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CRITERIA OF STABILITY BY THE FIRST APPROXIMATION FOR DISCRETE NONLINEAR SYSTEMS*

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The instability problem for discrete systems is considered. The criteria of instability by the first approximation for the flows of solutions are extended to cascades. The criteria of instability by Lyapunov and by Krasovsky are obtained.

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The present work is the extension of the work [1]. The results on instability by the first approximation [2–5] are extended here to discrete systems.

1. The Perron effects for discrete systems

We shall say that the sign reversal of the characteristic exponent of solutions of system by the first approximation and of the original system for the same initial data is the Perron effect [6]. The Perron effects for continuous systems are considered in [2–4, 6].

We give here a discrete analog of the assertion of Perron concerning to a sign reversal of characteristic exponents [1, 8]. In these works a nonlinear system, for which the linearization is exponentially stable and the solution of the original system is unstable by Lyapunov, is considered.

In the present work the changing of sign of characteristic exponents "on the contrary" is considered. We shall show that the positiveness of characteristic exponents of a system of first approximation is not a sufficient condition of instability for nonlinear systems.

Consider a system

$$\begin{aligned} x(t+1) &= \frac{\exp[(t+2)\sin\ln(t+2) - 2a(t+1)]}{\exp[(t+1)\sin\ln(t+1) - 2at]}x(t) + z(t) - y(t)^2 \\ y(t+1) &= e^{-a}y(t), \quad 1 < 2a < 1 + \tfrac{1}{2}e^{-\pi}, \\ z(t+1) &= G(y(t), z(t)), \end{aligned} \tag{1}$$

$$G(y,z) = \begin{cases} e^{-2a}z, & \text{for } z = y^2, \\ e^{-2a}y^2, & \text{for } z \neq y^2. \end{cases}$$

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A general solution of system (1), namely

$$\begin{aligned} x(t) &= x_0 \exp[(t+1)\sin\ln(t+1) - 2at], \\ y(t) &= y_0 e^{-at}, \\ z(t) &= z_0 e^{-2at}, \\ z_0 &= {y_0}^2 \end{aligned} \tag{2}$$

is Lyapunov stable.

Consider the characteristic exponents of solutions (2):

$$\mathcal{X}[x(t)] = 1 - 2a < 0,\ \mathcal{X}[y(t)] = -a < 0,\ \mathcal{X}[z(t)] = -2a < 0.$$

The linearization of system (1) along the zero solution for $t = 2, 3, ...$ has the form

$$\begin{aligned} x(t+1) &= \frac{\exp[(t+2)\sin\ln(t+2) - 2a(t+1)]}{\exp[(t+1)\sin\ln(t+1) - 2at]} x(t) + z(t), \\ y(t+1) &= e^{-a} y(t), \\ z(t+1) &= e^{-2a} z(t). \end{aligned} \tag{3}$$

For the characteristic exponent of the nonzero solution $x(t)$ the following estimate

$$\mathcal{X}[x(t)] \geq 1 - 2a + \frac{1}{2} e^{-\pi} > 0$$

is satisfied [8]. Therefore the zero solution of system (3) is Lyapunov unstable.

Thus, here for unstable linearization the Perron effect occurs.

2. Criteria of instability by the first approximation of a cascade of solutions

Consider a discrete system

$$x(t+1) = F(t, x(t)), \quad x(t) \in \mathbb{R}^n, \quad t = 0, 1, 2, ..., \tag{4}$$

where $F(\cdot, \cdot)$ is a twice continuously differentiable vector-function with respect to the second argument.

Consider the linearizations of system (4) along the solutions with the initial data x_0 from the open bounded in $\mathbb{R}^n$ set Ω

$$y(t+1) = \left.\frac{\partial F(t,x)}{\partial x}\right|_{x=x(t,x_0)} y(t), \tag{5}$$

where $x(0, x_0) = x_0$. Consider also a fundamental matrix of linear system (5)

$$Y(t, x_0), \quad Y(0, x_0) = I.$$

We assume that for the certain vector-function $\xi(t)$ and the scalar function $\alpha(t)$ the following relations

$$\begin{aligned} &\left|\xi(t)\right| = 1, \qquad \forall t = 0, 1, 2..., \\ &\inf_{x(0)\in\Omega} \left|Y(t, x(0))\xi(t)\right| \geq \alpha(t) \end{aligned} \tag{6}$$

are satisfied.

The following theorem is the extension of the instability criterion by the first approximation for the flows of smooth dynamical systems [5].

Theorem 1. *Suppose, for the function $\alpha(t)$ the following relation*

$$\overline{\lim_{t\to+\infty}}\ \alpha(t) = +\infty \tag{7}$$

is valid. Then all solutions $x(t, x(0))$, $x(0)\in\Omega$ are Lyapunov unstable.

Proof. Hold the certain pair $x(0) \in \Omega$ and $t \in N_0$ fixed. Consider the δ-neighborhoods of the point x_0 and choose the vector $y(0)$ such that

$$x(0) - y(0) = \delta\xi(t).$$

Let the number δ be so much smaller that $\{w||w-x(0)|\leq\delta\}\subset\Omega$.

It is well known [9] that for any fixed numbers t, i, and the vectors x_0, y_0 there exists the vector $w_i \in \mathbb{R}^n$ such that

$$\begin{aligned} |x_0 - w_i| &\leq |x_0 - y_0|, \\ x^i(t, x_0) - x^i(t, y_0) &= Y^i(t, w_i)(x_0 - y_0). \end{aligned} \tag{8}$$

Here $Y^i(t, z)$ is the i-th row of the matrix $Y(t, z)$ and $x^i(t, x_0)$ is the i-th component of the vector $x(t, x_0)$.