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STABILITY BY THE FIRST APPROXIMATION FOR DISCRETE SYSTEMS*

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The problem of stability by the first-order approximation for a discrete system is considered. For a discrete system, possessing by a regular linearization with negative Lyapunov exponents, the stability theorem is proved. In the discrete case the analog of the Perron contrpexample, which implies that the regular condition is substantial, is constructed.

In the present work the discrete analog of the results of A. M. Lyapunov and O. Perron, concerning the problem of the stability by the first-order approximation, is obtain.

1. Characteristic exponents of functions

Consider the function $f(t)$, $t = 0, 1, 2 \dots$, and introduce the following notions [1]:

Definition 1. The number or the symbol $-\infty(+\infty)$, defined by formula

$$\mathcal{X}[f] = \overline{\lim_{t\to\infty}} \frac{1}{t} \ln |f(t)|, \tag{1}$$

is called a Lyapunov characteristic exponent.

The characteristic exponent of the function $f(t)$ (introduced by Lyapunov [3]), taken with the opposite sign, is the characteristic exponent α.

The obvious properties of finite characteristic exponents are the following:

1) $\mathcal{X}[f(t)] = \mathcal{X}[|f(t)|]$;

2) $\mathcal{X}[cf(t)] = \mathcal{X}[f(t)]$, $(c \neq 0)$;

3) $f(t) < F(t)\ t > T \Rightarrow \mathcal{X}[f(t)] \leq \mathcal{X}[F(t)]$;

4) $\mathcal{X}\left[\sum\limits_{k=1}^{m} f_k(t)\right] \leq \max\limits_{k} \mathcal{X}[f_k(t)]$;

5) $\mathcal{X}\left[\prod\limits_{k=1}^{m} f_k(t)\right] \leq \sum\limits_{k=1}^{m} \mathcal{X}[f_k(t)]$.

Definition 2. The characteristic exponent of $f(t)$ is said to be strict if there exists a finite limit such that

$$\mathcal{X}[f(t)] = \lim_{t\to\infty} \frac{1}{t} \ln |f(t)|.$$

Then it is obvious that $f(t) \neq 0$ for $t > T$. For the strict characteristic exponent of $f(t)$ we have

6) $\mathcal{X}\left[\frac{1}{f}\right] = -\mathcal{X}[f]$.

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$$|g(t,x)| \le \kappa |x|^m, \quad m > 1, \tag{19}$$

then the solution $x(t) \equiv 0$ *of system* (15) *is asymptotically stable.*

Proof. By (18), (19) we have

$$|x(t)| \le |Y(t)||x(0)| + |Y(t)| \sum_{n=0}^{t-1} |Y(n+1)^{-1}|\kappa |x(n)|^m.$$

By the data $a_n < -a < 0$. This implies that there exists the number c_Y such that

$$|Y(t)| \le c_Y e^{-at}. \tag{20}$$

Taking into account the estimate of the matrix for regular system, we get

$$|x(t)|e^{at} \le c_Y |x(0)| + \sum_{n=0}^{t-1} c_0 \kappa e^{\varepsilon(n+1)} e^{a(n+1)} |x(n)|^m.$$

Putting $|x(t)|e^{at} = y(t)$, we obtain

$$|y(t)| \le c_Y |y(0)| + \sum_{n=0}^{t-1} c_0 \kappa e^{(\varepsilon+a)} e^{n(\varepsilon - a(m-1))} |y(n)|^m. \tag{21}$$

Since (21) is valid for any $\varepsilon > 0$, $(m-1)$ and the values a are positive, the number ε exists such that the following inequality

$$(\varepsilon - a(m-1)) < 0$$

is satisfied. Then, using relation (14) for (21) in the case $r = e^{(\varepsilon - a(m-1))}$, $C_r = c_0 \kappa e^{(\varepsilon+a)}$, we obtain that $|y(t)| < 1$. This implies that $|x(t)| \to 0$, as $m \to +\infty$. ■

Remark. Replacing (16) by more limitary conditions

$$\|A(t)^{-1}\| \le C < +\infty, \quad \|A(t)\| \le C < +\infty,$$

we obtain the result, described in [2].

5. A discrete analog of the Perron contrexample

We prove that if the equation of the first-order approximation is implicitly dependent of t, then the negativeness of the Lyapunov exponents is a nonsufficient condition for the stability of the solution of the original system. For the continuous case such a contrexample is constructed in [4].

Consider a discrete system

$$\begin{aligned} x(t+1) &= e^{-a}x(t) \quad 3 < 2a < 1 + \frac{1}{2}e^{-\pi}, \\ y(t+1) &= \frac{\exp((t+2)\sin\ln(t+2) - 2a(t+1))}{\exp((t+1)\sin\ln(t+1) - 2at)} y(t) + x(t)^2 \end{aligned} \tag{22}$$

and the stability problem for a zero solution of this system. We prove that for a linearized discrete system the Lyapunov exponents are less than null, a zero solution is asymptotically stable, and the solution of the original system is unstable.

A linearized discrete system is as follows

$$\begin{aligned} x(t+1) &= e^{-a}x(t), \\ y(t+1) &= \frac{\exp((t+2)\sin\ln(t+2) - 2a(t+1))}{\exp((t+1)\sin\ln(t+1) - 2at)} y(t). \end{aligned} \tag{23}$$

The solution of system (23), namely

$$\begin{aligned} x(t) &= c_1 e^{-at}, \\ y(t) &= c_2 \exp((t+1)\sin\ln(t+1) - 2at), \end{aligned} \tag{24}$$

is asymptotically Lyapunov stable since $1 < 2a < 1 + \frac{1}{2}e^{-\pi}$ and the Lyapunov exponents of system (23), which are $-a$ and $1 - 2a$, are less than null. Prove now that the solution of system (22)

$$\begin{aligned} x(t) &= c_1 e^{-at}, \\ y(t) &= c_3 \exp((t+1)\sin\ln(t+1) - 2at) \left(\sum_{k=0}^{t-1} \exp(-(k+2)\sin\ln(k+2) + 2a) \right) \end{aligned} \tag{25}$$