The problem of stability by the first-order approximation for a discrete system is considered. For a discrete system, possessing by a regular linearization with negative Lyapunov exponents, the stability theorem is proved. In the discrete case the analog of the Perron contrпexample, which implies that the regular condition is substantial, is constructed.

In the present work the discrete analog of the results of A. M. Lyapunov and O. Perron, concerning the problem of the stability by the first-order approximation, is obtain.

1. Characteristic exponents of functions
Consider the function $f(t)$, $t = 0, 1, 2, \ldots$, and introduce the following notions [1]:

**Definition 1.** The number or the symbol $-\infty(+\infty)$, defined by formula

$$
\mathcal{X}[f] = \lim_{t \to \infty} \frac{1}{t} \ln |f(t)|,
$$

is called a Lyapunov characteristic exponent.

The characteristic exponent of the function $f(t)$ (introduced by Lyapunov [3]), taken with the opposite sign, is the characteristic exponent $\alpha$.

The obvious properties of finite characteristic exponents are the following:
1) $\mathcal{X}[c f(t)] = \mathcal{X}[f(t)]$;
2) $\mathcal{X}[c f(t)] = \mathcal{X}[f(t)]$, $(c \neq 0)$;
3) $f(t) < F(t)$ $t > T \Rightarrow \mathcal{X}[f(t)] \leq \mathcal{X}[F(t)]$;
4) $\mathcal{X}\left[\sum_{k=1}^{m} f_k(t)\right] \leq \max_{k} \mathcal{X}[f_k(t)]$;
5) $\mathcal{X}\left[\prod_{k=1}^{m} f_k(t)\right] \leq \sum_{k=1}^{m} \mathcal{X}[f_k(t)]$.

**Definition 2.** The characteristic exponent of $f(t)$ is said to be strict if there exists a finite limit such that

$$
\mathcal{X}[f(t)] = \lim_{t \to \infty} \frac{1}{t} \ln |f(t)|.
$$

Then it is obvious that $f(t) \neq 0$ for $t > T$. For the strict characteristic exponent of $f(t)$ we have

6) $\mathcal{X}\left[\frac{1}{f}\right] = -\mathcal{X}[f]$.

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we obtain the result, described in [2].

Then, using relation (14) for (21) in the case of a discrete system, we get

\[
|Y(t)| \leq c_Y e^{-at}.
\]

Taking into account the estimate of the matrix for regular system, we get

\[
|x(t)| e^{at} \leq c_Y |x(0)| + \sum_{n=0}^{t-1} c_0 c e^{(n+1)} e^{a(n+1)} |x(n)|^m.
\]

Putting \( |x(t)| e^{at} = y(t) \), we obtain

\[
|y(t)| \leq c_Y |y(0)| + \sum_{n=0}^{t-1} c_0 c e^{(n+1)} e^{a(n+1)} |y(n)|^m.
\]

Since (21) is valid for any \( \varepsilon > 0 \), \( m - 1 \) and the values \( a \) are positive, the number \( \varepsilon \) exists such that the following inequality

\[
(\varepsilon - a(m - 1)) < 0
\]

is satisfied. Then, using relation (14) for (21) in the case \( r = e^{(\varepsilon - a)(m-1)} \), \( C_r = c_0 c e^{(\varepsilon - a)} \), we obtain that \( |y(t)| < 1 \). This implies that \( |x(t)| \to 0 \), as \( m \to +\infty \).

**Remark.** Replacing (16) by more limitary conditions

\[
\|A(t)^{-1}\| \leq C < +\infty, \quad \|A(t)\| \leq C < +\infty,
\]

we obtain the result, described in [2].

5. **A discrete analog of the Perron contreexample**

We prove that if the equation of the first-order approximation is implicitly dependent of \( t \), then the negativeness of the Lyapunov exponents is a nonsufficient condition for the stability of the solution of the original system. For the continuous case such a contreexample is constructed in [4].

Consider a discrete system

\[
\begin{align*}
x(t + 1) &= e^{-a} x(t) \quad 3 < 2a < 1 + \frac{1}{2} e^{-\pi}, \\
y(t + 1) &= \frac{\exp((t + 2) \sin (t + 2) - 2a(t + 1))}{\exp(t + 1) \sin (t + 1) - 2at} y(t) + x(t)^2
\end{align*}
\]

and the stability problem for a zero solution of this system. We prove that for a linearized discrete system the Lyapunov exponents are less than null, a zero solution is asymptotically stable, and the solution of the original system is unstable.

A linearized discrete system is as follows

\[
\begin{align*}
x(t + 1) &= e^{-a} x(t), \\
y(t + 1) &= \frac{\exp((t + 2) \sin (t + 2) - 2a(t + 1))}{\exp(t + 1) \sin (t + 1) - 2at} y(t).
\end{align*}
\]

The solution of system (23), namely

\[
\begin{align*}
x(t) &= c_1 e^{-at}, \\
y(t) &= c_2 \exp((t + 1) \sin (t + 1) - 2at),
\end{align*}
\]

is asymptotically Lyapunov stable since \( 1 < 2a < 1 + \frac{1}{2} e^{-\pi} \) and the Lyapunov exponents of system (23), which are \(-a\) and \(1 - 2a\), are less than null. Prove now that the solution of system (22)

\[
\begin{align*}
x(t) &= c_1 e^{-at}, \\
y(t) &= c_3 \exp((t + 1) \sin (t + 1) - 2at) \left( \sum_{k=0}^{t-1} \exp(-(k + 2) \sin (k + 2) + 2a) \right)
\end{align*}
\]