}f_{13} + 234f_{21}g_{20}f_{40} + 36f_{21}f_{11}f_{22} + 378f_{21}f_{20}^2g_{02} + \\ & 288f_{21}f_{20}f_{13} - 9f_{21}^2g_{20}g_{11} + 2700g_{30}g_{20}f_{40} + 756g_{30}g_{02}g_{31} + 30g_{30}g_{11}^3g_{02} + 675g_{30}g_{20}g_{31} - 288g_{30}f_{20}^2f_{12} + 540g_{02}^2f_{20}g_{20} + \\ & 270g_{30}f_{11}g_{31} - 891g_{30}^2g_{20}g_{11} + 54f_{11}g_{30}g_{13} - 63f_{11}^4g_{02}f_{20} - 264f_{03}f_{20}g_{11}^2g_{02} - 225g_{11}^2f_{50} - 162f_{11}^2f_{30}g_{30} + 108f_{11}^2f_{20}^3g_{20} + \\ & 90f_{11}^2f_{20}g_{04} - 18f_{11}^2f_{20}g_{22} - 27g_{12}f_{02}f_{31} - 126g_{12}f_{02}g_{04} + 162g_{12}g_{20}g_{13} - 30f_{11}g_{21}^2f_{02} + 612g_{12}f_{20}^2g_{20}g_{11} - 720g_{02}f_{30}g_{22} + \\ & 423f_{03}g_{02}g_{31} + 972f_{20}f_{30}g_{13} - 27g_{12}f_{21}f_{02}f_{11} - 504g_{11}^2f_{02}g_{04} - 33f_{12}^2g_{02}g_{11} + 63g_{20}f_{11}g_{02}^2f_{12} + 135f_{03}g_{20}g_{13} + \\ & 744f_{20}f_{02}^2f_{11}g_{02} - 270g_{60}f_{20} - 1404f_{30}^2f_{20}g_{20} + 930f_{02}^2g_{02}^2f_{30} - 432f_{30}^2f_{02}f_{11} + 270g_{02}^3f_{22} + 180f_{20}^5g_{20} - 1188f_{20}f_{30}f_{04} - \\ & 504f_{02}g_{21}f_{04} + 1512f_{20}^2f_{02}g_{02}^2f_{11} + 1398f_{20}f_{02}^2f_{21}g_{02} - 54g_{20}g_{12}f_{04} + 450g_{50}g_{20}g_{11} - 414f_{40}^2f_{12} + 135g_{02}f_{11}g_{23} + \\ & 63g_{20}f_{11}^2g_{02}^2f_{02} + 252f_{03}f_{20}f_{13} + 177f_{12}^2f_{20}f_{11} + 495g_{50}g_{02}g_{11} + 18f_{02}f_{11}g_{32} - 5820f_{20}f_{02}g_{20}g_{02}f_{30} - 18g_{12}f_{21}g_{02}f_{02} + \\ & 6g_{11}^4f_{30} + 432f_{02}f_{30}f_{40} + 3g_{11}g_{21}f_{22} - 738g_{30}g_{20}f_{11}g_{21} - 414g_{02}f_{02}f_{41} + 318f_{20}f_{02}g_{02}^2g_{20}g_{11} - 550f_{20}f_{02}g_{11}^3g_{02} + \\ & 234g_{30}g_{02}^2f_{02}f_{11} + 108g_{30}f_{20}^2f_{30} - 2520f_{03}f_{20}f_{02}^2g_{02} - 18g_{30}f_{02}f_{11}g_{12} + 78g_{11}^4f_{20}g_{02} - 756g_{40}f_{40} + 189f_{02}^2g_{12}g_{21} - \\ & 189g_{04}g_{13} + 1350f_{02}^2f_{30}g_{30} + 63g_{11}f_{12}f_{04} + 360g_{02}g_{11}f_{05} - 351f_{20}g_{11}g_{23} + 54f_{21}f_{11}g_{31} - 450g_{20}f_{11}f_{20}g_{11}g_{21} - \\ & 135g_{50}g_{21} + 81g_{43} - 1620f_{20}f_{02}g_{20}^2g_{21} + 105g_{11}^3f_{20}f_{02}f_{11} - 81g_{30}f_{20}f_{13} - 135f_{21}g_{02}^2g_{21} - 483f_{20}^2f_{02}f_{21}f_{11} - 24g_{21}^2f_{12} - \\ & 351f_{11}^2g_{20}f_{40} - 3972g_{30}f_{20}^2g_{20}g_{11} + 126g_{20}g_{21}g_{22} - 1026g_{02}f_{12}g_{04} + 324f_{30}^2g_{02}g_{11} - 6f_{21}g_{11}^3g_{20} + 210f_{12}^2g_{02}f_{02} - \\ & 108f_{11}^2g_{20}g_{31} + 90g_{20}^2f_{12}f_{21} - 9g_{11}f_{12}f_{40} + 657f_{20}f_{02}f_{11}g_{13} + 81f_{13}f_{22} - 45f_{21}g_{11}^2f_{30} - 72g_{12}f_{21}f_{20}f_{11} - 54g_{20}f_{11}f_{04}g_{02} - \end{aligned}$$

$$\begin{aligned}
& 81f_{20}f_{11}g_{32} + 1440f_{03}g_{20}f_{40} + 144f_{02}^2f_{21}g_{21} - 270g_{02}^2f_{11}f_{04} + 135g_{12}g_{40}g_{11} + 234g_{12}f_{21}f_{20}g_{20} + 63f_{11}^2f_{02}f_{31} - \\
& 63f_{02}^2g_{12}f_{30} - 42f_{20}^2f_{11}g_{02}f_{12} - 705g_{11}^2g_{20}g_{02}f_{30} + 63f_{11}g_{21}g_{04} + 180g_{11}f_{05}f_{11} + 1188g_{02}f_{30}g_{40} + 285g_{02}^2f_{20}g_{11}f_{30} - \\
& 45g_{20}^3f_{11}^2f_{02} - 180g_{20}f_{12}g_{04} + 108f_{20}f_{02}g_{20}f_{22} + 36f_{03}f_{11}f_{22} - 45f_{03}g_{20}f_{11}^2f_{02} + 15f_{12}^2g_{20}g_{11} + 252f_{02}f_{30}f_{22} - \\
& 639g_{30}g_{02}^2g_{21} + 672f_{20}^2f_{02}f_{11}^2g_{02} + 117g_{02}f_{31}g_{21} - 156g_{20}f_{11}g_{11}f_{02}f_{30} + 54f_{12}g_{32} + 174g_{21}^2f_{20}g_{20} + 210f_{02}f_{12}f_{22} - \\
& 2792f_{20}^3f_{02}^2g_{20} + 1221g_{11}^2g_{02}^2f_{20}g_{20} + 9f_{11}^4f_{12} + 210f_{02}^3g_{20}g_{02}f_{11} + 222g_{02}f_{20}g_{21}^2 + 117g_{20}^2f_{20}f_{13} + 216f_{21}g_{20}g_{02}f_{12} - \\
& 972g_{02}f_{30}f_{31} + 126g_{11}f_{13}f_{03} - 15f_{11}^3g_{11}^2f_{02} + 63g_{02}g_{21}f_{13} + 18f_{11}^3g_{11}f_{03} - 15f_{11}^2g_{11}^2f_{12} + 111g_{21}f_{12}g_{20}g_{11} - 36g_{11}f_{12}g_{31} + \\
& 54f_{31}g_{30}g_{11} - 456f_{20}f_{02}^2f_{21}f_{11} - 1200f_{02}^2g_{30}g_{20}g_{11} - 63g_{12}f_{11}^2g_{02}f_{02} + 135f_{11}^2g_{20}^2g_{21} + 252f_{02}g_{11}f_{04}g_{02} + 30g_{20}f_{02}g_{21}^2 - \\
& 378f_{03}g_{02}^2f_{12} - 243g_{02}f_{11}f_{20}f_{31} - 26f_{20}^4f_{02}f_{11} + 90g_{12}f_{11}f_{40} + 384f_{20}f_{12}f_{22} - 54g_{12}f_{30}f_{21} + 648f_{02}f_{30}g_{13} + 117f_{12}f_{11}g_{04} + \\
& 2052f_{20}^4f_{02}g_{02} + 4g_{11}^5g_{02} + 144g_{02}g_{21}g_{22} + 522g_{02}g_{40}g_{21} - 504g_{02}f_{11}g_{40}g_{11} - 2700g_{02}f_{60} - 123g_{20}^2f_{20}g_{11}f_{12} + 81f_{03}g_{11}g_{22} + \\
& 18f_{11}^2g_{11}f_{13} + 81f_{21}f_{04}f_{11} + 90g_{20}^2g_{12}f_{12} - 30f_{11}g_{21}^2g_{11} - 198f_{21}^2g_{02}f_{02} - 45g_{20}f_{12}f_{13} - 132g_{20}^2g_{11}^3g_{02} - 162g_{02}g_{21}g_{04} + \\
& 54g_{11}f_{03}f_{31} + 342g_{02}g_{20}f_{02}f_{31} - 36f_{11}f_{30}g_{22} + 315f_{02}f_{11}f_{03}^2 - 552f_{03}f_{20}^3f_{11} + 306g_{12}g_{20}g_{02}f_{02}f_{11} - 1512f_{02}f_{30}f_{04} - \\
& 54g_{22}f_{40} + 504g_{20}g_{40}g_{21} + 54f_{11}^2g_{41} - 90g_{20}^3g_{02}f_{12} + 504g_{02}^2f_{40} - 54g_{20}f_{24} + 243f_{21}g_{02}g_{31} - 99g_{20}^2f_{31}g_{11} + 378f_{20}f_{02}g_{20}g_{11}g_{12} + \\
& 2946g_{20}^3f_{20}^2g_{11} + 57g_{11}f_{12}f_{22} + 54g_{04}f_{22} - 135g_{21}f_{05} - 90g_{20}^3f_{02}f_{21} + 630g_{02}f_{02}f_{03}^2 - 144f_{30}g_{21}f_{12} - 2052g_{02}f_{30}g_{04} + \\
& 180f_{02}f_{12}f_{40} + 315f_{02}f_{11}f_{05} + 18f_{11}g_{11}f_{23} + 99g_{20}f_{31}g_{21} - 486f_{20}g_{20}g_{32} + 9f_{11}^3f_{40} + 279g_{02}^3f_{11}f_{30} + 135f_{02}f_{51} + \\
& 48f_{03}g_{11}^3g_{20} + 441f_{20}f_{12}g_{13} - 9f_{12}f_{11}g_{22} + 189f_{04}f_{13} - 54g_{20}f_{21}f_{04} - 455f_{02}^3g_{11}^2g_{20} - 81g_{02}f_{11}f_{14} + 54g_{13}f_{13} + \\
& 315f_{02}^2f_{11}g_{13} + 9f_{11}^3f_{22} - 504f_{20}f_{02}g_{23} - 252f_{20}g_{11}f_{32} - 900f_{20}f_{02}g_{41} + 315f_{02}f_{13}f_{03} + 54g_{13}f_{03}f_{11} - 350f_{02}^3g_{11}^2g_{02} + \\
& 216f_{02}g_{11}g_{20}g_{13} - 414g_{02}f_{42} + 270g_{02}^3g_{20}g_{21} - 135g_{20}g_{15} + 432f_{11}f_{30}g_{40} + 1062g_{02}^2f_{20}g_{22} + 189f_{03}f_{04}f_{11} - 140f_{02}^3f_{20}f_{12} + \\
& 1086g_{12}f_{20}^2g_{02}g_{11} + 45g_{20}^2f_{11}^3f_{20} + 45f_{02}f_{12}g_{31} + 135g_{20}g_{11}f_{05} + 396g_{20}f_{11}g_{41} + 504f_{02}^2g_{11}^2f_{30} + 264f_{20}f_{02}g_{20}f_{11}f_{12} + \\
& 63g_{12}g_{30}f_{20}f_{11} - 120f_{20}^2f_{02}f_{11}^3 + 72g_{12}f_{11}g_{31} + 267f_{03}f_{20}^2g_{21} - 1755g_{20}^2g_{11}f_{02}f_{30} - 2880f_{20}^3g_{04} - 27f_{12}f_{11}g_{40} - \\
& 1890g_{20}f_{60} + 189f_{03}g_{20}^2f_{11}f_{20} + 9g_{20}g_{21}f_{13} - 1005f_{02}^2g_{30}g_{02}g_{11} - 45f_{20}f_{11}g_{12}^2 + 540g_{12}g_{02}^2f_{02}f_{11} - 396f_{21}g_{20}g_{02}g_{21} - \\
& 216f_{21}f_{04}g_{02} + 81f_{34} + 585g_{20}^3f_{11}g_{21} + 648f_{20}^5g_{02} - 252g_{20}f_{30}g_{22} + 140f_{02}^4f_{20}g_{02} + 2026f_{20}^3f_{02}g_{21} - 585g_{20}^4f_{11}g_{11} + \\
& 324f_{20}^2f_{11}g_{13} + 216f_{03}g_{30}g_{21} - 678f_{20}^2g_{02}^2g_{21} - 3087g_{20}^2f_{20}g_{11}f_{30} - 204g_{02}g_{20}f_{20}g_{11}f_{12} - 27f_{41}g_{21} - 210f_{02}^3f_{13} - \\
& 588f_{20}^2f_{02}g_{30}f_{11} + 180g_{11}f_{15} - 120g_{12}f_{20}^2f_{12} - 30g_{20}f_{11}f_{20}^3g_{02} - 360g_{12}f_{20}g_{04} - 612g_{12}g_{20}g_{02}g_{21} + 63f_{20}f_{12}g_{31} - \\
& 81g_{30}f_{32} - 99g_{11}^2g_{02}^2f_{20}f_{11} + 27g_{12}g_{11}^2f_{30} + 135f_{16} + 54g_{12}f_{11}^2f_{20}g_{20} - 776f_{20}^4f_{02}g_{20} + 66f_{21}g_{11}^2f_{02}g_{02} + 54f_{02}g_{42} - \\
& 2535g_{30}f_{20}^2g_{02}g_{11} + 540g_{12}g_{02}^2f_{12} - 27g_{12}f_{32} - 450g_{30}g_{02}^2f_{20}g_{20} - 81g_{22}g_{13} + 135f_{52} - 147f_{03}g_{11}^2f_{02}f_{11} - 2286g_{12}g_{02}^2f_{20}g_{20} + \\
& 279g_{12}f_{20}g_{11}g_{21} + 225f_{20}f_{11}g_{50} - 540g_{20}^5g_{11} + 108g_{13}f_{03}g_{02} + 333g_{20}^2f_{20}^2g_{11}f_{11} + 108f_{03}f_{11}^3f_{02} - 216g_{20}f_{11}f_{20}f_{13} + \\
& 612g_{12}g_{02}^2f_{20}f_{11} + 1386f_{20}^2f_{02}g_{20}g_{12} + 177g_{20}f_{11}g_{11}f_{02}f_{12} - 90f_{12}^2f_{30} - 324g_{20}^3f_{20}f_{11} + 198f_{20}f_{02}f_{12}f_{21} - 2916f_{20}f_{02}g_{30}g_{02}g_{11} - \\
& 270g_{02}^3f_{11}f_{12} - 720g_{02}^2f_{11}f_{40} + 384g_{21}f_{12}g_{02}f_{02} - 2457f_{20}^2f_{02}f_{31} - 126f_{02}^2g_{20}g_{11}g_{12} - 270g_{02}^3g_{20}^2g_{11} - 81f_{11}^2f_{04}g_{02} - \\
& 231g_{12}g_{11}f_{02}f_{12} - 9f_{11}^3g_{02}^2f_{02} + 36g_{30}g_{11}^2f_{12} + 153f_{21}f_{11}f_{40} + 24f_{02}^2f_{21}g_{20}g_{11} + 276f_{20}g_{11}f_{11}g_{31} + 267g_{20}^2g_{11}^2f_{02}f_{11} + \\
& 372f_{20}f_{02}g_{12}g_{21} + 1428f_{20}f_{02}g_{20}g_{31} + 297f_{11}^2g_{30}f_{20}g_{20} + 504f_{02}g_{20}g_{11}f_{04} + 18g_{12}g_{11}^3g_{20} - 15f_{11}^2g_{11}^2f_{30} - 72f_{20}f_{11}f_{41} - \\
& 270g_{02}^4f_{11}f_{20} + 420g_{20}^2g_{11}^2f_{02}^2f_{11} - 189g_{11}^2f_{02}f_{13} - 468g_{12}g_{30}g_{20}g_{11} - 90g_{20}g_{11}g_{14} + 12g_{12}g_{11}^3g_{02} - 171g_{11}g_{21}f_{04} + \\
& 45g_{11}g_{21}f_{40} - 576g_{20}f_{11}f_{20}^2f_{30} + 156g_{20}^2g_{11}^2g_{21} + 504f_{11}^2f_{20}f_{02}g_{20}g_{11} + 168f_{03}f_{20}^2f_{12} - 54g_{02}f_{02}f_{11}g_{40} - 6f_{02}g_{21}f_{22} - \\
& 93f_{21}g_{11}^2f_{02}g_{20} + 1512g_{20}f_{30}g_{40} + 102g_{20}^2f_{20}^2f_{12} + 336f_{03}f_{20}f_{02}g_{20}g_{11} - 210f_{02}^3f_{11}f_{03} + 54f_{21}f_{20}g_{04} + 792f_{20}^2g_{20}f_{04} + \\
& 150f_{21}g_{11}^2f_{20}g_{02} - 216g_{11}f_{30}f_{40} - 27g_{20}f_{11}f_{14} - 495f_{20}^2g_{20}g_{13} - 54g_{12}^2f_{12} - 108g_{12}g_{41} + 2940f_{20}^2f_{02}g_{20}^3 - 90g_{20}^3f_{02}g_{12} + \\
& 1050f_{02}^3g_{11}f_{30} + 54f_{11}^3f_{21}f_{02} - 54f_{11}g_{33} + 72f_{20}g_{11}f_{04}g_{02} + 276f_{20}g_{11}g_{02}f_{22} - 675g_{02}g_{15} - 144f_{03}f_{20}g_{02}f_{21} + 108f_{02}f_{30}g_{31} - \\
& 108g_{11}f_{30}g_{31} - 261f_{11}^2f_{21}g_{02}f_{02} + 27f_{21}f_{32} + 129g_{20}f_{11}f_{20}^2g_{02}g_{11} - 165f_{11}^2f_{20}g_{11}f_{30} - 2030f_{20}f_{02}^3g_{20}g_{11} - 6g_{12}g_{11}^2f_{12} - \\
& 1053g_{30}f_{11}g_{02}f_{30} - 2250f_{02}^2f_{50} - 4476f_{20}^2f_{02}g_{30}g_{20} + 156g_{20}f_{11}g_{11}^2f_{02}g_{02} - 90g_{30}f_{20}f_{02}f_{12} - 529f_{20}f_{02}g_{11}^3g_{20} + \\
& 36g_{12}f_{21}g_{21} - 81g_{30}f_{02}f_{31} - 27g_{11}g_{42} - 660g_{20}^2g_{11}f_{02}g_{21} - 288g_{20}^2g_{02}^2f_{12} + 99f_{30}f_{12}g_{02}g_{11} + 153f_{20}f_{11}g_{14} - 875f_{20}^3g_{11}f_{12} - \\
& 492f_{20}^3g_{11}f_{30} + 420f_{02}^2g_{02}^2f_{12} - 90f_{03}g_{20}g_{02}f_{02}f_{11} + 4230g_{02}g_{20}f_{50} - 144g_{02}g_{11}f_{23} - 234g_{02}f_{11}f_{02}f_{13} + 18g_{12}g_{11}f_{13} + \\
& 450g_{12}g_{02}f_{40} - 180f_{11}g_{51} - 90g_{20}^3g_{02}f_{02}f_{11} - 342f_{20}g_{21}f_{40} + 180g_{30}g_{02}f_{22} + 45f_{11}^2g_{02}^2f_{20} - 978f_{20}^2f_{02}g_{20}g_{02}f_{11} - \\
& 54f_{02}f_{11}g_{12}^2 - 261g_{20}^2f_{21}g_{21} - 1251f_{02}g_{11}f_{20}f_{31} + 315f_{02}^2g_{20}g_{13} - 189g_{40}g_{31} + 1008f_{20}f_{02}g_{20}f_{04} - 954g_{20}^2g_{40}g_{11} + \\
& 654f_{20}^3g_{02}^2f_{11} + 324f_{21}f_{20}^2f_{12} - 972f_{20}^3f_{31} - 1062g_{02}^3g_{20}f_{30} + 45f_{11}^2f_{12}f_{21} - 72f_{03}f_{11}^2f_{20}g_{20} - 147f_{02}^2g_{11}^3g_{20} - 230f_{20}^4g_{20}g_{11} + \\
& 315g_{02}^2f_{11}g_{30}g_{11} + 414g_{12}g_{02}g_{31} - 216f_{20}^2g_{21} - 195f_{21}g_{11}f_{02}^2f_{11} - 3450f_{20}^3f_{02}g_{20}g_{11} - 90g_{20}f_{12}g_{22} - 108f_{21}g_{41} - \\
& 216g_{30}f_{20}g_{22} - 2940f_{20}f_{02}g_{20}g_{02}g_{21} - 2328f_{02}g_{11}g_{20}g_{02}f_{30} - 117f_{03}f_{21}g_{20}g_{11} - 45f_{21}f_{11}g_{02}g_{21} + 6f_{11}^2g_{11}f_{02}g_{21} + \\
& 84g_{11}^3f_{20}^2f_{11} + 864g_{30}f_{11}f_{40} + 420f_{02}^2f_{11}g_{02}f_{12} + 225g_{30}g_{02}^2f_{20}f_{11} + 2592g_{30}g_{02}f_{40} - 603g_{20}f_{11}g_{40}g_{11} - 351g_{02}g_{33} - \\
& 378f_{03}g_{02}f_{02}g_{12} + 144g_{11}^2f_{04}g_{02} + 108f_{03}f_{20}f_{11}g_{12} - 1620g_{20}^4g_{02}f_{20} + 1164f_{20}^3f_{02}f_{30} + 618f_{20}f_{02}f_{11}g_{02}f_{12} - 18f_{11}^3g_{30}f_{20} + \\
& 810g_{02}^2f_{20}f_{31} - 504f_{03}g_{02}^2f_{30} + 204f_{02}g_{11}f_{11}g_{31} + 675f_{20}f_{51} + 2460f_{02}g_{11}g_{02}f_{40} + 1005f_{02}^2g_{02}g_{31} - 27g_{30}f_{30}f_{21} + \\
& 108f_{12}f_{11}f_{13} - 168f_{03}f_{20}f_{02}g_{21} - 698f_{20}^2f_{02}^2g_{02}g_{11} + 780g_{20}^2f_{20}f_{02}^2g_{02} - 72f_{21}g_{20}f_{22} + 81f_{04}f_{31} + 108f_{21}^2f_{20}f_{11} + \\
& 351g_{20}^3g_{11}g_{12} + 1224f_{20}^2f_{11}f_{40} + 441g_{21}g_{02}g_{11}f_{30} + 2g_{11}^4f_{02}f_{11} + 18f_{11}^2f_{20}g_{11}^2g_{20} + 459f_{30}f_{41} + 21g_{11}^3f_{02}^2f_{11} - 810g_{02}f_{20}g_{32} -
\end{aligned}$$

$$\begin{aligned}
& 90f_{21}f_{20}g_{22} - 936g_{02}^2g_{20}g_{31} - 540g_{30}f_{20}f_{31} + 906g_{02}^2f_{11}f_{20}f_{02}g_{11} - 804f_{20}^2g_{11}f_{13} + 54g_{11}g_{21}g_{13} - 99f_{20}g_{21}g_{31} + \\
& 1296g_{02}^2f_{20}g_{04} + 18g_{11}^3g_{40} - 2502g_{20}f_{11}g_{02}f_{40} - 714f_{20}f_{02}g_{20}f_{11}f_{30} - 163f_{20}^3f_{02}g_{11}f_{11} - 90f_{11}^2g_{02}g_{31} + 105f_{12}^2f_{02}f_{11} + \\
& 756g_{20}^2g_{02}g_{11}g_{12} + 18g_{02}^2g_{20}f_{22} + 54f_{21}g_{11}f_{03}f_{11} - 276g_{20}f_{11}f_{20}^2g_{21} + 108f_{21}g_{40}g_{11} + 228f_{02}g_{11}g_{02}^2f_{12} + 9f_{11}^2f_{32} + \\
& 189f_{02}^2g_{20}f_{11}g_{02}g_{11} - 54g_{30}f_{12}f_{21} + 63f_{11}^3g_{20}g_{02}f_{02} + 33f_{11}^2g_{11}^2f_{02}g_{02} - 828f_{03}g_{02}^2f_{02}f_{11} + 270g_{02}^3f_{21}f_{02} + 180g_{12}g_{20}g_{02}f_{20}f_{11} - \\
& 198f_{21}g_{02}f_{22} + 1284f_{20}f_{02}f_{11}g_{02}f_{30} + 39g_{11}^3f_{20}f_{12} - 252f_{02}^2g_{11}f_{31} - 27f_{03}g_{12}f_{30} + 90f_{02}g_{20}f_{12}f_{30} - 54g_{30}g_{12}f_{12} + \\
& 27f_{11}f_{24} - 54f_{11}^4g_{02}f_{02} + 1209g_{20}^3f_{20}g_{11}^2 + 189g_{20}^2f_{21}f_{20}f_{11} - 270g_{12}f_{04}g_{02} + 144g_{30}f_{02}g_{20}g_{12} - 144f_{21}g_{02}g_{11}f_{03} - \\
& 198f_{03}f_{11}^2g_{02}f_{02} + 750g_{20}^3f_{02}^2g_{11} - 45g_{11}^2f_{32} - 180g_{20}f_{42} + 105f_{02}^2f_{11}f_{22} - 1328f_{20}^2f_{02}g_{11}f_{12} - 420g_{12}f_{20}g_{11}^2g_{20} - \\
& 423f_{03}g_{30}g_{02}g_{11} + 18g_{20}^2f_{11}^3f_{02} + 135f_{21}g_{02}^2f_{30} + 576f_{21}g_{20}g_{02}f_{20}f_{11} - 315f_{02}^2g_{11}f_{13} + 2089f_{20}^2f_{02}g_{11}g_{21} - 909g_{20}f_{11}g_{02}g_{31} + \\
& 18g_{20}^2f_{11}^2f_{12} - 450f_{20}g_{21}f_{04} - 18g_{12}f_{11}^2f_{30} + 1431g_{20}^2g_{30}f_{20}f_{11} - 30g_{11}^3f_{13} - 15f_{11}^3g_{11}^2f_{20} - 270f_{21}f_{50} + 648g_{30}f_{21}f_{20}g_{20} - \\
& 351g_{30}f_{21}g_{20}g_{11} + 756g_{04}f_{04} + 366g_{02}f_{12}f_{20}g_{21} - 140f_{02}^4g_{11}g_{02} - 168f_{02}^2g_{11}^3g_{02} - 180g_{02}^2g_{11}f_{13} - 45g_{30}f_{21}f_{02}f_{11} + \\
& 207f_{03}f_{20}^2f_{30} - 213g_{11}^2g_{20}g_{02}f_{12} - 60f_{21}g_{11}^2f_{20}f_{11} - 126g_{20}^2f_{20}f_{11}g_{12} - 135g_{02}^2g_{11}g_{22} + 768f_{20}f_{02}g_{20}^2f_{11}g_{11} - 231g_{12}g_{11}f_{02}^2f_{11} + \\
& 1935g_{02}f_{11}f_{50} - 45f_{11}^2g_{02}^2f_{20}g_{20} + 72f_{21}^2f_{02}f_{11} + 102g_{21}^2g_{02}f_{02} + 2250g_{20}^2f_{50} - 450f_{21}g_{02}^2f_{20}g_{20} + 81f_{03}f_{21}g_{21} + \\
& 18g_{20}f_{11}g_{02}f_{22} - 1155f_{20}f_{02}^2f_{13} + 216f_{21}g_{20}g_{31} + 153g_{20}g_{11}f_{12}f_{30} + 9g_{30}f_{11}f_{22} + 405g_{02}^2g_{23} + 345f_{02}^2f_{11}g_{31} - 2250g_{20}^2f_{02}^2f_{30} - \\
& 45g_{30}g_{11}f_{02}f_{12} + 228g_{02}^2f_{11}g_{11}f_{02}^2 - 27f_{11}^4f_{20}g_{20} + 84f_{03}g_{11}^2f_{02}g_{20} - 291f_{20}f_{02}f_{11}^2f_{30} + 216f_{03}g_{02}^2g_{20}g_{11} - 245f_{20}f_{02}^3g_{11}f_{11} + \\
& 81f_{03}g_{12}g_{21} - 4g_{11}^4g_{21} + 315f_{02}f_{12}g_{13} + 6g_{21}^3 - 2240f_{20}^2f_{02}^2g_{20} + 216f_{03}g_{40}g_{11} + 42g_{20}f_{11}g_{11}^2g_{21} + 1269g_{30}f_{20}g_{11}f_{30} - \\
& 72g_{11}^3g_{02}^2g_{20} - 720f_{20}g_{20}g_{14} - 1362f_{20}^2f_{02}g_{22} + 36f_{03}f_{11}^3f_{30} + 210f_{02}^4g_{21} + 63f_{03}f_{11}^2f_{12} - 18g_{12}g_{02}f_{22} - 1080f_{03}g_{20}^2f_{30} - \\
& 84g_{11}^2g_{02}^2f_{12} + 558g_{30}f_{20}g_{11}g_{21} - 885g_{30}g_{11}^2f_{02}g_{02} - 819f_{20}f_{02}f_{11}g_{30}g_{11} + 18f_{11}g_{11}f_{30}f_{12} + 990f_{20}^2g_{20}g_{31} + 234g_{30}g_{02}^2f_{12} + \\
& 18g_{12}g_{11}f_{03}f_{11} + 294f_{20}^2g_{02}^2f_{12} + 1080f_{20}^2g_{02}f_{40} - 90f_{21}g_{20}f_{11}g_{21} + 270g_{02}f_{02}g_{12}^2 + 1215g_{20}^3f_{11}f_{30} - 315f_{03}f_{02}^2g_{21} - \\
& 78f_{11}^2f_{20}g_{11}f_{12} - 45g_{02}f_{11}g_{11}g_{22} + 168f_{02}^2g_{02}g_{11}g_{12} - 108f_{21}f_{20}^3f_{11} + 1278g_{20}^2f_{11}g_{30}g_{11} - 138f_{03}f_{20}^3g_{20} + 5352f_{20}f_{02}g_{20}f_{40} + \\
& 105f_{02}^2f_{11}^2f_{12} + 9g_{11}^4f_{04}f_{11} + 1080f_{20}g_{11}g_{02}g_{13} + 1368f_{20}^2g_{40}g_{11} + 18f_{11}^2g_{11}f_{02}^2g_{02} - 1008g_{30}g_{02}^3f_{20} + 36g_{11}f_{30}f_{22} - \\
& 87g_{20}^2g_{11}^2f_{12} - 168f_{21}f_{20}g_{11}f_{12} + 54g_{11}^2g_{02}g_{13} - 27f_{11}^3g_{20}f_{30} - 42f_{20}f_{02}g_{12}f_{12} + 117g_{20}^2f_{30}f_{21} + 988f_{20}^3g_{11}g_{21} + \\
& 144g_{30}g_{20}f_{22} - 162g_{12}f_{20}f_{31} - 309g_{02}f_{11}g_{11}f_{02}f_{30} + 36g_{20}^2f_{02}f_{11}g_{12} - 1068f_{20}f_{02}g_{02}^2g_{21} + 90f_{12}f_{23} + 21g_{11}^3f_{02}f_{12} - \\
& 36g_{11}^2g_{23} + 396g_{12}g_{20}g_{02}f_{12} + 261g_{12}g_{20}g_{31} - 81g_{30}g_{23} - 270g_{02}^2f_{02}g_{20}g_{12} - 207f_{21}g_{20}f_{11}f_{30} - 414g_{20}^2g_{11}^3f_{30} + 504f_{03}g_{11}g_{04} + \\
& 81g_{20}^2f_{21}f_{02}f_{11} + 846g_{30}f_{20}^2g_{21} - 36g_{20}^2g_{11}g_{04} - 252f_{02}g_{11}f_{32} + 648f_{03}f_{30}g_{30} + 72g_{30}g_{20}f_{11}^2f_{02} - 60g_{20}^3g_{11}^3 + 414g_{12}g_{30}f_{20}g_{20} + \\
& 135g_{20}^2g_{23} - 54g_{40}f_{22} + 54g_{12}f_{04}f_{11} - 360g_{02}f_{02}f_{23} - 504f_{03}f_{20}f_{02}f_{30} + 1566f_{20}f_{02}g_{40}g_{11} + 3696f_{20}f_{02}g_{20}^2g_{02}g_{11} - \\
& 180g_{12}f_{11}g_{02}f_{30} + 135f_{02}f_{11}f_{23} - 288g_{20}^2g_{02}^2f_{02}f_{11} - 12f_{11}^2g_{11}f_{02}f_{12} - 4410f_{20}f_{02}g_{20}^2f_{30} + 135g_{30}f_{21}g_{21} - 18f_{11}^2g_{20}f_{22} + \\
& 1134g_{20}^2g_{11}^2f_{02}g_{02} + 174f_{21}f_{20}g_{11}g_{21} - 2412f_{20}^2g_{20}g_{02}g_{21} - 621f_{20}^3f_{13} - 306g_{20}f_{11}f_{20}g_{04} - 225g_{02}g_{11}g_{14} - 54g_{30}g_{11}^2g_{21} - \\
& 9f_{11}^2f_{02}g_{22} + 414f_{11}^2g_{20}g_{02}f_{30} - 4230f_{20}f_{02}f_{50} - 510f_{02}^2g_{20}f_{11}f_{30} + 2124g_{02}g_{20}f_{20}g_{04} + 336f_{03}f_{20}^2f_{02}g_{20} + 558f_{20}^4g_{21} - \\
& 102g_{21}^2g_{20}g_{11} + 2286g_{20}^2f_{20}g_{11}^2g_{02} - 345g_{02}f_{11}f_{20}g_{11}g_{21} - 270f_{20}^4f_{30} - 702g_{20}f_{11}g_{40}f_{20} + 912f_{02}^2f_{11}g_{02}f_{30} + 3066f_{20}g_{11}g_{20}f_{40} + \\
& 699f_{20}f_{02}f_{11}g_{31} - 4880f_{20}^2f_{02}^2g_{20}g_{11} + 630g_{02}f_{02}f_{05} + 2g_{11}^4f_{12} - 99g_{02}g_{20}g_{11}f_{13} + 360f_{20}g_{20}g_{12}^2 - 432f_{20}g_{02}f_{21}^2 - \\
& 84f_{03}g_{11}^2f_{02}g_{02} - 288g_{02}g_{20}g_{11}g_{22} + 1152g_{20}^2f_{20}f_{31} + 228g_{11}^2g_{20}g_{02}g_{21} - 342f_{20}f_{02}^2g_{20}g_{02}f_{11} - 672f_{03}f_{20}^2g_{11}f_{11} + \\
& 18g_{11}f_{02}f_{11}g_{13} + 9f_{11}^4f_{30} - 153f_{03}g_{30}f_{20}f_{11} + 2g_{11}^4f_{20}f_{11} + 216f_{03}f_{21}f_{20}g_{20} + 210f_{03}f_{20}f_{02}f_{12} + 168f_{03}f_{20}g_{11}f_{12} - \\
& 45g_{20}^3f_{11}f_{12} - 684f_{21}f_{20}g_{11}f_{02}f_{11} + 360f_{03}g_{20}g_{31} + 441f_{03}f_{11}f_{40} + 2184f_{20}f_{02}^2g_{02}g_{12} + 180g_{30}f_{21}g_{02}f_{02} + 45f_{11}^2g_{02}^2f_{30} + \\
& 207f_{03}g_{20}^2f_{11}g_{11} - 9g_{20}^2g_{11}f_{13} - 1437g_{30}g_{11}^2f_{20}g_{02} - 945f_{03}f_{02}^2f_{30} - 90g_{20}f_{02}g_{32} + 9g_{11}^2f_{14} + 12f_{11}^2f_{20}g_{11}g_{21} - 3714g_{02}g_{20}f_{20}g_{11}f_{12} + \\
& 45f_{11}f_{42} - 432g_{11}f_{30}f_{04} - 210f_{03}g_{11}f_{02}^2f_{11} - 63f_{03}f_{11}g_{02}f_{12} - 450g_{02}^2f_{02}f_{13} - 420f_{03}g_{11}f_{02}^2g_{02} - 1800g_{02}^2g_{20}f_{40} + \\
& 180f_{03}f_{21}f_{20}f_{11} + 252g_{11}^3f_{20}f_{30} + 576g_{20}^3f_{20}^3 - 243f_{21}g_{30}g_{02}g_{11} - 78g_{20}f_{11}g_{11}^2f_{30} + 243f_{21}f_{20}f_{31} + 1620g_{02}^2f_{50} + \\
& 2256f_{02}g_{11}g_{20}f_{40} + 18f_{11}g_{11}g_{32} + 900f_{20}g_{20}g_{50} + 1992f_{20}f_{02}g_{02}g_{31} + 2970g_{20}^3g_{02}f_{30} - 81f_{20}^2f_{11}g_{02}g_{21} + 396f_{03}f_{20}g_{11}f_{30} + \\
& 171f_{21}f_{02}f_{13} + 6f_{21}g_{11}^2g_{21} - 840f_{02}^3g_{04} + 504f_{21}g_{02}^3f_{20} + 558f_{03}g_{20}^2g_{02}f_{20} + 105f_{02}^3g_{20}f_{11}^2 - 90f_{20}^5f_{11} + 3228f_{20}^2f_{02}g_{02}g_{12} - \\
& 720g_{20}^3g_{02}f_{20}f_{11} - 45g_{20}g_{11}f_{23} - 18f_{21}f_{02}g_{22} + 126g_{12}f_{02}f_{13} - 438f_{20}f_{02}g_{11}g_{22} + 396f_{20}^2f_{11}g_{31} - 1152f_{03}g_{30}f_{20}g_{20} - \\
& 504f_{20}f_{02}f_{04}g_{02} + 216f_{11}f_{30}g_{04} - 18g_{20}^2g_{12}f_{30} + 90f_{02}g_{20}g_{21}f_{30} + 36f_{12}f_{21}^2 - 180g_{12}f_{11}g_{02}g_{21} - 1110f_{02}^2f_{20}g_{02}g_{30} - \\
& 3576f_{03}f_{20}^2g_{02}f_{02} + 2076f_{20}f_{02}f_{11}f_{40} - 4320g_{20}^2g_{02}f_{40} - 1935f_{20}g_{11}f_{50} + 72f_{03}g_{20}f_{11}g_{02}g_{11} - 1308f_{20}^3g_{02}^2g_{20} + 378f_{03}g_{02}^3f_{20} - \\
& 18f_{11}f_{31}g_{20}g_{11} + 135g_{25} - 90g_{12}f_{11}g_{30}g_{11} + 144g_{11}^2g_{02}^3f_{20} + 330f_{20}f_{02}f_{11}f_{22} + 9g_{20}f_{11}g_{11}^2f_{12} + 72g_{30}g_{20}f_{11}f_{12} + \\
& 945f_{70} + 522g_{02}^2g_{41} + 630g_{20}^3g_{11}^2f_{02} + 27f_{11}^2f_{14} - 18f_{11}^3g_{20}f_{12} + 36g_{12}f_{02}g_{20}f_{21} - 450g_{02}f_{11}g_{40}f_{20} + 135g_{20}^2f_{02}f_{31} - \\
& 81g_{30}f_{21}f_{20}f_{11} + 18f_{11}f_{31}g_{21} + 54f_{12}f_{11}f_{31} + 144g_{30}g_{20}g_{02}f_{02}f_{11} + 315f_{03}f_{02}^2g_{20}g_{11} + 3612g_{20}^2f_{20}^2g_{02}g_{11} + 81f_{11}^2f_{20}f_{31} - \\
& 108f_{03}f_{11}g_{02}g_{21} - 639g_{30}f_{11}g_{02}g_{21} + 327g_{20}f_{11}f_{20}g_{11}^2g_{02} - 315g_{20}g_{51} + 195g_{12}f_{20}^2g_{21} - 135g_{02}g_{11}g_{12}^2 + 24g_{30}g_{11}^2g_{20} + \\
& 1665g_{20}f_{11}f_{50} + 888g_{12}f_{20}^2g_{20} - 540g_{20}^2f_{11}g_{31} - 27f_{11}^2f_{02}g_{40} - 378g_{20}^3f_{20}g_{12} - 288f_{03}f_{20}g_{02}g_{12} + 720g_{02}f_{20}g_{50} - \\
& 90g_{20}f_{11}f_{02}f_{13} - 2352g_{02}g_{20}f_{20}g_{11}g_{21} - 84g_{11}^2g_{02}^2f_{02}f_{11} + 1692g_{12}f_{20}^3g_{02} - 216f_{30}^2f_{12} + 135g_{20}^2f_{32} - 4176f_{20}f_{02}g_{30}g_{20}g_{11} - \\
& 291g_{30}g_{11}^2f_{20}f_{11} + 396f_{21}g_{20}^2g_{02}g_{11} + 243f_{20}f_{02}f_{11}^2f_{12} - 30g_{20}^2g_{11}f_{02}f_{12} + 546f_{02}g_{11}f_{11}f_{40} + 720g_{02}f_{20}f_{05} - 210f_{02}^3g_{22} - \\
& 135g_{11}^2f_{20}f_{31} + 117f_{20}g_{11}f_{14} - 153f_{20}^2g_{23} + 1575g_{30}g_{02}^2g_{20}g_{11} - 108g_{12}f_{02}g_{22} + 3462f_{20}f_{02}^2g_{11}f_{30} - 135f_{03}f_{20}g_{11}^2g_{20} +
\end{aligned}$$

$$\begin{aligned}
& 210f_{02}^2g_{20}f_{22}+350f_{20}^2f_{02}^3f_{11}-162f_{11}^2f_{21}f_{20}g_{20}+243g_{21}g_{20}g_{11}f_{30}-2658f_{20}^2g_{11}g_{04}+504f_{03}g_{20}^2g_{02}g_{11}+225g_{12}g_{20}f_{11}g_{02}g_{11}- \\
& 27g_{21}g_{32}-576f_{03}g_{30}g_{20}g_{11}+504f_{02}g_{11}^2g_{02}^2g_{20}-342g_{12}f_{20}g_{11}f_{12}-246f_{02}g_{11}g_{20}g_{02}f_{12}+54g_{22}f_{04}+891g_{30}g_{11}f_{02}f_{30}+ \\
& 12g_{11}^2f_{11}g_{31}+1038f_{20}f_{02}g_{11}^2f_{30}-936f_{20}^2g_{20}g_{02}f_{30}+81g_{20}f_{11}g_{23}+423f_{11}^2f_{20}^2g_{20}g_{11}+1488f_{21}f_{20}^2g_{02}g_{11}-36f_{20}f_{02}g_{12}f_{30}- \\
& 135f_{11}^2g_{20}^3g_{11}-153g_{12}g_{20}f_{11}g_{21}-24g_{12}g_{11}^2g_{21}+3330g_{20}^3g_{30}f_{20}+2052f_{20}f_{30}f_{40}-969g_{30}g_{11}^2f_{02}g_{20}-153g_{21}f_{12}f_{20}f_{11}- \\
& 189f_{11}^2f_{20}g_{11}f_{02}-108g_{12}g_{40}f_{20}-12g_{20}^2f_{11}g_{11}^3-54g_{30}f_{20}^3g_{02}-345f_{02}^2f_{11}g_{30}g_{11}+1098g_{02}f_{20}g_{21}f_{30}-810g_{02}f_{20}f_{23}+ \\
& 126f_{03}f_{20}g_{20}g_{12}+72g_{20}^2f_{20}^2f_{30}-54f_{31}g_{31}+3105g_{20}^2g_{30}g_{02}g_{11}+951f_{20}g_{11}g_{20}g_{31}-191f_{20}f_{02}^2g_{11}^2f_{11}-84g_{11}^3g_{04}- \\
& 12f_{20}f_{02}g_{02}f_{22}-90g_{20}f_{02}f_{23}-1260f_{03}f_{20}f_{02}^2f_{11}-135f_{11}^2g_{20}^2g_{02}g_{11}+288f_{03}g_{02}^2f_{20}g_{20}+648f_{30}g_{30}^2-279g_{21}f_{02}f_{11}f_{30}- \\
& 27g_{12}f_{30}g_{30}+36f_{30}f_{12}f_{20}g_{20}-216f_{21}^2f_{20}g_{20}-750g_{20}^2f_{02}^2g_{21}+126f_{02}g_{11}f_{04}f_{11}-9g_{12}f_{11}f_{22}-81g_{20}g_{11}g_{12}^2+27f_{20}g_{21}g_{13}+ \\
& 54g_{12}f_{14}+36g_{20}g_{21}g_{04}-9g_{12}f_{11}^2f_{12}-105f_{02}^2g_{11}^2f_{11}-534f_{20}^2g_{20}f_{22}+144g_{20}f_{11}f_{20}g_{22}-360f_{02}^2g_{02}^2g_{21}-1062g_{02}f_{11}f_{20}g_{04}- \\
& 252f_{20}f_{02}f_{04}f_{11}+135f_{02}f_{33}-36f_{11}^3f_{20}^3-117g_{02}f_{31}g_{20}g_{11}-552f_{21}f_{20}^2g_{11}f_{11}+360f_{03}g_{20}^3g_{11}-70f_{02}^4g_{11}f_{11}+1422f_{03}g_{02}f_{40}- \\
& 27f_{02}g_{11}f_{11}f_{22}+225f_{11}^2f_{50}-72f_{21}g_{20}f_{11}f_{12}+1665g_{20}^3g_{30}g_{11}+2484f_{20}f_{02}f_{30}g_{30}-288f_{03}f_{21}g_{02}f_{02}+864g_{21}f_{20}g_{20}f_{30}+ \\
& 36f_{21}g_{02}^2f_{12}-198g_{20}^2g_{02}f_{22}-450f_{02}^2g_{41}-180f_{03}f_{11}g_{02}f_{30}+63f_{11}^2g_{20}g_{02}f_{12}-270g_{02}^3g_{20}f_{02}f_{11}-9g_{02}^2f_{11}f_{22}+ \\
& 1404f_{30}^2g_{02}f_{02}-111f_{21}g_{11}f_{02}f_{12}-105f_{02}^3f_{21}f_{11}-396f_{03}g_{20}f_{11}f_{30}-594f_{20}^2g_{41}+81g_{30}g_{11}g_{04}+36f_{20}f_{02}f_{30}f_{21}- \\
& 48f_{03}g_{11}^2g_{21}+63f_{11}^3f_{21}f_{20}+1308f_{20}f_{02}^2g_{20}f_{21}-612g_{12}f_{20}g_{11}f_{30}+126f_{03}f_{02}f_{11}g_{12}+1404f_{21}f_{20}f_{02}g_{02}g_{11}+234g_{30}g_{02}f_{02}g_{12}- \\
& 945f_{11}f_{60}-135g_{60}g_{11}-210f_{11}^2f_{20}f_{02}g_{21}-30f_{21}g_{11}^2f_{12}+63f_{02}g_{11}f_{14}-315f_{02}^2g_{23}-1080f_{20}g_{02}g_{30}^2+168f_{11}^2f_{20}^2f_{12}+ \\
& 126f_{20}f_{12}f_{04}-63g_{12}f_{11}g_{02}f_{12}-108g_{20}^2f_{11}g_{02}f_{12}+82f_{20}^2f_{02}g_{11}^2f_{11}-306g_{20}^2g_{12}g_{21}+816f_{20}f_{02}f_{21}g_{20}g_{11}-54f_{11}^3g_{02}f_{12}- \\
& 432g_{20}f_{30}g_{04}-1233g_{20}^2f_{20}g_{11}g_{21}-1362f_{20}f_{02}^2g_{11}^2g_{02}+162f_{20}^2g_{02}g_{13}-279g_{12}f_{20}^2f_{30}-324f_{30}^2f_{20}f_{11}+1548g_{30}g_{20}f_{11}g_{02}g_{11}- \\
& 1077f_{20}f_{02}g_{11}f_{13}-1260f_{02}^2g_{11}g_{04}+9g_{12}g_{11}^2f_{20}f_{11}+99g_{11}^2f_{02}g_{22}+9f_{11}^2g_{11}^2f_{02}g_{20}+144f_{11}^3g_{20}g_{02}f_{20}-162f_{03}g_{20}g_{11}g_{12}+ \\
& 27f_{21}g_{02}^2f_{02}f_{11}-156f_{20}f_{02}f_{11}g_{02}g_{21}-54f_{30}g_{21}^2-81g_{40}g_{13}-297f_{02}g_{11}g_{23}-1650g_{30}f_{20}g_{11}^2g_{20}-630f_{20}f_{02}^3g_{11}g_{02}- \\
& 531g_{12}g_{30}g_{02}g_{11}-300g_{02}f_{11}f_{20}g_{11}f_{30}+525g_{02}^2f_{11}f_{20}g_{11}+360f_{11}^2f_{20}^2g_{02}g_{11}-135g_{12}^2f_{30}-135g_{20}^3g_{13}-770f_{20}^2f_{02}^2f_{12}+ \\
& 54g_{12}g_{11}g_{22}-333f_{11}^2g_{02}f_{40}-675f_{03}g_{02}^2f_{20}f_{11}+66g_{11}^2g_{02}f_{22}+45f_{03}g_{11}^2f_{30}-642g_{11}^2f_{20}g_{04}+90g_{20}^2f_{02}g_{22}-1560g_{20}^2f_{20}^2g_{21}+ \\
& 344f_{02}^2f_{30}^2f_{11}-1026g_{02}f_{11}f_{02}g_{04}-54g_{30}f_{02}g_{22}+9g_{21}f_{23}+144g_{30}f_{02}g_{20}f_{21}+171f_{11}^2f_{20}f_{13}+900f_{02}^2g_{40}f_{20}+414g_{12}g_{20}f_{40}+ \\
& 135f_{06}f_{11}+42f_{03}g_{11}^2f_{12}+972f_{20}f_{02}g_{30}g_{21}-216g_{30}g_{41}+180f_{02}f_{11}g_{14}-1764f_{20}f_{02}^2f_{31}-9g_{20}f_{11}f_{20}f_{31}+315g_{20}^2f_{11}^2f_{30}+ \\
& 180f_{11}^2g_{30}g_{20}g_{11}-720f_{02}^2g_{02}^3f_{20}+156g_{12}g_{11}f_{02}g_{21}+24f_{03}g_{11}^3g_{02}+1008f_{20}f_{02}^2g_{20}g_{12}-888g_{02}^3f_{20}g_{11}-108g_{20}^2f_{11}^2g_{02}f_{02}+ \\
& 27f_{21}g_{20}g_{13}-9f_{11}^2g_{02}^2f_{12}-2036f_{20}f_{02}^2g_{11}^2g_{20}-54f_{11}^2g_{02}f_{22}-171g_{20}^2g_{11}g_{22}+675g_{20}f_{11}g_{02}^2f_{30}-63f_{11}^3g_{02}f_{30}+ \\
& 9f_{21}g_{11}g_{22}+210f_{02}^3g_{20}f_{21}-60g_{20}^2f_{20}f_{02}f_{12}-495g_{20}^2f_{11}g_{02}^2g_{11}-54g_{02}g_{20}f_{14}-6564f_{20}^2f_{02}g_{04}+684f_{20}f_{02}g_{02}^2f_{12}- \\
& 81g_{04}g_{31}-1710g_{20}^2f_{11}f_{40}+90f_{03}g_{20}f_{22}+1464f_{20}f_{02}^2g_{02}^2f_{11}+720g_{12}f_{20}g_{02}g_{30}+216f_{21}f_{30}^2g_{20}-255g_{12}g_{11}^2f_{02}g_{20}- \\
& 1782g_{30}^2f_{20}g_{20}-360f_{11}^2f_{20}g_{02}f_{21}+72g_{11}^2g_{02}^2g_{21}-162f_{21}g_{40}f_{20}+144g_{30}g_{20}g_{02}f_{12}+162g_{02}g_{20}g_{11}g_{04}-540g_{20}^3g_{31}- \\
& 156g_{02}f_{11}g_{11}f_{02}g_{21}-18f_{11}^4g_{20}f_{02}+117f_{11}^2f_{02}g_{04}+240f_{21}g_{11}f_{02}^2g_{02}-54g_{02}f_{12}g_{40}-261g_{02}f_{11}f_{02}f_{31}+135f_{11}^2g_{20}g_{02}g_{21}- \\
& 420f_{02}^3g_{02}f_{03}+420f_{02}^3g_{02}^2f_{11}+1692f_{20}^2f_{02}g_{40}-90g_{02}g_{20}f_{02}f_{13}-405g_{20}^2f_{11}^2g_{02}f_{20}-414f_{20}f_{02}f_{32}-45g_{12}f_{21}g_{20}g_{11}+ \\
& 225f_{21}g_{02}^2f_{20}f_{11}-1620f_{20}^2f_{50}+945g_{11}g_{06}+270g_{02}^3f_{11}^2f_{02}+540g_{20}^3g_{02}^2f_{20}-531g_{02}^2f_{20}f_{13}+180g_{12}f_{20}g_{02}f_{21}+ \\
& 1104f_{21}f_{20}^2g_{20}g_{11}-360g_{20}^2g_{02}^2f_{20}f_{11}-216g_{02}f_{24}-1080g_{20}^3f_{11}g_{02}g_{11}+750f_{02}^2g_{20}g_{31}-1080g_{20}^3g_{02}^2g_{11}+630f_{02}^4f_{30}- \\
& 648g_{20}f_{30}f_{31}-4620f_{20}f_{02}^2g_{04}+420f_{02}^3g_{02}g_{12}+1330f_{20}f_{02}^3g_{21}-237g_{20}f_{11}g_{11}f_{02}g_{21}+351f_{20}f_{33}+1140g_{20}^2g_{11}f_{02}^2g_{02}+ \\
& 2730f_{20}f_{02}g_{20}^3g_{11}+189f_{30}g_{32}-468f_{02}g_{11}g_{02}^3f_{20}+99g_{30}f_{11}^2g_{02}f_{02}-672f_{20}f_{02}g_{20}g_{02}f_{12}-93g_{12}f_{20}^3f_{11}+468g_{20}^2f_{20}g_{04}+ \\
& 108g_{30}g_{11}g_{22}+105f_{03}f_{20}g_{11}g_{21}+792f_{20}^2g_{20}f_{40}+33g_{11}^2f_{11}g_{02}f_{12}-90f_{11}^2f_{02}g_{20}f_{21}-81g_{22}g_{31}+342g_{02}g_{20}f_{32}- \\
& 204g_{12}f_{20}^2g_{11}f_{11}+105f_{02}^2g_{20}f_{11}f_{12}+297f_{11}^2f_{20}f_{02}g_{02}g_{11}-270g_{12}g_{02}^2g_{21}+9f_{21}^2g_{21}-72f_{11}^2g_{30}g_{21}-42g_{11}^2g_{20}g_{31}+ \\
& 504g_{11}^2f_{20}^2f_{30}-405f_{30}f_{05}-51f_{02}^2f_{11}g_{02}g_{21}+4g_{11}^5g_{20}+216f_{20}f_{02}f_{21}g_{21}-504f_{03}g_{20}g_{02}g_{21}-9g_{30}f_{11}^3f_{02}-2400f_{02}^2g_{30}f_{20}g_{20}+ \\
& 684g_{21}g_{02}f_{02}f_{30}-45f_{21}g_{11}^2f_{02}f_{11}-1560f_{02}^2g_{02}^2f_{20}g_{20}+45g_{21}g_{14}+90g_{20}^2f_{20}f_{02}^2f_{11}+3960g_{20}^2f_{20}g_{02}g_{30}-1980g_{02}g_{20}g_{40}f_{20}- \\
& 18g_{02}g_{20}f_{20}f_{13}+864g_{30}f_{20}g_{02}f_{21}+117g_{02}g_{11}g_{32}-54f_{11}^2g_{02}g_{11}f_{03}-189f_{30}f_{12}f_{02}f_{11}-216f_{03}g_{02}g_{11}g_{12}-348g_{12}g_{11}^2f_{02}g_{02}- \\
& 15g_{02}f_{11}g_{11}f_{02}f_{12}-504f_{02}g_{11}g_{02}^2g_{21}-595f_{20}f_{02}^2g_{11}f_{12}+531g_{02}^2f_{20}g_{11}f_{12}+558g_{11}^2g_{40}f_{20}-297f_{21}g_{11}f_{02}f_{30}-1062f_{03}g_{20}g_{02}f_{30} \\
& 18g_{11}^2g_{41}-1161f_{20}^2g_{11}f_{31}-288g_{02}f_{12}g_{22}-495g_{02}g_{51}+72g_{20}g_{11}g_{32}-270g_{12}f_{20}g_{22}+105f_{03}g_{11}f_{02}f_{12}-1344f_{03}f_{20}^2g_{02}g_{11}- \\
& 54f_{11}^2g_{40}g_{11}+630g_{20}^2g_{02}^2f_{30}+3360f_{20}f_{02}^3f_{30}+210f_{02}^2g_{02}f_{22}+432g_{02}g_{20}g_{23}-18g_{12}f_{12}f_{21}-432g_{20}^2g_{02}g_{13}+455f_{02}^3g_{11}g_{21}- \\
& 48f_{11}^3g_{11}f_{02}^2-270g_{02}^2f_{14}+84g_{20}^2f_{20}^3g_{02}-27g_{20}f_{11}^2f_{04}+6g_{11}^3g_{22}-207f_{03}g_{20}f_{11}g_{21}+573f_{20}g_{11}f_{11}f_{40}-90f_{11}^2g_{40}f_{20}- \\
& 546f_{20}^2g_{20}g_{02}f_{12}-93f_{11}^2f_{02}^2g_{21}+36f_{21}g_{02}f_{40}-399f_{20}^2g_{11}g_{22}-54g_{12}f_{02}g_{40}-993f_{03}f_{20}g_{11}f_{02}f_{11}+180g_{12}g_{20}^2f_{11}g_{11}+ \\
& 171g_{02}^2f_{02}f_{31}+495f_{20}f_{15}-1440f_{03}f_{20}^3g_{02}+18g_{11}f_{12}g_{13}+168g_{11}^2f_{20}^3f_{11}+180f_{12}g_{14}-756g_{30}^2g_{02}g_{11}+9f_{11}^5f_{02}- \\
& 84f_{20}f_{02}^2f_{11}^3-36f_{02}g_{21}f_{40}-612g_{20}f_{11}g_{02}^3f_{20}-54f_{02}f_{21}g_{40}-1824f_{20}^2f_{02}g_{02}g_{30}-648f_{20}^2f_{04}g_{02}+45f_{11}g_{21}g_{22}+ \\
& 504g_{02}f_{11}g_{41}-6g_{12}g_{11}^2f_{02}f_{11}-234f_{03}f_{11}^2g_{02}f_{20}+980f_{20}^2f_{02}^3g_{02}+126g_{11}f_{03}^3f_{11}+1098g_{02}f_{12}f_{20}f_{30}-297f_{03}f_{20}f_{31}- \\
& 249g_{11}^2g_{02}^2f_{30}-1080f_{03}f_{50}-240f_{02}^2g_{30}f_{20}f_{11}+360f_{20}f_{11}f_{05}-180g_{20}f_{02}g_{14}-810g_{12}g_{02}^3f_{20}+99g_{30}f_{11}g_{02}f_{12}+ \\
& 18f_{11}g_{11}f_{30}g_{21}-558g_{20}^3f_{21}f_{20}+1614f_{20}^2f_{02}f_{21}g_{20}-1350g_{20}^4g_{02}g_{11}+720f_{20}f_{30}f_{22}-30g_{11}^2g_{02}g_{31}+2256f_{20}^2f_{02}g_{20}^2g_{02}-
\end{aligned}$$

$$\begin{aligned}
& 396g_{02}f_{11}f_{20}f_{13} + 6f_{12}f_{11}g_{11}g_{21} - 270g_{02}^4f_{30} - 54f_{21}f_{11}g_{30}g_{11} - 315f_{02}^3f_{31} + 72f_{20}f_{02}g_{20}g_{13} + 81f_{20}^2f_{32} - 1539g_{30}g_{20}f_{11}f_{30} - \\
& 198g_{20}^2f_{21}g_{02}f_{02} + 459f_{30}g_{14} + 45f_{21}g_{20}f_{11}g_{02}g_{11} + 54f_{11}^2f_{30}f_{21} + 210f_{02}^2f_{11}^2g_{02} + 414f_{20}g_{24} - 135g_{20}g_{33} - 27f_{03}f_{30}f_{21} - \\
& 234g_{02}f_{02}g_{32} + 123g_{11}^2f_{20}g_{22} + 630f_{02}^2g_{02}g_{13} - 360f_{20}g_{20}f_{23} + 226f_{20}^3f_{02}g_{02}g_{11} - 1764g_{02}f_{20}f_{41} + 132g_{21}f_{12}f_{20}g_{20} + \\
& 2136f_{20}^2f_{02}g_{11}f_{30} - 54g_{02}f_{11}g_{11}f_{13} + 90f_{11}^2g_{30}g_{02}g_{11} - 1710g_{20}^2f_{20}g_{02}g_{12} - 288g_{20}^2g_{02}f_{02}g_{12} + 171f_{03}f_{21}f_{02}f_{11} + \\
& 81f_{31}f_{22} - 270f_{11}g_{30}^2g_{11} + 261g_{20}^3f_{21}g_{11} - 3930f_{20}f_{02}g_{11}g_{04} - 522g_{02}^2g_{40}g_{11} - 54f_{21}f_{20}g_{11}^2g_{20} + 99f_{03}f_{11}^3f_{20} - 204g_{30}g_{11}^2f_{02}f_{11} - \\
& 63f_{20}f_{02}^2f_{11}g_{12} + 108f_{11}f_{30}f_{13} - 396g_{30}f_{20}^3f_{11} + 5448f_{20}f_{02}g_{02}f_{40} - 495f_{21}f_{20}g_{11}f_{30} + 3120f_{02}^2g_{02}f_{40} - 558g_{12}g_{11}^2f_{20}g_{02} + \\
& 1260f_{03}f_{02}g_{04} - 264g_{11}^2f_{20}^2f_{12} + 612f_{20}g_{02}g_{12}^2 - 1023g_{02}^2f_{20}g_{11}g_{21} - 15g_{11}^2f_{11}f_{22} - 123g_{21}f_{12}f_{02}f_{11} + 1620g_{20}^2g_{02}^3f_{20} + \\
& 450g_{20}^2g_{41} - 81g_{20}^2f_{11}g_{13} + 18g_{02}^2f_{02}g_{20}f_{21} - 1774f_{20}^2f_{02}g_{11}^2g_{02} - 81g_{20}f_{11}f_{20}^2f_{12} + 108f_{21}f_{02}f_{31} - 1800g_{30}g_{20}g_{02}g_{21} + \\
& 486f_{20}f_{02}g_{20}f_{11}g_{02}g_{11} + 171g_{02}^2f_{32} - 270g_{12}f_{50} - 45g_{11}^2f_{02}f_{31} + 2250g_{02}g_{20}f_{20}f_{31} + 123f_{21}g_{11}f_{02}g_{21} + 405g_{12}g_{02}^2g_{20}g_{11} - \\
& 180g_{20}f_{11}f_{02}g_{04} - 30g_{21}f_{12}^2 + 30g_{20}f_{02}g_{21}f_{12} - 2781f_{20}^2f_{02}g_{11}^2g_{20} - 900g_{02}f_{02}g_{14} + 609f_{02}g_{11}g_{20}g_{31} - 93f_{11}^2g_{11}f_{02}f_{30} - \\
& 141f_{11}^3f_{20}g_{11} + 153f_{11}^2f_{02}f_{13} - 18g_{11}^2g_{20}g_{13} + 4188f_{20}^2f_{02}^2f_{30} + 144f_{03}f_{11}g_{31} - 1440g_{20}^2f_{20}g_{02}f_{21} + 27g_{11}^2g_{20}f_{22} - \\
& 180g_{02}^2f_{11}g_{11}f_{03} - 1152g_{30}f_{20}^3g_{20} + 189f_{31}f_{40} + 648f_{20}^3g_{40} - 1080g_{30}f_{50} - 261g_{21}f_{20}f_{11}f_{30} + 1755g_{20}^2f_{11}g_{02}f_{30} + \\
& 90f_{21}g_{20}^2f_{11}g_{11} + 1794f_{20}^2f_{02}g_{02}f_{21} - 630f_{20}^2g_{02}f_{22} + 135f_{21}g_{02}^2g_{20}g_{11} + 1080g_{20}^2f_{11}g_{02}g_{21} + 351g_{30}g_{02}g_{13} - 216f_{03}g_{02}^2g_{21} + \\
& 102f_{11}^2f_{30}g_{02} - 72g_{20}f_{11}g_{11}g_{22} + 168f_{20}^2f_{02}g_{20}^2f_{11} - 72f_{20}^2f_{11}g_{02}f_{30} - 258f_{03}g_{11}^2f_{20}f_{11} + 117g_{12}f_{20}f_{13} + 1620g_{20}^4f_{30} + \\
& 288g_{20}^2f_{20}^3f_{11} + 54g_{11}f_{33} + 453g_{20}^2g_{11}^2f_{20}f_{11} + 54f_{21}f_{02}g_{04} - 1125g_{20}^2g_{30}g_{21} - 297f_{21}f_{11}g_{02}f_{30} + 63f_{02}f_{11}f_{41} + 450f_{02}^2g_{40}g_{11} + \\
& 135g_{02}^3g_{30}g_{11} - 147g_{20}f_{11}f_{20}g_{11}f_{30} + 210f_{02}^3g_{20}g_{12} + 936f_{20}f_{02}g_{02}g_{13} - 468g_{12}g_{11}f_{02}f_{30} - 2436f_{20}^2f_{02}g_{02}^2g_{20} - 27f_{20}g_{11}f_{04}f_{11} + \\
& 216g_{30}g_{20}g_{13} + 405g_{30}g_{40}g_{11} + 216g_{30}^2g_{21} + 1700f_{20}f_{02}^2g_{11}g_{21} + 189f_{30}f_{23} + 834f_{20}f_{02}g_{02}^2f_{30} - 72f_{20}g_{11}f_{11}f_{22} + \\
& 126f_{20}f_{02}f_{14} - 270g_{02}^3g_{20}f_{12} + 720f_{03}g_{30}^2f_{20} - 164f_{40}^2g_{11}f_{11} - 135g_{02}^3g_{31} + 369g_{30}g_{11}f_{02}g_{21} + 195f_{02}g_{11}g_{02}^2f_{30} - \\
& 27f_{21}g_{23} + 18g_{12}g_{20}f_{11}f_{12} + 27g_{12}^2g_{21} + 648f_{21}f_{20}^2f_{30} - 489g_{30}f_{20}^2g_{11}f_{11} + 885f_{02}g_{11}g_{02}g_{31} + 279f_{11}^2g_{02}^3f_{20} + 3216f_{20}g_{11}g_{02}f_{40} + \\
& 135g_{61} + 945g_{07} - 48g_{11}^3f_{02}g_{21} - 405f_{30}g_{50} - 4050g_{30}g_{20}g_{02}f_{30} - 3375g_{20}^2f_{30}g_{30} + 174f_{02}g_{11}g_{20}f_{22} - 99g_{20}f_{11}f_{02}f_{31} - \\
& 1080g_{20}^5f_{20} + 360f_{02}g_{11}^2g_{40} + 810g_{30}g_{40}f_{20} - 9g_{30}f_{11}^2f_{12} - 1464f_{20}^2f_{02}g_{02}^3 + 198f_{20}^2f_{14} - 174f_{20}^2f_{02}^2g_{11}f_{11} + 162f_{11}^2f_{02}^2g_{20}g_{11} + \\
& 210f_{02}^2g_{20}g_{02}f_{12} - 108g_{20}f_{30}f_{13} + 261f_{11}^2f_{20}g_{02}g_{30} + 213g_{21}f_{12}g_{02}g_{11} + 2700f_{20}g_{06} + 558g_{20}^2f_{20}g_{22} + 135g_{12}g_{30}g_{21} + \\
& 45g_{11}g_{24} - 198g_{02}f_{11}f_{02}g_{22} - 498g_{12}f_{20}g_{11}f_{02}f_{11} - 207f_{21}f_{11}g_{02}f_{12} - 531f_{03}f_{20}^2g_{20}g_{11} + 180f_{02}g_{24} + 270g_{11}^2f_{20}^2g_{21} - \\
& 108f_{03}f_{20}g_{22} + 315f_{02}f_{15} - 702f_{20}^3g_{22} - 36g_{12}g_{20}f_{11}f_{30} + 46g_{11}^4f_{02}g_{02} - 1578f_{20}^2f_{02}f_{11}f_{03} + 81f_{30}f_{21}^2 - 405g_{02}^2g_{20}g_{13} + \\
& 534f_{20}f_{02}^2g_{20}f_{11}^2 - 360f_{02}g_{11}g_{41} - 72g_{30}f_{20}g_{11}f_{12} - 648f_{20}^3g_{02}^3 - 12g_{20}f_{11}g_{11}^3g_{02} - 18g_{20}f_{11}g_{11}f_{13} - 1920f_{02}^2g_{20}g_{02}f_{30} + \\
& 81f_{30}f_{12}f_{20}f_{11} - 249f_{02}^2f_{11}^3f_{30} - 1494g_{02}g_{20}g_{40}g_{11} - 48f_{20}g_{21}f_{22} + 57g_{02}f_{11}f_{20}g_{11}f_{12} + 1620g_{02}g_{20}f_{20}g_{22} - 372f_{20}^2g_{02}^2g_{20}g_{11} - \\
& 162g_{30}f_{20}g_{04} + 81f_{13}f_{40} + 18g_{12}g_{20}f_{11}^2f_{02} + 12g_{11}^2f_{11}g_{02}g_{21} - 1578f_{20}^2f_{02}f_{13} + 397f_{20}f_{02}g_{11}^2g_{21} + 225g_{20}f_{11}f_{20}g_{11}f_{12} - \\
& 576f_{03}f_{20}g_{02}g_{30} - 129f_{11}^2f_{20}^2g_{21} - 84f_{03}g_{11}f_{02}g_{21} + 147f_{02}^2g_{11}^2g_{21} + 495g_{20}f_{11}g_{02}^2g_{21} + 240f_{02}g_{11}g_{02}f_{22} + 450f_{02}^2g_{30}g_{21} - \\
& 420f_{02}^2g_{20}f_{11}g_{21} - 36g_{20}f_{11}f_{02}g_{22} - 18f_{11}^2g_{20}g_{11}f_{03} + 504f_{20}g_{02}f_{03}^2 + 42f_{11}^2f_{20}g_{11}^2g_{02} + 2572f_{20}^2f_{02}^2g_{21} - 132f_{20}^2f_{02}f_{11}g_{12} + \\
& 243g_{30}g_{11}^2f_{30} - 2160g_{20}^3f_{40} + 234f_{21}g_{20}g_{02}f_{02}f_{11} - 1140f_{02}^2g_{20}g_{02}g_{21} - 405g_{30}g_{02}^2f_{30} - 105f_{02}^2g_{11}^2f_{12} - 135f_{03}f_{32} - \\
& 1764g_{02}f_{20}g_{14} + 216f_{20}g_{42} - 81g_{12}g_{23} + 558f_{21}g_{20}g_{02}f_{30} - 1236f_{02}g_{11}g_{20}g_{02}g_{21} + 405g_{12}g_{02}g_{13} + 630g_{12}g_{20}g_{02}f_{30} + \\
& 255g_{11}^2g_{20}f_{40} - 216f_{03}g_{41} - 359f_{20}f_{02}g_{11}^2f_{12} + 27f_{11}^3f_{04} - 135g_{20}f_{11}g_{02}g_{13} - 18g_{12}f_{11}^3f_{20} - 90g_{20}^3f_{22} + 630f_{03}f_{02}^2f_{20}g_{20} - \\
& 12g_{11}^3f_{11}g_{30} + 792f_{02}g_{11}g_{02}g_{13} + 252g_{02}g_{11}f_{03}^2 + 378f_{03}f_{04}g_{02} + 1080g_{20}^2g_{02}^2g_{21} + 63g_{12}g_{11}g_{04} + 324f_{20}g_{11}g_{20}g_{13} + \\
& 738f_{20}g_{20}g_{11}f_{04} - 438g_{11}^3f_{20}g_{20} + 207f_{11}g_{40}g_{21} - 84g_{11}^3f_{20}g_{21} - 9g_{12}f_{11}^3f_{02} + 288g_{02}g_{20}f_{02}g_{22} + 1512f_{30}f_{12}g_{02}f_{02} - \\
& 210f_{02}^4g_{20}g_{11} - 360f_{03}g_{20}^2g_{21} - 966f_{20}f_{02}^2g_{22} + 270f_{06}g_{02} - 1776f_{03}f_{20}f_{02}g_{02}g_{11} - 315f_{02}^2f_{32} - 63g_{20}f_{11}g_{11}g_{04} - \\
& 9f_{02}g_{21}g_{31} - 420f_{02}^4f_{20}g_{20} + 144g_{02}f_{12}^2f_{20} + 36g_{02}f_{11}f_{32} - 1324g_{11}^2f_{20}^3g_{20} + 108f_{12}f_{41} - 297g_{02}f_{12}f_{31} - 630g_{20}^4f_{11}f_{20} + \\
& 1299f_{20}g_{11}g_{02}g_{31} + 72g_{11}g_{21}g_{31} - 252f_{03}g_{11}f_{02}f_{30} + 267f_{20}^2f_{11}f_{22} + 54f_{21}f_{14} - 72f_{02}g_{20}f_{21}^2 + 36f_{21}g_{11}f_{13} - 24g_{11}^2f_{11}g_{02}f_{30} + \\
& 192f_{20}g_{11}g_{20}f_{22} + 36g_{20}f_{11}f_{32} - 468f_{20}g_{11}g_{41}
\end{aligned}$$

1.5 The Lienard equation

Assuming in (24)

$$f(x, y) \equiv 0, \quad \frac{dg(x, y)}{dy} = g_{x1}(x), \quad g(x, 0) = g_{x0}(x), \quad \frac{dg_{x0}}{dx}(0) = 0,$$

we obtain the following system

$$\begin{aligned}
\dot{x} &= -y, \\
\dot{y} &= x + g_{x1}(x)y + g_{x0}(x),
\end{aligned} \tag{25}$$

or the equivalent Lienard equation

$$\ddot{x} + x + \dot{x}g_{x1}(x) + g_{x0}(x) = 0.$$

Let be $g_{x1}(x) = g_{01} + g_{11}x + \dots$, $g_{x0}(x) = g_{10}x + g_{11}x^2 + \dots$. Then

$$L_1 = -\frac{\pi}{4}(g_{20}g_{11} - g_{21}).$$

If $g_{21} = g_{20}g_{11}$, then $L_1 = 0$ and

$$L_2 = \frac{\pi}{24}(3g_{41} - 5g_{20}g_{31} - 3g_{40}g_{11} + 5g_{20}g_{30}g_{11}).$$

If $g_{41} = \frac{5}{3}g_{20}g_{31} + g_{40}g_{11} - \frac{5}{3}g_{20}g_{30}g_{11}$, then $L_2 = 0$ and

$$L_3 = -\frac{\pi}{576}(70g_{20}^3g_{30}g_{11} + 105g_{20}g_{51} + 105g_{30}^2g_{11}g_{20} + 63g_{40}g_{31} - 63g_{11}g_{40}g_{30} - 105g_{30}g_{31}g_{20} - 70g_{20}^3g_{31} - 45g_{61} - 105g_{50}g_{11}g_{20} + 45g_{60}g_{11}).$$

The certain methods for the computation of the Lyapunov quantities and their symbolic expressions for different forms of the Lienard system can also be found in the survey S. Lynch [Lynch, 2005].

2 Transformation of quadratic system to the Lienard system

As may be seen from the above relations for the Lyapunov quantities, the most compact expressions are obtained for truncated systems. Therefore before the analysis of system and the computation of the Lyapunov quantities this system is often reduced to more simple form. One of routine forms, to which the polynomial systems of different forms are reduced (see, for example, the survey [Albarakati et. al., 2000]), is the Lienard equation.

Describe the reduction of quadratic system to the Lienard system following, in substance, the works [Leonov, 1998; 2006; 2007].

Consider a quadratic system

$$\begin{aligned}\dot{x} &= p(x, y) = a_1x^2 + b_1xy + c_1y^2 + \alpha_1x + \beta_1y, \\ \dot{y} &= q(x, y) = a_2x^2 + b_2xy + c_2y^2 + \alpha_2x + \beta_2y.\end{aligned}\tag{26}$$

We assume that in the stationary point $(x_{sp} = 0, y_{sp} = 0)$ the matrix of first approximation $A_{(x_{sp}, y_{sp})}$

$$A_{(0,0)} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$$

has two purely imaginary eigen numbers, i.e. it is satisfied the conditions

$$\alpha_1 + \beta_2 = 0, \quad \Delta = \alpha_1\beta_2 - \beta_1\alpha_2 > 0 \Rightarrow \beta_1\alpha_2 < -\alpha_1^2.\tag{27}$$

Hence $\beta_2 = -\alpha_1$, $\beta_1\alpha_2 \neq 0$

Proposition 1. *Without loss of generality we can assume that $c_1 = 0$.*

Let us make a nonsingular change of variables

$$x = \tilde{x} + \nu y.\tag{28}$$

Then

$$\begin{aligned}\dot{\tilde{x}} &= (a_1 - \nu a_2)\tilde{x}^2 + (-2\nu^2 a_2 + (2a_1 - b_2)\nu + b_1)\tilde{x}y + \\ &+ (-a_2\nu^3 + (a_1 - b_2)\nu^2 + (b_1 - c_2)\nu + c_1)y^2 + \\ &+ (-\nu\alpha_2 + \alpha_1)\tilde{x} + (\nu\alpha_1 + \beta_1 - \nu(\nu\alpha_2 + \beta_2))y = \\ &= \tilde{a}_1\tilde{x}^2 + \tilde{b}_1\tilde{x}y + \tilde{c}_1y^2 + \tilde{\alpha}_1\tilde{x} + \tilde{\beta}_1y.\end{aligned}$$

If $a_2 \neq 0$, then the cubic equation $(-a_2\nu^3 + (a_1 - b_2)\nu^2 + (b_1 - c_2)\nu + c_1)$ always has a real root. Therefore there exists $\nu \in \mathbb{R}, \nu \neq 0$ such that $\tilde{c}_1 = 0$. If $a_2 = 0$, we make the change $\tilde{x} = y, \tilde{y} = x$ and obtain $\tilde{c}_1 = a_2 = 0$.

Suppose below, $c_1 = 0$.

Proposition 2. *Without loss of generality, it can be assumed that $\tilde{\alpha}_2 = -\frac{2\tilde{\alpha}_1}{\tilde{\beta}_1}$.*

Having performed a nonsingular change of variables

$$y = \tilde{y} + hx, \quad (29)$$

we obtain

$$\begin{aligned}\dot{\tilde{x}} &= (b_1h + a_1)x^2 + b_1x\tilde{y} + (\alpha_1 + \beta_1h)x + \beta_1\tilde{y}, \\ \dot{\tilde{y}} &= ((c_2 - b_1)h^2 + (b_2 - a_1)h + a_2)x^2 + ((2c_2 - b_1)h + b_2)xy + c_2y^2 + \\ &+ (-\beta_1h^2 - 2\alpha_1h + \alpha_2)x + (-\beta_1h - \alpha_1)y.\end{aligned}$$

Then for

$$h = h^* = -\frac{\alpha_1}{\beta_1} + \frac{(-\alpha_1^2 - \beta_1\alpha_2)^{1/2}}{\beta_1} \quad (30)$$

we have $\tilde{\alpha}_2 = -\frac{2\tilde{\alpha}_1^2}{\tilde{\beta}_1}$.

Proposition 3. *Let be $\alpha_1 \neq 0$ and $\alpha_2 = -\frac{2\alpha_1^2}{\beta_1}$. Then, without loss of generality, we can assume that $\beta_1 = 1, \alpha_2 = -2$ and b_1 is equal to 0 or 1.*

Consider the certain nonsingular changes

$$\begin{aligned}\text{for } b_1 \neq 0 : \quad x &= \frac{\tilde{x}\beta_1}{b_1}, y = \frac{\tilde{y}\alpha_1}{b_1}, \\ \text{for } b_1 = 0 : \quad y &= \frac{\tilde{y}\alpha_1}{\beta_1}.\end{aligned} \quad (31)$$

Then for $b_1 \neq 0$ we obtain

$$\begin{aligned}\frac{\beta_1}{b_1}\dot{\tilde{x}} &= \dots + b_1\frac{\beta_1}{b_1}\frac{\alpha_1}{b_1}\tilde{x}\tilde{y} + \alpha_1\frac{\beta_1}{b_1}\tilde{x} + \beta_1\frac{\alpha_1}{b_1}\tilde{y}, \\ \frac{\alpha_1}{b_1}\dot{\tilde{y}} &= \dots - \frac{2\alpha_1^2}{\beta_1}\frac{\beta_1}{b_1}\tilde{x} - \alpha_1\frac{\alpha_1}{b_1}\tilde{y};\end{aligned}$$

and for $b_1 = 0$ we get

$$\begin{aligned}\dot{\tilde{x}} &= \dots + \alpha_1\tilde{x} + \beta_1\frac{\alpha_1}{\beta_1}\tilde{y}, \\ \frac{\alpha_1}{\beta_1}\dot{\tilde{y}} &= \dots - \frac{2\alpha_1^2}{\beta_1}\tilde{x} - \alpha_1\frac{\alpha_1}{\beta_1}\tilde{y}.\end{aligned}$$

Having performed in both cases the change $t = \frac{t}{\alpha_1}$, finally, we obtain $\tilde{\alpha}_1 = \tilde{\beta}_1 = 1, \tilde{\alpha}_2 = -2$, in which case $\tilde{b}_1$ is equal to 0 or 1.

Thus, system (26) with degenerated center in the stationary point $(0, 0)$ can be reduced to the form

$$\begin{aligned}\dot{x} &= a_1x^2 + b_1xy + x + y, \quad b_1 \in \{0, 1\}, \\ \dot{y} &= a_2x^2 + b_2xy + c_2y^2 - 2x - y,\end{aligned} \quad (32)$$

i.e., without loss of generality, we can assume that

$$c_1 = 0, \quad \alpha_1 = -\beta_2 = 1, \quad \beta_1 = 1, \alpha_2 = -2. \quad (33)$$

Using nonsingular changes, we reduce further the quadratic system of such form to the Lienard equation.

For computational experiments it is often convenient to reduce the quadratic system to more simple form.

Proposition 2'. *Without loss of generality, suppose $\alpha_1 = \beta_2 = 0$.*

Assuming in (29)

$$h = h^{**} = -\frac{\alpha_1}{\beta_1} \quad (30'),$$

we obtain $\tilde{\alpha}_1 = \tilde{\beta}_2 = 0$.

Proposition 3'. *Let be $\alpha_1 = \beta_2 = 0$. Then, without loss of generality, we can assume that $\beta_1 = -\alpha_2 = 1$ and b_1 is equal to 0 or 1.*

Having performed the change of variables

$$\begin{aligned} \text{for } b_1 \neq 0 : \quad x &= \frac{\tilde{x}\beta_1}{b_1}, y = \frac{\tilde{y}\sqrt{-\alpha_2\beta_1}}{b_1}, \\ \text{for } b_1 = 0 : \quad x &= \tilde{x}\beta_1, y = \tilde{y}\sqrt{-\alpha_2\beta_1}, \end{aligned} \quad (31')$$

we obtain

$$\begin{aligned} \dot{\tilde{x}} &= \dots + \tilde{b}_1 \tilde{x} \tilde{y} + \sqrt{-\alpha_2\beta_1} \tilde{y}, \\ \dot{\tilde{y}} &= \dots - \sqrt{-\alpha_2\beta_1} \tilde{x}, \end{aligned}$$

where $\tilde{b}_1 = \sqrt{-\alpha_2\beta_1}$ for $b_1 \neq 0$ and $\tilde{b}_1$ is equal to 0 otherwise.

Having performed the change $t = \frac{t}{\sqrt{-\alpha_2\beta_1}}$, we obtain finally $\tilde{\beta}_1 = -\tilde{\alpha}_2 = 1$, in which case $\tilde{b}_1$ is equal to 0 or 1.

Thus, system (26) with degenerated center in the stationary point $(0, 0)$ can be reduced to the form

$$\begin{aligned} \dot{x} &= a_1 x^2 + b_1 x y + y, \quad b_1 \in \{0, 1\}, \\ \dot{y} &= a_2 x^2 + b_2 x y + c_2 y^2 - x, \end{aligned} \quad (32')$$

i.e., without loss of generality, we can assume that

$$c_1 = 0, \quad \alpha_1 = \beta_2 = 0, \quad \beta_1 = -\alpha_2 = 1. \quad (33')$$

Remark 1. *For $b_1 \neq 0$ the straight line $\beta_1 + b_1 x = 0$ is invariant or transversal for systems (32) and (32').*

The proof follows from the relation

$$(\beta_1 + b_1 x)^\bullet = b_1 [(\beta_1 + b_1 x)y + a_1 x + \alpha_1 x]$$

for $x = -\frac{\beta_1}{b_1}$ (Here the sign $\bullet$ means a derivative in virtue of system).

This implies that the trajectories of system (32) and (32') for $c_1 = 0$ and $b_1 \neq 0$ are also the trajectories of system

$$\begin{aligned} \dot{x} &= y + \frac{a_1 x^2 + \alpha_1 x}{\beta_1 + b_1 x}, \\ \dot{y} &= \frac{a_2 x^2 + b_2 x y + c_2 y^2 + \alpha_2 x + \beta_2 y}{\beta_1 + b_1 x}. \end{aligned} \quad (34)$$

Consider the following transformation

$$\begin{aligned}\tilde{y} &= y + \frac{a_1 x^2 + \alpha_1 x}{\beta_1 + b_1 x}, \\ \tilde{x} &= x.\end{aligned}$$

In these new phase variables (here we omit the sign $\sim$ over variables: $\tilde{x} \rightarrow x$, $\tilde{y} \rightarrow y$) the system (34) can be written as

$$\dot{x} = y, \quad \dot{y} = -Q(x)y^2 - R(x)y - P(x), \quad (35)$$

where

$$\begin{aligned}Q(x) &= \frac{-c_2}{\beta_1 + b_1 x}, \\ R(x) &= -\frac{(b_1 b_2 - 2a_1 c_2 + a_1 b_1)x^2 + (b_2 \beta_1 + b_1 \beta_2 - 2\alpha_1 c_2 + 2a_1 \beta_1)x + \alpha_1 \beta_1 + \beta_1 \beta_2}{(\beta_1 + b_1 x)^2}, \\ P(x) &= -\left(\frac{a_2 x^2 + \alpha_2 x}{\beta_1 + b_1 x} - \frac{(b_2 x + \beta_2)(a_1 x^2 + \alpha_1 x)}{(\beta_1 + b_1 x)^2} + \frac{c_2(a_1 x^2 + \alpha_1 x)^2}{(\beta_1 + b_1 x)^3} \right).\end{aligned}$$

Remark 2 implies that the trajectories of system (35) are also the trajectories of the following system

$$\dot{x} = ye^{p(x)}, \quad \dot{y} = [-Q(x)y^2 - R(x)y - P(x)]e^{p(x)},$$

where $p(x)$ is a certain integral of the function $Q(x)$.

From this system, using the change $\tilde{x} = x$, $\tilde{y} = ye^{p(x)}$, we obtain

$$\dot{x} = y, \quad \dot{y} = -F_q(x)y - G_q(x), \quad (36)$$

or, in the form of one equation, $\ddot{x} + F_q(x)\dot{x} + G_q(x) = 0$. Here again after the mentioned above transformation the bars over the variables x and y are omitted.

Thus, for $b_1 \neq 0$ system (26) can be reduced by using the above nonsingular changes to the Lienard equation (36) with the functions

$$\begin{aligned}F_q(x) &= R(x)e^{p(x)} = R(x)|\beta_1 + b_1 x|^q, \\ G_q(x) &= P(x)e^{2p(x)} = P(x)|\beta_1 + b_1 x|^{2q}.\end{aligned}$$

Here $q = -\frac{c_2}{b_1}$ and the expressions for $R(x)$ and $P(x)$ can be simplified with provision for (33)

$$\begin{aligned}F_q &= \frac{(-b_2 + 2c_2 a_1 - a_1)x + (1 - b_2 + 2c_2 - 2a_1)}{(x+1)^2} x|x+1|^q, \\ G_q &= -\frac{(-b_2 a_1 + c_2 a_1^2 + a_2)x^3 + (-2b_2 a_1 + a_1 - b_2 + 2a_2 + 2c_2 a_1)x^2 + (a_2 + c_2 + a_1 - 3 - b_2)x - 1}{(x+1)^3} x|x+1|^{2q}.\end{aligned} \quad (37)$$

3 The reverse conversion of the Lienard system to quadratic system

Consider the Lienard equation

$$\ddot{x} + F(x)\dot{x} + G(x) = 0 \quad (38)$$

and the equivalent system

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -F(x)y - G(x),\end{aligned}\tag{39}$$

where

$$\begin{aligned}F(x) &= (Ax + B)x|x + 1|^{q-2}, \\ G(x) &= (C_1x^3 + C_2x^2 + C_3x + 1)x \frac{|x + 1|^{2q}}{(x + 1)^3}.\end{aligned}\tag{40}$$

Note that in the stationary point $(x_{sp} = 0, y_{sp} = 0)$ the matrix of first approximation $A_{(x_{sp}, y_{sp})}$ of system (39)

$$A_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -\frac{dG}{dx}(0) & F(0) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

has two purely imaginary eigen numbers.

Here there arises the question of the case when equation (38) can be reduced by a nonsingular change to quadratic system (32), for which relations (33) are satisfied.

Proposition 4. *Suppose, for the coefficients A, B, C_1, C_2, C_3, q of equation (38) the relations*

$$\frac{(B - A)}{(2q - 1)^2} ((1 - q)B + (3q - 2)A) = 2C_2 - 3C_1 - C_3,\tag{41}$$

$$\frac{(B - A)}{(2q - 1)^2} (B + 2(q - 1)A) = C_2 - 2C_1 - 1\tag{42}$$

are satisfied. Then by a nonsingular change, equation (38) can be reduced to the quadratic system with the coefficients $b_1 = 1, \alpha_1 = 1, \beta_1 = 1, c_2 = -q, \alpha_2 = -2, \beta_2 = -1,$

$$\begin{aligned}a_1 &= 1 + \frac{B - A}{2q - 1}, \\ a_2 &= -(q + 1)a_1^2 - Aa_1 - C_1, \\ b_2 &= -A - a_1(2q + 1).\end{aligned}\tag{43}$$

These relations follow from the equating of functions in (40) and (37).

4 Investigation of equilibria of the Lienard system

Equation (38) can have two specific equilibria

$$\begin{aligned}x_{sp}^1 &= 0, \\ x_{sp}^2 &= -\frac{B}{A}, \text{ if } G(x_{sp}^2) = 0 \text{ and } \frac{dG}{dx}(x_{sp}^2) > 0,\end{aligned}\tag{44}$$

in which the real parts of eigen numbers are equal to zero.

Now we consider the nondegenerate case

$$A \neq B, \quad AB \neq 0, \quad q \neq \frac{1}{2}.$$

Then by the above relations for the Lyapunov quantities L_1 and L_2 , we obtain the following lemmas.

Consider the zero equilibrium $x_{sp}^1 = 0$.

Lemma 1.

Let the the following condition be valid

$$L_1(x_{sp}^1) = 0, \quad 5A - 2Bq - 4B = 0. \quad (45)$$

Then

1. $L_2(x_{sp}^1) = 0$ and for C_1, C_2, C_3 , which are determined from equations (41) and (42) and condition (45), we obtain the following formulas

$$\begin{aligned} C_1 &= (q + 3) \frac{B^2}{25} - \frac{(1 + 3q)}{5}, \\ C_2 &= (15(1 - 3q) + 3B^2) \frac{1}{25}, \\ C_3 &= \frac{3(3 - q)}{5}; \end{aligned}$$

2. For $L_3(x_{sp}^1)$, determined via the obtained C_1, C_2, C_3 , we have the formula

$$L_3(x_{sp}^1) = - \frac{\pi B(q + 2)(3q + 1)[5(q + 1)(2q - 1)^2 + B^2(q - 3)]}{10000};$$

3. From $L_3(x_{sp}^1) = 0$ we get $L_4(x_{sp}^1) = 0$.

Thus, if for equation (38) conditions (45) are satisfied, then by small disturbances we can obtain three small cycles round one equilibrium.

We now obtain the Lyapunov quantities in the case of the second degenerated equilibrium x_{sp}^2 (i.e. in the case when conditions (44) are satisfied).

Lemma 2. *The following conditions are equivalent*

- 1) $L_1(x_{sp}^1) = 0$,
- 2) $L_1(x_{sp}^2) = 0$,
- 3) $L_1(x_{sp}^1) = 0, L_2(x_{sp}^1) = 0$,
- 4) $L_1(x_{sp}^2) = 0, L_2(x_{sp}^2) = 0$,

and this imply that

$$L_3(x_{sp}^1) = L_3(x_{sp}^2) = 0.$$

Thus, if for equation (38) conditions (44) are satisfied, then the above algorithm permits us to obtain only two small cycles: one at a time round either of equilibria [Leonov, 2008¹].

5 Computer experiments

V.I. Arnold writes [Arnold, 2005]: "To estimate the number of limit cycles of square vector fields on plane, A.N. Kolmogorov distributes a few hundreds of such fields (with the randomly chosen coefficients of quadratic expressions) among a few hundreds of students of mechanics and mathematics faculty of MGU as a mathematical practice. Each student has to find the number of limit cycles of his field. The result of this experiment was absolutely unhopd: never a one field had a one limit cycle! It is known that for the small change of field coefficients the limit cycle persists. Therefore the system with one, two, three (and even, as has become known later, four) limit cycles make up an open set in space of coefficients and so for the random choice of polynomial coefficients the expectancy of hitting in it is positive. The fact that this did not occur suggests that the above-mentioned probabilities are, apparently, small."

Our computational test show that it is practically impossible to find "small" cycles in the neighborhood of equilibrium, where the zero and first Lyapunov quantities are equal to zero. However we can distinctly

observe the "large" cycles in a series of computer experiments. For example, for the quadratic system of the form

$$\begin{aligned}\dot{x} &= xy - x + y, \\ \dot{y} &= -x^2 - xy + \frac{1}{4}y^2 - 1000x + 2y\end{aligned}$$

we have a "large" cycle, inside which there is the stationary point (0,0) (Fig. 1), what is confirmed by theoretical results.

Theorem 1 (Leonov, 2006). *Let for the coefficients of quadratic system (26) it be satisfied the following conditions*

$$\begin{aligned}b_1 &> 0, \\ 0 < 2c_2 < b_1, \beta_1 &> 0, \\ \frac{a_1\beta_1}{b_1} &> \alpha_1, \\ \frac{a_1(2c_2 - b_1)}{b_1} &> b_2, \\ \frac{b_1}{a_1(b_1b_2 - a_1c_2)} &> a_2,\end{aligned}$$

and in the half-plane $x < -\frac{\beta_1}{b_1}$ system (26) has a unique single Lyapunov unstable focal equilibrium. Then the system has at least one cycle in the half-plane $\{x < -\beta_1/b_1\}$.

...
...
...

6 Example of algorithm realization in Matlab

```

1 function [L,T,xt,yt] = fLQ_k1(fxy,gxy,N)
2 syms x y h t 'real'
3 NL=2*N+1; Nfg=NL;
4 xt_s(1:Nfg-1)=0*h; yt_s(1:Nfg-1)=0*h; xth_s=0*t; yth_s=0*t;
5 for n=1:Nfg
6     xt_s(n)=sym(['xt_',int2str(n)],'real');
7     xth_s=xth_s+xt_s(n)*h^n;
8     yt_s(n)=sym(['yt_',int2str(n)],'real');
9     yth_s=yth_s+yt_s(n)*h^n;
10 end
11 disp(['NL=',int2str(NL)]);
12 sT_h_cur=0;
13 for i=1:NL-1
14     sT_h(i,1)=sym(['T_',int2str(i)],'real');
15     sT_h_cur=sT_h_cur + sT_h(i,1)*h^i;
16 end;
17 ugt(1:Nfg)=0*t; xt(1:Nfg)=0*t; yt(1:Nfg)=0*t;
18 xt(1)=-sin(t); yt(1)=cos(t); xt_cur=xt(1)*h; yt_cur=yt(1)*h;
19 for i=2:NL
20     uft_s=subs(diff(subs(fxy, [x y], [xth_s yth_s]),h,i)
21         /factorial(i),h,0);
22     uft(i)= subs(uft_s, [xt_s yt_s], [xt yt]);
23     ugt_s=subs(diff(subs(gxy, [x y], [xth_s yth_s]),h,i)
24         /factorial(i),h,0);
25     ugt(i)= subs(ugt_s, [xt_s yt_s], [xt yt]);
26     uIt=diff(ugt(i),t)+uft(i);
27     Iucos=int(cos(t)*uIt,t); Iucos_t0=(Iucos - subs(Iucos,t,0));
28     Iusin=int(sin(t)*uIt,t); Iusin_t0=(Iusin - subs(Iusin,t,0));
29     ug0=subs(ugt(i),t,0);
30     xt(i)=simplify(cos(t)*ug0+Iucos_t0*cos(t)
31         +Iusin_t0*sin(t)-ugt(i));
32     yt(i)=simplify(sin(t)*ug0+Iucos_t0*sin(t)-Iusin_t0*cos(t));
33     xt_cur=xt_cur+xt(i)*h^i; yt_cur=yt_cur+yt(i)*h^i;
34 end;
35
36 xh_cur=subs(xt_cur,t,2*pi);
37 for k=1:NL
38     xh_cur=xh_cur + subs(diff(xt_cur,k,t),t,2*pi)*sT_h_cur^k
39     /factorial(k);
40 end;

```

```

41 for k=1:NL
42     xh(k,1)=subs(diff(xh_cur,k,h)/factorial(k),h,0);
43 end;
44
45 xh_temp=xh; T_cur=0; T(1,1)=0*x;
46 for k=2:NL
47     T(k-1,1)=solve(xh_temp(k,1),sT_h(k-1,1));
48     T_cur=T_cur + T(k-1,1)*h^(k-1);
49     xh_temp=subs(xh_temp,sT_h(k-1,1),T(k-1,1));
50 end;
51
52 yh_cur=subs(yt_cur,t,2*pi);
53 for k=1:NL
54     yh_cur=yh_cur + subs(diff(yt_cur,k,t),t,2*pi)*T_cur^k
55     /factorial(k);
56 end;
57 for k=1:NL
58     yh(k,1)=subs(diff(yh_cur,k,h)/factorial(k),h,0);
59 end;
60
61 for k=1:N
62     L(k)=factor(yh(2*k+1))
63 end;

```

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