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Computation of the first Lyapunov quantity for the second-order dynamical system

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Abstract. A new direct method for the computation of Lyapunov quantities (Lyapunov values or coefficients, Poincare-Lyapunov constants, focus values) for the second-order dynamical system, permitting us to narrow the requirements on a smoothness of system, is obtained.

Keywords: Lyapunov value or coefficient, Poincare-Lyapunov constant, focus value, symbolic computations, small limit cycle, 16th Hilbert problem.

1 Introduction

The classical method for the computation of Lyapunov quantities involves the introduction of the polar coordinates and the reduction of original system to normal form [Lyapunov, 1892; Bautin, 1962; Lloyd & Pearson, 1990; Yu, 1998; Lynch, 2005]. In the present work the substantially different method, not requiring the direct reduction to normal form, is proposed. The quality of this method is ideological simplicity and visualization. In this method a less smoothness of the right-hand sides of differential equations in comparison with classical consideration is required. We follow here ideas, developed in [Leonov 2006, 2007].

2 Computation of Lyapunov quantity

Consider a system

$$\begin{aligned} \dot{x} &= -y + u_f(t), \\ \dot{y} &= x + u_g(t). \end{aligned} \tag{2.1}$$

The solution of this system with initial data $x(0) = 0, y(0) = 0$ is as follows

$$\begin{aligned} x = {} & u_g(0)\cos(t) + \\ & + \cos(t)\int\limits_0^t \cos(\tau)(u_g'(\tau) + u_f(\tau))\mathrm{d}\tau + \\ & + \sin(t)\int\limits_0^t \sin(\tau)(u_g'(\tau) + u_f(\tau))\mathrm{d}\tau - u_g(t) \\ y = {} & u_g(0)\sin(t) + \\ & + \sin(t)\int\limits_0^t \cos(\tau)(u_g'(\tau) + u_f(\tau))\mathrm{d}\tau - \\ & - \cos(t)\int\limits_0^t \sin(\tau)(u_g'(\tau) + u_f(\tau))\mathrm{d}\tau \end{aligned} \tag{2.2}$$

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[2]PDF slides http://www.math.spbu.ru/user/nk/PDF/Limit-cycles-Focus-values-Lyapunov-quantity-16th-Hilbert.pdf

Consider now the equations

$$\begin{aligned} \dot{x} &= -y + f(x,y) \\ \dot{y} &= x + g(x,y) \end{aligned} \tag{2.3}$$

Here $f(0,0) = g(0,0) = 0$ and in a certain neighborhood of the point $(x,y) = (0,0)$ the functions $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ have partial derivatives up to the order 2 and $f'_x(0,0) = f'_y(0,0) = g'_x(0,0) = g'_y(0,0) = 0$.

Further we shall use a smoothness of the functions f and g and follow the first Lyapunov method on finite time interval [Lefschetz, 1957; Cesari, 1959].

Since the functions f and g are assumed to be smooth, we can write

$$\begin{aligned} f(x,y) &= f_{20}x^2 + f_{11}xy + f_{02}y^2 + o((|x|+|y|)^2) = \\ &= f_2(x,y) + o((|x|+|y|)^2), \\ g(x,y) &= g_{20}x^2 + g_{11}xy + g_{02}y^2 + o((|x|+|y|)^2) = \\ &= g_2(x,y) + o((|x|+|y|)^2). \end{aligned} \tag{2.4}$$

in a certain neighbourhood of the point (0,0).

Consider the solution

$$x(t,h) = x(t,x(0),y(0)), y(t,h) = y(t,x(0),y(0))$$

of system (2.3) with the initial data

$$\begin{aligned} x(0,x(0),y(0)) &= 0, \\ y(0,x(0),y(0)) &= h. \end{aligned} \tag{2.5}$$

From the equations

$$\begin{aligned} \dot{x}_1 &= -y_1, \quad x_1(0,h) = 0, \\ \dot{y}_1 &= x_1, \quad y_1(0,h) = h \end{aligned} \tag{2.6}$$

for the first approximation $x_1(t,h), y_1(t,h)$ of the solution $x(t,x(0),y(0)), y(t,x(0),y(0))$ we have

$$x_1(t,h) = -h\sin(t), y_1(t,h) = h\cos(t).$$

By the assumption that f, g is smooth, we obtain that the right-hand side of system (2.3) has 2 continuous partial derivatives with respect to x and y. Then [Hartman, 1964] the solution of system (2.3), i.e. $x(t,h), y(t,h)$, has partial derivatives up to the order 2 with respect to the initial data h.

We shall seek sequential approximations for $x(t,h), y(t,h)$ in the form of the sums

$$\begin{aligned} x_2(t,h) &= x_1(t)h + x_2(t)h^2, \quad x_2(0) = 0, \\ y_2(t,h) &= y_1(t)h + y_2(t)h^2, \quad y_2(0) = 0. \end{aligned} \tag{2.7}$$

Here, in according to the local Taylor formula, at the fixed moment of time $t = t^*$ the following representation holds

$$\begin{aligned} x(t^*,h) &= x_2(t^*,h) + o(h^2), \\ y(t^*,h) &= y_2(t^*,h) + o(h^2). \end{aligned} \tag{2.8}$$

Substituting (2.7) in (2.4) and then in (2.3) and determining the coefficients $u_2^f(t)$ and $u_2^g(t)$ of h^2 in $f(x_1(t,h), y_1(t,h))$ and $g(x_1(t,h), y_1(t,h))$, respectively, we obtain the approximations

$$\begin{aligned} u_2^f(t,h) &= u_2^f(t)h^2, \\ u_2^g(t,h) &= u_2^g(t)h^2. \end{aligned} \tag{2.9}$$

For determining $x_2(t), y_2(t)$ we have the equations

$$\begin{aligned} \dot{x}_2 &= -y_2 + u_2^f(t) \\ \dot{y}_2 &= x_2 + u_2^g(t). \end{aligned} \tag{2.10}$$

Taking into account (2.2), we can find solutions of these equations. They take the form

$$
\begin{aligned}
&x_2(t) = \\
&= \frac{1}{3}[-(\cos(t)-1)^2 g_{20} - \sin(t)(\cos(t)-1)g_{11} + \\
&+(\cos(t)+\cos(t)^2-2)g_{02} - (\sin(2t)-2\sin(t))f_{20} - \\
&-(\cos(t)-\cos(2t))f_{11} + (\sin(2t)+\sin(t))f_{02}] \\
&y_2(t) = \\
&= \frac{1}{3}[-(\sin(2t)-2\sin(t))g_{20} - (\cos(t)-\cos(2t))g_{11} + \\
&+(\sin(2t)+\sin(t))g_{02} + (\cos(t)-1)^2 f_{20} + \\
&+\sin(t)(\cos(t)-1)f_{11} - (\cos(t)-2+\cos(t)^2)f_{02}]
\end{aligned}
$$

Here $x_2(0) = y_2(0) = x_2(2\pi) = y_2(2\pi) = 0$.

Lemma. *Suppose*

$$
\begin{aligned}
&x_1(2\pi) = 0, \ y_1(2\pi) = 1, \\
&x_2(2\pi) = y_2(2\pi) = 0.
\end{aligned}
\tag{2.11}
$$

Then on a phase plane for sufficiently small h, the solution $x(t,h), y(t,h)$ crosses the half-line $(x = 0, y > 0)$ at time

$$T = 2\pi + o(h). \tag{2.12}$$

Proof.

Since $x_2(2\pi, h) = 0$ and $y_2(2\pi, h) = h$, we conclude that on a phase plane at time $t = 2\pi$ the trajectory $(x(t,h), y(t,h))$ lies in the neighborhood, of radius $o(h^2)$ of the point $(x = 0, y = h)$ (2.8).

At the fixed moment of time $t = t^*$, we have [Hartman, 1964 and (2.8)]

$$\dot{x}(t^*, h) = -h\cos t^* + o(h).$$

Since $\dot{x}(t,h)$ is bounded with respect to h and t in a certain neighborhood of $(x = 0, y = h)$ and $t = 2\pi$, respectively, for t in a certain neighborhood of the point 2π and sufficiently small h we obtain the following inequality

$$\dot{x}(t,h) \le -ch$$

for a certain $c > 0$.

Hence

$$T = 2\pi + o(h).$$

■

Consider now a function

$$V(x,y) = x^2 + y^2. \tag{2.13}$$

We remark that for the derivative of the function V along the solutions of system (2.3), the relation

$$\dot{V}(x,y) = 2xf(x,y) + 2yg(x,y) \tag{2.14}$$

is valid.

Introduce the following notation

$$L = V\big(x(T,h), y(T,h)\big) - V\big(x(0,h), y(0,h)\big). \tag{2.15}$$

Integrating (2.14) from 0 to $T = 2\pi + o(h)$, we obtain

$$
\begin{aligned}
L = &\int_0^T \dot{V}(x(t,h), y(t,h))\mathrm{d}t = \\
= &\int_0^{2\pi} \dot{V}(x(t,h), y(t,h))\mathrm{d}t + o(h^4).
\end{aligned}
$$

Substituting (2.14) in this relation, we have

$$
\begin{aligned}
L = &\int_0^{2\pi} 2x_2(t,h) f_2(x_2(t,h), y_2(t,h)) + \\
&+ 2y_2(t,h) g_2(x_2(t,h), y_2(t,h))\mathrm{d}t + o(h^4).
\end{aligned}
\tag{2.16}
$$

Substituting $x_2(t,h), y_2(t,h)$ in $f_2(x,y)$, and $g_2(x,y)$ and then in (2.16) and using terms grouping up to the order h^4, we find

$$
L = L_1 h^4 + o(h^4), \tag{2.17}
$$

where $L_1/2$ is the 1th Lyapunov quantity $\mathbf{L_1}$:

$$
\mathbf{L_1} = \frac{\pi}{4}(
f_{11}f_{02} + 2f_{02}g_{02} - 2f_{20}g_{20} - g_{11}g_{20} - g_{11}g_{02} + f_{11}f_{20})
$$

Here the sign $\mathbf{L_1}$ characterizes an unwinding or a twisting of trajectory of the system $(x(t,h), y(t,h))$ on a phase plane.

We stress that for the computation of L_1 it is sufficient that in the neighborhood of considered stationary point the relation $f, g \in \mathbb{C}^2$ is satisfied, what is one less than conventional assumptions on a smoothness [Marsden & McCracken, 1976].

3 Conclusion

In conclusion, we note that there is a wide class of polynomial systems, for which by the proposed technique small cycles can be constructed (see, for example, [Bautin, 1952; Leonov, 1998; Lloyd & Pearson, 1997; Lynch, 2005; Yu & Han, 2005] and others).

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