Lyapunov quantities, limit cycles and strange behavior of trajectories in two-dimensional quadratic systems

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Draft [1][2]

Introduction

The computation of Lyapunov quantities is closely connected with the important in engineering mechanics question of dynamical system behavior near to boundary of the stability domain. Followed by the work of Bautin [1], one differs "safe" or "dangerous" boundaries, a slight shift of which implies a small (invertible) or noninvertible changes of system status, respectively. Such changes correspond, for example, to scenario of "soft" or "hard" excitations of oscillations, considered by Andronov [2].

In classical works of Poincare [3] and Lyapunov [4] for the analysis of system behavior near boundary of the stability domain was developed the method of computation of so-called Lyapunov quantities (or Poincare-Lyapunov constants, Lyapunov coefficients, focus values), which determine a system behavior in the neighborhood of boundary. This method also permits us effectively to study the bifurcation of birth of small cycles [1, 6–15], which correspond in mechanics to small vibrations.

In the present work the method of Lyapunov quantities is applied to investigation of small limit cycles. A new method for computation of Lyapunov quantities, developed for the Euclidian coordinates and in the time domain, is suggested. The general formula for computation of the third Lyapunov quantity for Lienard system is obtained.

Also, the computer modeling of large (normal amplitude) limit cycles are carried out. The transformation of quadratic system to a special type of Lienard system is used for investigation of large limit cycles. For this type of Lienard systems there is obtained a domain on the plane of two parameters of system, which the systems with three small and one large cycles correspond to (around two different stationary point). In our computer experiments the effects of trajectories "flattening", that make the computational modeling difficult, are observed.

Methods of calculation of Lyapunov quantities

For computation of Lyapunov quantities one usually consider a sufficiently smooth two-dimensional system with two purely imaginary eigenvalues of lineal part of

$$\frac{dx}{dt} = -y + f(x, y),$$
$$\frac{dy}{dt} = x + g(x, y). \tag{1}$$

Here $x, y \in R$ and the functions $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$ have continuous partial derivatives of (n)-st order in the open neighborhood U of radius R_U of the point $(x, y) = (0, 0)$. Suppose, the expansion of the functions f, g begins with the terms not lower than the second order and therefore we have

$$f(0, 0) = g(0, 0) = 0,$$
$$\frac{df}{dx}(0, 0) = \frac{df}{dy}(0, 0) = \frac{dg}{dx}(0, 0) = \frac{dg}{dy}(0, 0) = 0. \tag{2}$$

By assumption on smoothness in the neighborhood U we have

$$f(x, y) = \sum_{k+j=2}^{n} f_{kj} x^k y^j + o((|x| + |y|)^n) =$$
$$= f_n(x, y) + o((|x| + |y|)^n),$$
$$g(x, y) = \sum_{k+j=2}^{n} g_{kj} x^k y^j + o((|x| + |y|)^n) =$$
$$= g_n(x, y) + o((|x| + |y|)^n). \tag{3}$$

The study of limit cycles and Lyapunov quantities of two-dimensional dynamical systems was stimulated by as purely mathematical problems (the center-and-focus problem, Hilbert's sixteenth problem, and isochronous centers problem) as many applied problems (the oscillations of electronic generators and electrical machines, the dynamics of populations) [1–21]. The problems of greater dimension (when there are two purely imaginary roots and the rest are negative) can be reduced to two-dimensional problems with the help of procedure, proposed by Lyapunov [4].

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At present, there exist different methods for determining Lyapunov quantities and the computer realizations of these methods, which permit us to find Lyapunov quantities in the form of symbolic expressions, depending on expansion coefficient of the right-hand sides of equations of system (see., for example, [3-10, 15] and others). These methods differ in complexity of algorithms and compactness of obtained symbolic expressions. The first method for finding Lyapunov quantities was suggested by Poincare [3]. This method consists in sequential constructing time-independent holomorphic integral for approximations of the system. Further, different methods for computation, which use the reduction of system to normal forms, was developed (see, for example, [6, 10]).

Another approach to computation of Lyapunov quantities is related with finding approximations of solution of the system. So, a classical approach [4] it is used changes for reduction of turn time of all trajectories to a constant (as, for example, in the polar system of coordinates) and procedures for recurrent construction of solution approximations.

In the works [12,15] a new method of computation of Lyapunov quantities is suggested which based on constructing approximations of solution (as a finite sum in powers of degrees of initial data) in the original Euclidean system of coordinates and in the time domain. The advantages of given method are due to its ideological simplicity and visualization power. This approach can also be applied to the problem of distinguishing of isochronous center since it permits us to find out approximation of time of trajectory "turn" (time constants) depend upon initial data [7,9,21].

The first and second Lyapunov quantities have been computed in the 40-50s of last century [1,23]. The third Lyapunov quantity was computed in terms of f_{ij} and g_{ij} in [14,15] and its expression occupies more then four pages and the expression for the fourth Lyapunov quantity occupies 45 pages.

Note that for reduction of symbolic expression and simplification of analysis of system, special transformations of system to complex variables [6,10,21] are often used.

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Direct Method for Computation of Lyapunov Quantities (in Euclidean Coordinates and in the Time Domain)

Here the new method, developed for the Euclidian coordinates and in time domain, not requiring the reduction to normal form, is described. The advantages of this method are due to its ideological simplicity and a visualization power.

The first steps in the development of this method were made in the works [11–15] and some related with it results can be found in the work [22].

Here we assume that

$$f(\cdot,\cdot), g(\cdot,\cdot): \mathbb{R}\times\mathbb{R} \to \mathbb{R} \in \mathbb{C}^{(n+1)}(U). \qquad (7)$$

The existence condition of $(n+1)$th partial derivatives with respect to x and y for f and g is used for simplicity of exposition and can be weakened.

Approximation of solutions. Further we will use a smoothness of the functions f and g and will follow the first Lyapunov method on finite time interval (see f.e. classical works [23, 24] and others).

Let $x(t,x(0),y(0))$, $y(t,x(0),y(0))$ be a solution of system (**1**) with the initial data

$$x(0)=0, \ y(0)=h \qquad (8)$$

Denote

$$x(t,h)=x(t,0,h), \ y(t,h)=y(t,0,h).$$

$$x(t,h) = h\frac{\partial x(t,\eta)}{\partial \eta}\Big|_{\eta=0} + \frac{h^2}{2}\frac{\partial^2 x(t,\eta)}{\partial \eta^2}\Big|_{\eta=h\theta_x(t,h)}$$
$$0 \le \theta_x(t,h) \le 1,$$
$$y(t,h) = h\frac{\partial y(t,\eta)}{\partial \eta}\Big|_{\eta=0} + \frac{h^2}{2}\frac{\partial^2 y(t,\eta)}{\partial \eta^2}\Big|_{\eta=h\theta_y(t,h)} \qquad (10)$$
$$0 \le \theta_y(t,h) \le 1,$$

Note that, the functions

$$\frac{h^2}{2}\frac{\partial^2 x(t,\eta)}{\partial \eta^2}\Big|_{\eta=h\theta_x(t,h)}, \quad \frac{h^2}{2}\frac{\partial^2 y(t,\eta)}{\partial \eta^2}\Big|_{\eta=h\theta_y(t,h)}$$

and their time derivatives are smooth functions of t and have the order of smallness $o(h)$ uniformly with respect to t on a considered finite time interval $[0,4\pi]$.

Introduce the following denotations

$$\tilde{x}_{h^k}(t) = \frac{\partial^k x(t,\eta)}{\partial^k \eta}\Big|_{\eta=0}, \quad \tilde{y}_{h^k}(t) = \frac{\partial^k y(t,\eta)}{\partial^k \eta}\Big|_{\eta=0}.$$

We shall say that the sums

$$x_{h^m}(t,h) = \sum_{k=1}^{m} \tilde{x}_{h^k}(t)h^k = \sum_{k=1}^{m}\frac{\partial^k x(t,\eta)}{\partial \eta^k}\Big|_{\eta=0}\frac{h^k}{k!},$$

$$y_{h^m}(t,h) = \sum_{k=1}^{m} \tilde{y}_{h^k}(t)h^k = \sum_{k=1}^{m}\frac{\partial^k y(t,\eta)}{\partial \eta^k}\Big|_{\eta=0}\frac{h^k}{k!}$$

are the mth approximation of solution of system with respect to h. Substitute representation (10) in system (**1**). Then, equating the coefficients of h^1 and taking into account (**2**), we obtain

$$\frac{d\tilde{x}_{h^1}(t)}{dt} = -\tilde{y}_{h^1}(t),$$
$$\frac{d\tilde{y}_{h^1}(t)}{dt} = \tilde{x}_{h^1}(t). \qquad (11)$$

Hence, by conditions on initial data (**8**) for the first approximation with respect to h of the solution $(x(t,h),y(t,h))$, we have

$$x_{h^1}(t,h) = \tilde{x}_{h^1}(t)h = -h\sin(t),$$
$$y_{h^1}(t,h) = \tilde{y}_{h^1}(t)h = h\cos(t). \qquad (12)$$

Similarly, to obtain the second approximation $(x_{h^2}(t,h), y_{h^2}(t,h))$, we substitute representation

$$x(t,h) = x_{h^2}(t,h) + \frac{h^3}{3!}\frac{\partial^3 x(t,\eta)}{\partial \eta^3}\Big|_{\eta=h\theta_x(t,h)},$$

$$y(t,h) = y_{h^2}(t,h) + \frac{h^3}{3!}\frac{\partial^3 y(t,\eta)}{\partial \eta^3}\Big|_{\eta=h\theta_y(t,h)}. \qquad (13)$$

in formula (3) for $f(x,y)$ and $g(x,y)$. Note that in expressions for f and g (denote their by $u_{h^2}^f$ and $u_{h^2}^g$, respectively) in

virtue of (2) the coefficients of h^2 depend only on $\tilde{x}_{h^1}(t)$ and $\tilde{y}_{h^1}(t)$, i.e., by (12) they are known functions of time and are independent of the unknown functions $\tilde{x}_{h^2}(t)$. Thus, we have

$$f(x_{h^2}(t,h)+o(h^2),\, y_{h^2}(t,h)+o(h^2)) = u^f_{h^2}(t)h^2 + o(h^2),$$

$$g(x_{h^2}(t,h)+o(h^2),\, y_{h^2}(t,h)+o(h^2)) = u^g_{h^2}(t)h^2 + o(h^2).$$

Substituting (13) in system (1), for the determination of $\tilde{x}_{h^2}(t)$ and $\tilde{y}_{h^2}(t)$ we obtain

$$\frac{d\tilde{x}_{h^2}(t)}{dt} = -\tilde{y}_{h^2}(t) + u^f_{h^2}(t),$$

$$\frac{d\tilde{y}_{h^2}(t)}{dt} = \tilde{x}_{h^2}(t) + u^g_{h^2}(t). \qquad (14)$$

Lemma 3 *For solutions of the system*

$$\frac{d\tilde{x}_{h^k}(t)}{dt} = -\tilde{y}_{h^k}(t) + u^f_{h^k}(t),$$

$$\frac{d\tilde{y}_{h^k}(t)}{dt} = \tilde{x}_{h^k}(t) + u^g_{h^k}(t). \qquad (15)$$

with the initial data
$$\tilde{x}_{h^k}(0) = 0,\, \tilde{y}_{h^k}(0) = 0 \qquad (16)$$
we have

$$\tilde{x}_{h^k}(t) = u^g_{h^k}(0)\cos(t) +$$
$$+\cos(t)\int_0^t \cos(\tau)((u^g_{h^k}(\tau))' + u^f_{h^k}(\tau))d\tau +$$
$$+\sin(t)\int_0^t \sin(\tau)((u^g_{h^k}(\tau))' + u^f_{h^k}(\tau))d\tau - u^g_{h^k}(t),$$

$$\tilde{y}_{h^k}(t) = u^g_{h^k}(0)\sin(t) + \qquad (17)$$
$$+\sin(t)\int_0^t \cos(\tau)((u^g_{h^k}(\tau))' + + u^f_{h^k}(\tau))d\tau -$$
$$-\cos(\tau)\int_0^t \sin(t)((u^g_{h^k}(\tau))' + u^f_{h^k}(\tau))d\tau.$$

Repeating this procedure for the determination of the coefficients $\tilde{x}_{h^k}$ and $\tilde{y}_{h^k}$ of the functions $u^f_{h^k}(t)$ and $u^g_{h^k}(t)$, by formula (17) we obtain sequentially the approximations $(x_{h^k}(t,h), y_{h^k}(t,h))$ for $k=1,...,n$. We have

$$x(t,h) =$$
$$= x_{h^n}(t,h) + \frac{h^{n+1}}{(n+1)!}\frac{\partial^{n+1}x(t,\eta)}{\partial\eta^{n+1}}\Big|_{\eta=h\theta_x(t,h)} =$$
$$= x_{h^n}(t,h) + o(h^n) = \sum_{k=1}^n \tilde{x}_{h^k}(t)\frac{h^k}{k!} + o(h^n),$$

$$y(t,h) = \qquad (18)$$
$$= y_{h^n}(t,h) + \frac{h^{n+1}}{(n+1)!}\frac{\partial^{n+1}y(t,\eta)}{\partial\eta^{n+1}}\Big|_{\eta=h\theta_y(t,h)} =$$
$$= y_{h^n}(t,h) + o(h^n) = \sum_{k=1}^n \tilde{y}_{h^k}(t)\frac{h^k}{k!} + o(h^n),$$
$$0 \le \theta_x(t,h) \le 1, \quad 0 \le \theta_y(t,h) \le 1.$$

$$x_{h^k}(0,h) = x(0,h) = 0,\ y_{h^k}(0,h) = y(0,h) = h, \quad k = 1,...,n.$$

Computation of Lyapunov quantities in the time domain. Consider for the initial datum $h \in (0,H]$ the time $T(h)$ of first crossing of the solution $(x(t,h),y(t,h))$ of the half-line $\{x=0,\ y>0\}$. Complete a definition (by continuity) of the function $T(h)$ in zero: $T(0)=2\pi$. Since by (12) the first approximation of solution crosses the half-line $\{x=0,y>0\}$ at the time 2π, then the crossing time can be represented as

$$T(h)=2\pi+\Delta T(h),$$

where $\Delta T(h)=O(h)$. We shall say that $\Delta T(h)$ is a residual of crossing time.

By definition of $T(h)$ we have

$$x(T(h),h)=0. \qquad (20)$$

$x(\cdot,\cdot)$ has continuous partial derivatives with respect to either arguments up to the order n inclusive and $\dot{x}(t,h)=\cos(t)h+o(h))$, by the theorem on implicit function, the function $T(\cdot)$ is n times differentiable. It is possible to show that $T(h)$ is also differentiable n times in zero. By the Taylor formula we have

$$T(h) = 2\pi + \sum_{k=1}^n \tilde{T}_k h^k + o(h^n), \qquad (21)$$

where $\tilde{T}_k = \frac{1}{k!}\frac{d^kT(h)}{dh^k}$ (called period constant [7,9,20]). We shall say that the sum

$$\Delta T_k(h) = \sum_{j=1}^k \tilde{T}_j h^j \qquad (22)$$

is the kth approximation of the residual of the time $T(h)$ of the crossing of the solution $(x(t,h),y(t,h))$ of the half-line $\{x=0,\ y>0\}$. Substituting relation (21) for $t=T(h)$ in the right-hand side of the first equation of (18) and denoting the coefficient of h^k by $\tilde{x}_k$, we obtain the series $x(T(h),h)$ in terms of powers of h:

$$x(T(h),h) = \sum_{k=1}^n \tilde{x}_k h^k + o(h^n). \qquad (23)$$

In order to express the coefficients $\tilde{x}_k$ by the coefficients $\tilde{T}_k$ of the expansion of residual of crossing time we assume that in (18) $t=2\pi+\tau$:

$$x(2\pi+\tau,h) = \sum_{k=1}^n \tilde{x}_{h^k}(2\pi+\tau)\frac{h^k}{k!} + o(h^n). \qquad (24)$$

We have

$$\tilde{x}_{h^k}(2\pi+\tau) = \tilde{x}_{h^k}(2\pi) + \sum_{m=1}^n \tilde{x}_{h^k}^{(m)}(2\pi)\frac{\tau^m}{m!} + o(\tau^n),$$
$$k = 1,...,n.$$

Substitute this representation in (24) for the solution $x(2\pi+\tau,h)$ for $\tau = \Delta T(h)$, and bring together the

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coefficients of the same exponents h. By (**20**) and taking into account (**21**) for $T(h)$, we obtain

$$h : 0 = \tilde{x}_1 = \tilde{x}_{h^1}(2\pi),$$

$$h^2 : 0 = \tilde{x}_2 = \tilde{x}_{h^2}(2\pi) + \tilde{x}'_{h^1}(2\pi)\tilde{T}_1,$$

$$h^3 : 0 = \tilde{x}_3 = \tilde{x}_{h^3}(2\pi) + \frac{1}{2}\tilde{x}'_{h^2}(2\pi)\tilde{T}_1 + \tilde{x}'_{h^1}(2\pi)\tilde{T}_2 +$$

$$+ \frac{1}{2}\tilde{x}''_{h^1}(2\pi)\tilde{T}_1^2,$$

....

$$h^n : 0 = \tilde{x}_n = \dots$$

From the above we sequentially find $\tilde{T}_j$. The coefficients $T_{k=1,\dots,n-1}$ can be determined sequentially since the expression for $\tilde{x}_k$ involves only the coefficients $T_{m<k}$ and the factor $\tilde{x}'_{h^1}(2\pi)$ multiplying T_{k-1} is equal to -1.

We apply a similar procedure to determine the coefficients $\tilde{y}_k$ of the expansion

$$y(T(h),h) = \sum_{k=1}^{n} \tilde{y}_k h^k + o(h^n).$$

Substitute the representation

$$\tilde{y}_{h^k}(2\pi + \Delta T(h)) = \tilde{y}_{h^k}(2\pi) + \sum_{k=1}^{n} \tilde{y}_{h^k}^{(m)}(2\pi)\frac{\Delta T(h)^m}{m!} +$$

$$+ o(h^n), \quad k = 1, \dots, n$$

in the expression

$$y(2\pi + \Delta T(h),h) = \sum_{k=1}^{n} \tilde{y}_{h^k}(2\pi + \Delta T(h))\frac{h^k}{k!} + o(h^n).$$

Equating the coefficients of the same exponents h, we obtain the following relations

$$h : \tilde{y}_1 = \tilde{y}_{h^1}(2\pi),$$

$$h^2 : \tilde{y}_2 = \tilde{y}_{h^2}(2\pi) + \tilde{y}'_{h^1}(2\pi)\tilde{T}_1,$$

$$h^3 : \tilde{y}_3 = \tilde{y}_{h^3}(2\pi) + \frac{1}{2}\tilde{y}'_{h^2}(2\pi)\tilde{T}_1 + \tilde{y}'_{h^1}(2\pi)\tilde{T}_2 +$$

$$+ \frac{1}{2}\tilde{y}''_{h^1}(2\pi)\tilde{T}_1^2,$$

....

$$h^n : \tilde{y}_n = \dots$$

for the sequential determination of $\tilde{y}_{i=1,\dots,n}$. Here $\tilde{y}_{h_{k=1,\dots,n}}(\cdot)$ and $\tilde{T}_{k=1,\dots,n-1}$ are the obtained above quantities.

Thus, for $n=2m+1$ we sequentially obtained the approximations of the solution $(x(t,h),y(t,h))$ at the time $t=T(h)$ of the first crossing of the half-line $\{x=0,y>0\}$ accurate to $o(h^{2m+1})$ and the approximation of the time $T(h)$ itself accurate to $o(h^{2m})$. If in this case $\tilde{y}_k = 0$ for $k=2,..,2m$, then $\tilde{y}_{2m+1}$ is called the mth Lyapunov quantity L_m. Note, that, according to the Lyapunov theorem, the first nonzero coefficient of the expansion $\tilde{y}_i$ is always of an odd number and for sufficiently small initial data h the sign of $\tilde{y}_i$ (of the Lyapunov quantity) designates a qualitative behavior (winding or unwinding) of the trajectory $(x(t,h),y(t,h))$ on plane [4].

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